Relative monodromy of abelian logarithms for finite covers of universal families

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Abstract

Let's fix a complex abelian scheme endowed with a non-torsion section and with a finite surjective modular map onto a universal family of abelian varieties (with some fixed level-ℓ-structure and without locally constant parts). We show that the relative monodromy group of the abelian logarithm is non-trivial and of full rank. As a consequence we deduce a new proof of Manin's kernel theorem and of the algebraic independence of the coordinates of abelian logarithms with respect to the coordinates of periods.

1 Introduction

A g-dimensional abelian scheme $\phi\colon \mathcal{A} \to S$ over $\mathbb{C}$ can be seen as family of g-dimensional complex abelian varieties $\{\mathcal{A}_s\}_{s\in S(\mathbb{C})}$ (the fibres of ϕ) that "varies algebraically accordingly to ϕ" over the parameterizing space $S(\mathbb{C})$. Each fibre $\mathcal{A}_s$ can be analytically identified with a complex torus $\mathbb{C}^g/\Lambda_s$. Locally, on simply connected subsets $U \subset S(\mathbb{C})$ it is possible to define a holomorphic period map $\mathfrak{P} : U \to \mathrm{Lie}(\mathcal{A})^{2g}$ where $\mathrm{Lie}(\mathcal{A})$ is the complex lie algebra bundle of $\mathcal{A}(\mathbb{C})$. The period map associates to any point $s \in U$ a $\mathbb{R}$-basis $\mathfrak{P}(s)$ of the lattice Λ_s and in general it cannot be globally defined on $S(\mathbb{C})$. The obstruction to the existence of the analytic prolongation of $\mathfrak{P}$ along a loop around $s \in S(\mathbb{C})$ is measured by the monodromy group $\mathrm{Mon}(\mathcal{A}, s)$. Given an algebraic section of the abelian scheme, which is a $\mathbb{C}$-morphism $\sigma\colon S \to \mathcal{A}$, again locally on simply connected open subsets $U' \subset S(\mathbb{C})$ it is possible to lift σ through the exponential map $\exp\colon \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}(\mathbb{C})$ to a holomorphic map $\log_\sigma\colon U' \to \mathrm{Lie}(\mathcal{A})$. This map is called an abelian logarithm of the section σ. On a common open set $V \subset S(\mathbb{C})$ where both $\log_\sigma$ and $\mathfrak{P}$ are defined, we can express $\log_\sigma(s) \in \mathbb{C}^g$ in terms of the basis $\mathfrak{P}(s)$ of the lattice Λ_s. The coordinates form a real-analytic function $\beta_\sigma\colon V \to \mathbb{R}^{2g}$ which is nowadays called the Betti map (associated to σ) and that turned out to be of crucial importance in the theory of unlikely intersections. Such map was implicitly considered by Manin in [23], to get a proof of Mordell's conjecture in the function field case. The aforementioned abelian logarithm $\log_\sigma$ is in general defined only locally, thus it is interesting to measure the obstruction to the existence of a global abelian logarithm: the simultaneous monodromy of the period map and abelian logarithm of σ gives rise to the monodromy group of the section M_σ. The knowledge of $\mathrm{Mon}(\mathcal{A}, s)$ and the monodromy group of the logarithm of a section gives relevant information on the structure of the abelian scheme and on the section, respectively: in fact, no non-zero periods can be globally defined when the abelian scheme has no fixed part (see [15, Lemma 5.6]) and the monodromy of a logarithm is trivial only when the section is the zero-section of the scheme. It is usually important to understand more on the structure of monodromy groups. In [8] an [26] the authors construct examples where the monodromy group $\mathrm{Mon}(\mathcal{A}, s)$ is of infinite index in the arithmetic group attached to its Zariski closure, i.e. thin. Also in [30] it is pointed out that thin groups often arise in the context of monodromy groups. Given a finitely generated group in $\mathrm{GL}_n(\mathbb{Z})$, usually it's not too difficult to compute its Zariski closure, but deciding if the group is thin can be very subtle and difficult. Thus, in general it is very interesting to determine under which hypotheses the monodromy groups are big in their Zariski closures.

Main results of this paper. We investigate the obstruction to have a globally defined abelian logarithm along loops inducing a trivial monodromy action on the period map. This obstruction is measured by the relative monodromy group M_σ^{rel}. It is easy to check that M_σ^{rel} is a $\mathbb{Z}$-submodule of $\mathbb{Z}^{2g}$, so it is interesting to understand how its rank is related to σ. The group M_σ^{rel} gives relevant information on the section, in particular its non-triviality should be related to the property of σ of being non-torsion, and its rank should give information on the minimal group subscheme containing the image of σ.

In the case of non-isotrivial elliptic schemes Corvaja and Zannier proved in [9] that when σ is a non-torsion section M_σ^{rel} is nontrivial and moreover it is isomorphic to $\mathbb{Z}^2$ (so it has full rank). In the same paper, they provided an example showing that the relative monodromy group could be smaller than expected (even trivial) when compared to the Zariski closures of non-relative monodromy groups: therefore, there are some groups which cannot arise as monodromy groups in this setting. Moreover, we point out that the second author of the present paper in [33] studied the relative monodromy for products of two elliptic schemes. In [34] he also gives an explicit proof of the results of Corvaja and Zannier.

In general it is conjectured that for any abelian scheme $\phi\colon \mathcal{A} \to S$ without fixed part the rank of M_σ^{rel} is $2g'$ where g' is the relative dimension of the minimal group subscheme of $\mathcal{A} \to S$ containing the image of the section σ. In this paper we give a positive answer to this conjecture in the case of an abelian scheme admitting a finite base change endowed with a surjective finite map onto a universal family $\mathfrak{A}_g \to \mathbb{A}_g$ with some fixed level-ℓ-structure and without locally constant parts. Specifically we prove the following bigness results about relative monodromy groups:

Theorem 1.1. *Let $\phi : \mathcal{A} \to S$ be an abelian scheme such that (up to a finite base change) the modular map $p : S \to \mathbb{A}_g$ is a finite surjective morphism. If $\sigma : S \to \mathcal{A}$ is a non-torsion section, then the relative monodromy group M_σ^{rel} is non-trivial.*

Theorem 1.2. *Let $\phi : \mathcal{A} \to S$ be an abelian scheme such that (up to a finite base change) the modular map $p : S \to \mathbb{A}_g$ is a finite surjective morphism. If $\sigma : S \to \mathcal{A}$ is a non-torsion section, then the relative monodromy group M_σ^{rel} is isomorphic to $\mathbb{Z}^{2g}$.*

Notice that in our statements we don't need to distinguish the cases when the image of the section σ is contained in proper group subschemes since having a finite map onto a universal family prevents our abelian scheme from having any non-trivial proper subscheme.

Application I: properties of the Betti map. These theorems have immediate implications on the arithmetic properties of the sections of abelian schemes, in particular on some properties of the Betti map. On one hand periods and abelian logarithms in general cannot be globally defined, but on the other hand it's also true that they can be always be globally defined after going to the universal cover $\widetilde{S}$ of the parameter base S, so the Betti map is also globally defined on $\widetilde{S}$. Let us denote by $S_* \to S$ the minimal unramified cover where periods are globally defined and by $S_\sigma \to S_*$ the minimal unramified cover where abelian logarithm of σ is globally defined: then, $S_\sigma \to S_*$ is also the minimal unramified cover of S_* where the Betti map is globally defined and we have the following situation

$$\widetilde{S} \to S_\sigma \to S_* \to S\,.$$

The map $S_* \to S$ is a Galois cover whose Galois group is $\mathrm{Mon}(\mathcal{A}, s)$, while the map $S_\sigma \to S$ is a Galois cover with Galois group M_σ. Our main theorems determine properties of the Galois group M_σ^{rel} of the Galois cover $S_\sigma \to S_*$, giving precise information about the global definition of the Betti map which is useful to know for several applications.

For instance, it is straightforward to see that if σ is a torsion section then the Betti map of σ is constant and thus globally defined on the base. The converse statement is also true but more difficult to prove, it is widely known as Manin's kernel theorem. As a consequence of Theorem 1.1, we obtain a new proof of a stronger version of Manin's kernel theorem (under our hypotheses). We call it "strong version" because in its statement we just assume the Betti coordinates to be globally defined instead of rational constant a priori.

Application II: the functional transcendence step in the Pila-Zannier method. The Betti map is used in the work [28] of Pila and Zannier together with a result of Pila and Wilkie in transcendental Diophantine geometry (see [27]) to give a new proof of the Manin-Mumford conjecture. Such a strategy is referred as "the Pila-Zannier method"; in the case of abelian schemes over $\overline{\mathbb{Q}}$ it works in the following way: the goal is to have a control on the distribution of torsion values of σ (i.e. elements of $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$) with bounded heights. First one gives lower bounds on the number of such torsion values using Galois conjugates and the height inequality of Dimitrov-Gao-Habegger. On the other hand the Betti map β_σ transforms the torsion values in rational points of a definable set in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. At this point a "Pila-Wilkie type" result gives a control on the rational points on the transcendental part of such definable set. Still we have no information on the rational points in the algebraic part of the definable set, but here the crucial point is to use some functional transcendence theorem of "Ax-Schanuel type" to control the algebraic part of the definable set. Let's take a closer look at the last step involving the functional transcendence arguments: a classical way to tackle this step is by using a result of André (see [1, Theorem 3]). It says that the coordinates of the abelian logarithm $\log_\sigma$ are pairwise algebraically independent over the extension of $\mathbb{C}(S)$ generated by the coordinates of the period map $\mathfrak{P}$. Such result has often very strong implications, for instance in [13], thanks to such result, it is shown that the algebraic part of the definable set constructed with the Pila-Zannier method is empty.

Such type of functional transcendence results are related to the study of the relative monodromy. In fact André's result can be interpreted in terms of differential Galois theory. More precisely, considering directional derivatives along a tangent vector field ∂ with respect to the Gauss-Manin connection, the transcendence degree of the extension

$$\mathbb{C}(S)(\mathfrak{P}, \partial\mathfrak{P})(\log_\sigma, \partial \log_\sigma)/\mathbb{C}(S)(\mathfrak{P}, \partial\mathfrak{P})$$

is equal to the dimension of the kernel of the homomorphism from the differential Galois group of $\mathbb{C}(S)(\mathfrak{P}, \partial\mathfrak{P})(\log_\sigma, \partial \log_\sigma)/\mathbb{C}(S)$ to the differential Galois group of $\mathbb{C}(S)(\mathfrak{P}, \partial\mathfrak{P})/\mathbb{C}(S)$. Recall that by general theory, the differential Galois group of a Picard-Vessiot extension is exactly the Zariski closure of the corresponding monodromy group. In [9] it is pointed out that André's theorem gives no information about the relative monodromy group M_σ^{rel}. Thus, Theorem 1.2 can be interpreted as a strengthening of André's theorem (under our hypotheses). We shall in fact prove how André's result is a consequence of Theorem 1.2, and this in turns leads to a new proof of the algebraic independence of the coordinates of $\log_\sigma$ with respect to periods.

Further developments. After the works of André, Corvaja, Masser and Zannier (see for instance [2], [7], [9], [37]) the Betti map became a standard tool in Diophantine geometry. Remarkably, Dimitrov, Gao and Habbegger in [12] used the Betti map and a novel height inequality (then generalized by Yuan and Zhang in [36]) to prove a uniform version of the Mordell-Lang conjecture. The Pila-Zannier method was in turns employed by Gao and Habbegger in combination with new ideas involving some "degeneracy loci", to solve the relative Manin-Mumford conjecture (see [17]). Other very recent applications of the Betti map can be found in the works of Xie and Yuan in [35] towards the geometric Bombieri-Lang conjecture.

Our hope for future developments is twofold. First of all, the main theorems of this paper could be applied for the solution of some "special points problems" when in the functional transcendence step of the Pila-Zannier method André's theorem fails to give a control on rational points of the algebraic part of the constructed definable set. It is is not clear yet if such cases exist and if they are interesting. Secondly, it would be nice to remove the main hypothesis on the surjectivity of the modular map $p\colon S \to \mathbb{A}_g$. Such assumption is necessary in order to apply the generalization of Shioda's theorem due to Mok and To (see [25, Main Theorem]). This result has in fact been proved only for Kuga families. One could think about a different proof of our main theorems that doesn't invoke [25], but at the moment the required strategy is not entirely understood.

Strategy of the proofs. The approach of our proofs was originally inspired by [9], and it is based on the interpretation of the obstruction to globally define a logarithm as a Galois cohomology class with respect to the monodromy action of the fundamental group of $S(\mathbb{C})$ on $\mathbb{Z}^{2g}$. The new key ideas consist of using the Lefschetz hyperplane theorem for quasi-projective varieties (see [18, 2.2]) and a theorem of Mok and To (see [25, Main Theorem]). In the first step of the proof of Theorem 1.1 we construct an intermediate abelian scheme $\mathcal{A}' \to S'$ which factorizes the universal diagram induced by the modular map $p : S \to \mathbb{A}_g$. This construction is carried out by means of topological techniques in such a way that $\mathcal{A}' \to S'$ is a finite unramified base change of the universal family (a Kuga family). In the second step we find a suitable curve inside $S(\mathbb{C})$ which preserves the complete monodromy action of the fundamental group of $S(\mathbb{C})$ on periods and logarithms. The curve is obtained after iterating the aforementioned Lefschetz theorem and turns out to be a Stein space. This last property allows us to construct holomorphic sections of the Kuga family coming from a Galois cohomology class describing the obstruction of the global logarithms. Then the proof follows by a contradiction argument: we assume the triviality of the relative monodromy group for a non-torsion section σ, then by using the previous apparatus and the properties of trace and pull-back operators, we deduce that σ has a global logarithm. This finally contradicts Lang-Néron theorem on the Mordell-Weil group of the generic fiber. This proves the non-triviality of M_σ^{rel}.

In order to prove Theorem 1.2 on the maximality of the rank of M_σ^{rel} the main key ingredient is to exploit the irreducibility of the monodromy action on the lattice of periods.

Overview of the paper. In Section 2 we define all the main objects and we collect the results coming from the general theory. In particular, we give a detailed account of the various monodromy groups. In Section 3 we carry out the main proofs after some technical results about "the invariance" of our main theorems. Section 4 is devoted to some useful applications of our results.

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2 Monodromy on abelian schemes

2.1 Setting

In general if X is a quasi-projective complex algebraic variety, with $X(\mathbb{C})$ we denote the set of its closed points which has a structure of complex manifold.

Let S be a regular, irreducible, quasi-projective variety defined over $\mathbb{C}$. Let $\phi : \mathcal{A} \to S$ be an abelian scheme over S of relative dimension $g \geq 1$, namely a proper smooth group S-scheme admitting a "zero section" $\sigma_0 : S \to \mathcal{A}$ and whose fibers are g dimensional abelian varieties. For any $s \in S(\mathbb{C})$ the fiber over s is denoted by $\mathcal{A}_s$. We denote with Σ^{an} the sheaf over $S(\mathbb{C})$ of complex analytic sections of ϕ whereas $\Sigma(S) \subset \Sigma^{\text{an}}(S(\mathbb{C}))$ is the abelian group of (global) algebraic $\mathbb{C}$-sections.

In Section 3.1 we will show that our main theorems are insensitive to finite base changes of S and to isogenies. Hence, by using for instance [12, Proof of Theorem B.1 Devissages $(iv) - (vi)$] we can assume that the abelian scheme carries principal polarization and it has level-ℓ-structure.

In this paper, sometimes we also need to assume an analytic viewpoint on families of abelian varieties: we look at $S(\mathbb{C})$ and $\mathcal{A}(\mathbb{C})$ as normal quasi-projective varieties with a surjective holomorphic map $\phi : \mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$ whose fibers are abelian varieties, thus the zero section $\sigma_0 : S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ and the fiberwise group operations are complex analytic. The previous family can be regarded as a geometric model of a polarized abelian variety A defined over the function field $\mathbb{C}(S)$ of meromorphic functions on $S(\mathbb{C})$: from the latter perspective a rational point σ of A over $\mathbb{C}(S)$ is equivalently a complex analytic section of $\phi : \mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$. Moreover, we can consider normal projective compactifications $S(\mathbb{C}) \hookrightarrow \overline{S}(\mathbb{C})$ and $\mathcal{A}(\mathbb{C}) \hookrightarrow \overline{\mathcal{A}}(\mathbb{C})$ such that ϕ extends to a complex analytic map $\overline{\phi} : \overline{\mathcal{A}}(\mathbb{C}) \to \overline{S}(\mathbb{C})$; suppose furthermore that the zero section extends meromorphically to $\overline{S}(\mathbb{C})$ and that the fiberwise group operations over $S(\mathbb{C})$ extends to meromorphic maps on $\overline{\mathcal{A}}(\mathbb{C})$. Then, $\overline{\phi} : \overline{\mathcal{A}}(\mathbb{C}) \to \overline{S}(\mathbb{C})$ can be regarded as a geometric model of a polarized abelian variety A defined over a function field $\mathbb{C}(\overline{S})$. The rational points $A(\mathbb{C}(\overline{S}))$ of A over $\mathbb{C}(\overline{S})$ are then in one-to-one correspondence with the set of meromorphic sections of $\overline{\mathcal{A}}$ over $\overline{S}$ which are complex analytic over $S(\mathbb{C})$: in fact, $\sigma|_S : S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ is necessarily holomorphic outside a set of complex codimension ≥ 2 on $S(\mathbb{C})$; on the other hand, over a small simply connected open set U on $S(\mathbb{C})$, $\phi^{-1}(U)$ is uniformized by $U \times \mathbb{C}^d$. It follows by lifting $\sigma|_U : U \to \phi^{-1}(U)$ to $U \times \mathbb{C}^d$ and the Hartogs Extension Theorem for holomorphic functions that in fact σ is everywhere holomorphic in $S(\mathbb{C})$. Notice that in particular the set of meromorphic sections of $\overline{\mathcal{A}}(\mathbb{C})$ over $\overline{S}(\mathbb{C})$ which are holomorphic over $S(\mathbb{C})$

agrees with the set of algebraic sections of $\phi : \mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$. By the Mordell-Weil Theorem over complex function fields, the group $A(\mathbb{C}(\overline{S}))$ is finitely generated provided that it has no constant parts.

Fix an integer $\ell \geq 3$. Now we use two points of view to describe the moduli space of g-dimensional principally polarized abelian varieties, with level-ℓ-structure. First we use scheme theory. There exist an irreducible smooth quasi-projective variety $\mathbb{A}_g$ over $\overline{\mathbb{Q}}$ and a principally polarized abelian scheme $\mathfrak{A}_g \to \mathbb{A}_g$ of relative dimension g with symplectic level-ℓ-structure with the following property: if S is any scheme over $\mathbb{C}$ and $\mathcal{A} \to S$ is a principally polarized abelian scheme of relative dimension g with symplectic level-ℓ-structure, then there exists a unique $\mathbb{C}$-morphism $p : S \to \mathbb{A}_g$ such that $\mathcal{A}$ is isomorphic to the pull-back $\mathfrak{A}_g \times_{\mathbb{A}_g} S$ (i.e. $\mathbb{A}_g$ is a fine moduli space). Let $\pi : \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety and write $p_{\mathcal{A}}$ for the induced $\mathbb{C}$-morphism $\mathcal{A} \to \mathfrak{A}_g$. Also $\mathfrak{A}_g$ is an irreducible, smooth, quasi-projective variety definable over a number field. In other words, we are supposed to have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{p_{\mathcal{A}}} & \mathfrak{A}_g \\ \phi\downarrow & & \downarrow\pi \\ S & \xrightarrow{p} & \mathbb{A}_g. \end{array}$$

Now, let's assume a more analytic viewpoint on moduli spaces introducing modular families of abelian varieties. We denote by $\mathbb{H}_g$ the Siegel upper-half plane:

$$\mathbb{H}_g := \{\tau \in \mathrm{Sym}(g, \mathbb{C}) \mid \mathrm{Im}\, \tau > 0\}.$$

The isomorphism classes of principally polarized abelian varieties are in one-to-one correspondence with $\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$ where $\mathrm{Sp}_{2g}(\mathbb{Z})$ is a discrete subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$. To any torsion-free subgroup $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ of finite index one can associate a universal family $\pi_\Gamma : \mathcal{A}_\Gamma(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ of principally polarized abelian varieties: following Satake we call such a family π_Γ a Kuga family of principally polarized abelian varieties. Without loss of generality, we always assume that Γ is torsion-free. In the case of Kuga families $\pi_\Gamma : \mathcal{A}_\Gamma(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ we will always take $\overline{S}_\Gamma(\mathbb{C})$ to be the Satake-Baily-Borel compactification: when $S_\Gamma(\mathbb{C})$ is a noncompact curve and the Kuga family is irreducible then $\pi_\Gamma : \mathcal{A}_\Gamma(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ is isomorphic to a modular elliptic scheme, which one can compactify as an elliptic modular surface; in all other irreducible cases $\overline{S}_\Gamma(\mathbb{C}) - S_\Gamma(\mathbb{C})$ are of codimension ≥ 2 in $\overline{S}_\Gamma(\mathbb{C})$ so that the choice of a completion $\overline{\pi}_\Gamma : \overline{\mathcal{A}}_\Gamma(\mathbb{C}) \to \overline{S}_\Gamma(\mathbb{C})$ is unimportant. With respect to any such compactification, the zero section σ_0 and the fiberwise group operations extend meromorphically. Thus, associated to each Kuga family $\pi : \mathcal{A}_\Gamma(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ we have a well-defined modular polarized abelian variety A_Γ over the modular function field $\mathbb{C}(\overline{S}_\Gamma)$. Given a principally polarized abelian scheme $\phi : \mathcal{A} \to S$ of relative dimension g with level ℓ-structure, there is a congruence subgroup

$$\Gamma = \Gamma(\ell) \subset \mathrm{Sp}_{2g}(\mathbb{R})$$

and a unique modular map $p : S \to \mathbb{H}_g/\Gamma = S_\Gamma$ such that $\phi : \mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$ is isomorphic to the pull-back of the Kuga family $\pi : \mathcal{A}_\Gamma(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ by p. Since Γ is torsion-free applying the extension theorem of Borel, the modular map $p : S(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ always extends to a holomorphic map $\overline{p} : \overline{S}(\mathbb{C}) \to \overline{S}_\Gamma(\mathbb{C})$, where $\overline{S}_\Gamma(\mathbb{C})$ denotes the Satake-Baily-Borel compactification of $S_\Gamma(\mathbb{C})$; in particular, this defines a compactification $\overline{\phi} : \overline{\mathcal{A}}(\mathbb{C}) \to \overline{S}(\mathbb{C})$, which is a geometric model for a polarized abelian variety A over the complex function field $\mathbb{C}(\overline{S})$.

For our purposes we are interested in some vanishing results of the Mordell-Weil rank. Now, we fix once and for all a suitable level $\ell \geq 3$ such that the Kuga family $\overline{\pi}_\Gamma : \overline{\mathcal{A}}_\Gamma(\mathbb{C}) \to \overline{S}_\Gamma(\mathbb{C})$ has no locally constant parts, i.e. there is no constant part and the same is true even when Γ is replaced by any subgroup of finite index; in particular, observe that a base change of such a family performed via an unramified cover of the base is still a Kuga family with same properties. In [25, Main Theorem], the authors provided a finiteness result about the number of meromorphic sections of such Kuga families, which we restate in the following form:

Theorem 2.1 (Mok-To, 1993)**.** *Let $\pi_\Gamma : \mathcal{A}_\Gamma(\mathbb{C}) \to S_\Gamma(\mathbb{C})$ be a Kuga family of polarized abelian varieties without locally constant parts. Every algebraic section $\sigma : S_\Gamma(\mathbb{C}) \to \mathcal{A}_\Gamma(\mathbb{C})$ is torsion.*

Remark 2.2. Particular instances of Theorem 2.1 are provided in [29] for special Kuga families and in [31, Theorem 5.1] in the case of elliptic modular surfaces. If $A(\mathbb{C}(\overline{S}))$ has additional structure, such as having extra endomorphisms, the classifying space S_Γ should be replaced by some other modular variety; and some analogous results to Theorem 2.1 have been obtained by Silveberg in the setting of Shimura varieties (see [32]). For an alternative proof of [31, Theorem 5.1] see [9, Theorem 2.5]).

One can study the $\mathbb{Z}$-rank of $A(\mathbb{C}(\overline{S}))$ in terms of the modular map $p : S \to S_\Gamma$ for any Kuga family $\pi_\Gamma : \mathcal{A}_\Gamma \to S_\Gamma$ of polarized abelian varieties. In particular, notice that in the case when $p(S(\mathbb{C}))$ is totally geodesic and the mapping $p : S(\mathbb{C}) \to p(S(\mathbb{C}))$ is unramified, then $\phi : \mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$ is itself a Kuga family and we still have the vanishing of the rank of the Mordell-Weil group. The general expectation is that the Mordell-Weil rank can be estimated in terms of the second fundamental form and/or the ramification locus of the modular map. In the case of a ramified covering $\overline{p} : \overline{S} \to \overline{S}_\Gamma$, Mok announced an upper bound of the Mordell-Weil rank in terms of the ramification divisor (see [24, Chapter 3, Theorem 2′]).

2.2 Periods and abelian logarithms

Let us consider an abelian scheme $\phi : \mathcal{A} \to S$ with a zero-section as above. Each fiber $\mathcal{A}_s(\mathbb{C})$ is analytically isomorphic to a complex torus $\mathbb{C}^g/\Lambda_s$ and for any subset $T \subseteq S(\mathbb{C})$ we denote $\Lambda_T := \bigsqcup_{s\in T} \Lambda_s$. The space $\mathrm{Lie}(\mathcal{A}) := \bigsqcup_{s\in S(\mathbb{C})} \mathrm{Lie}(\mathcal{A}_s)$ has a natural structure of g-dimensional holomorphic vector bundle $f\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ (it is actually a complex Lie algebra bundle). By using the fiberwise exponential maps one can define a global map $\exp\colon \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}$. Let $\Theta_0 \subset \mathcal{A}$ be the image of the zero section of the abelian scheme, then obviously

$$\exp^{-1}(\Theta_0) = \Lambda_S. \tag{1}$$

Clearly $S(\mathbb{C})$ can be covered by finitely many open simply connected subsets where the holomorphic vector bundle $f\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ trivializes. Let $U \subseteq S(\mathbb{C})$ be any of such subsets and consider the induced holomorphic map $f\colon \Lambda_U \to U$; it is actually a fiber bundle with structure group $\mathrm{GL}(n,\mathbb{Z})$. Since U is simply connected, by [11, Lemma 4.7] we conclude that $f\colon \Lambda_U \to U$ is a topologically trivial fiber bundle. Thus we can find $2g$ continuous sections of f:

$$\omega_i : U \to \Lambda_U\,, \quad i = 1, \ldots 2g \tag{2}$$

such that $\{\omega_1(s), \ldots, \omega_{2g}(s)\}$ is a basis of periods for Λ_s for any $s \in U$. Since $\Lambda_U \subseteq \mathrm{Lie}(\mathcal{A})_{|U}$, we can put periods into the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Lie}(\mathcal{A})_{|U} \\ & \overset{\omega_i}{\nearrow} & \big\downarrow \exp_{|U} \\ S(\mathbb{C}) \supset U & \xrightarrow{\ \sigma_0|U\ } & \mathcal{A}_{|U}, \end{array}$$

where σ_0 is the zero section. Since σ_0 is holomorphic and exp is a local biolomorphism, then the period functions defined in Equation (2) are holomorphic. The map

$$\mathfrak{P} = (\omega_1, \ldots, \omega_{2g}) \tag{3}$$

is called a *period map*; roughly speaking it selects a $\mathbb{Z}$-basis for Λ_s which varies holomorphically for $s \in U$.

Let us consider now a non-torsion section $\sigma : S \to \mathcal{A}$ of the abelian scheme. The set $U \subseteq S(\mathbb{C})$ is simply connected, therefore we can choose a holomorphic lifting $\ell_\sigma : U \to \mathrm{Lie}(\mathcal{A})$ of the restriction $\sigma_{|U}$; ℓ_σ is often called an *abelian logarithm*. Thus for any $s \in U$ we can write uniquely

$$\ell_\sigma(s) = \beta_1(s)\omega_1(s) + \ldots + \beta_{2g}\omega_{2g}(s) \tag{4}$$

where $\beta_i : U \to \mathbb{R}$ is a real analytic function for $i = 1, \ldots, 2g$. The map $\beta_\sigma : U \to \mathbb{R}^{2g}$ defined as $\beta_\sigma = (\beta_1, \ldots, \beta_{2g})$ is called the *Betti map associated to the section* σ, whereas the β_i's are the *Betti coordinates*. Observe that the Betti map depends both on the choices of the period map $\mathcal{P}$ and of the abelian logarithm ℓ_σ, but this is irrelevant for our aims.

As already mentioned, in this paper we are going to make also use of the sheaf of holomorphic sections Σ^{an}. Let $\mathscr{Lie}(\mathcal{A})$ denote the locally free sheaf on $S(\mathbb{C})$ associated to the vector bundle $\mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$. Then we have a morphism of sheaves $\psi\colon \mathscr{Lie}(\mathcal{A}) \to \Sigma^{\mathrm{an}}$ defined in the following way on any open set $U \subseteq S(\mathbb{C})$:

$$\begin{array}{rcl} \psi_U\colon \mathscr{Lie}(\mathcal{A})(U) & \to & \Sigma^{\mathrm{an}}(U) \\ t & \mapsto & \exp \circ\, t. \end{array}$$

Observe that ψ is surjective because of the local existence of abelian logarithms. Moreover the sheaf of periods Λ_S is exactly the kernel of ψ, therefore we obtain the following short exact sequence of sheaves of abelian groups on S:

$$0 \to \Lambda_S \to \mathscr{Lie}(\mathcal{A}) \to \Sigma^{\mathrm{an}} \to 0. \tag{5}$$

2.3 Monodromy of periods

Let $\mathcal{F}$ be a sheaf of abelian groups over a topological space X and denote by Γ the functor of "global sections" which to $\mathcal{F}$ associates $\Gamma(X,\mathcal{F}) = \mathcal{F}(X)$, with values in the category of abelian groups. Since the category of sheaves of abelian groups has sufficiently many injective objects, then the derived functors $R^k\Gamma$ do exist. They are generally written

$$R^k\Gamma(\mathcal{F}) =: H^k(X,\mathcal{F}).$$

Let X be a locally contractible topological space and denote by $\mathbb{Z}_X$ the constant sheaf of abelian groups over X with constant stalk $\mathbb{Z}$. Then we have a canonical isomorphism

$$H^q_{\text{sing}}(X,\mathbb{Z}) \cong H^q(X,\mathbb{Z}_X),$$

where we are considering the cohomology of X with coefficients in the constant sheaf of stalk $\mathbb{Z}_X$ on the right, and the singular cohomology with coefficients in $\mathbb{Z}$ on the left. The same result holds with $\mathbb{Z}$ replaced by any commutative ring G.

Now, let $\phi : \mathcal{A} \to S$ be an abelian scheme as defined above and consider the constant sheaf $\mathbb{Z}_{\mathcal{A}}$. We want to define a sheaf on the base S containing information on periods of the abelian schemes, to this end let us consider the direct image functor ϕ_*. Since ϕ_* is left exact and the category of sheaves of abelian groups has enough injectivities, the right derived functors of ϕ_* are well defined from the category of sheaves on $\mathcal{A}$ to the category of sheaves on S: they are called *higher direct image functors* and will be denoted by $R^k\phi_*$. For any $k \geq 0$, define the sheaf

$$\mathcal{P}^k_S := R^k\phi_*\mathbb{Z}_{\mathcal{A}}.$$

By [20, Proposition 8.1], this is the sheaf on S associated to the presheaf

$$U \mapsto H^k(\phi^{-1}(U), \mathbb{Z}_{\mathcal{A}}|_{\phi^{-1}(U)}).$$

Since $\phi : \mathcal{A} \to S$ is a smooth surjective morphism of algebraic varieties over $\mathbb{C}$, by [20, Theorem 10.4] then ϕ is a submersion. We can apply Ehresmann's Lemma to conclude that the proper submersion ϕ is a C^∞-fiber bundle.

Since $S(\mathbb{C})$ is locally contractible, then for sufficiently small $U \subseteq S(\mathbb{C})$, the open sets $\mathcal{A}_U := \phi^{-1}(U)$ have the same homotopy type as the fibre $\mathcal{A}_s(\mathbb{C})$ with $s \in U$. Using the invariance under homotopy, i.e. the fact that if U is a contractible space then $H^k(\mathcal{A}_s(\mathbb{C}) \times U, \mathbb{Z}) = H^k(\mathcal{A}_s(\mathbb{C}), \mathbb{Z})$ for all $k \geq 0$, we deduce that the sheaves $\mathcal{P}^k_S = R^k p_*\mathbb{Z}_{\mathcal{A}}$ are locally constant sheaves (or equivalenty *local systems*) on S with stalk

$$(\mathcal{P}^k_S)_s = (R^k\phi_*\mathbb{Z}_{\mathcal{A}})_s = H^k(\mathcal{A}_s(\mathbb{C}), \mathbb{Z}).$$

Now, the fundamental group $\pi_1(S(\mathbb{C}), s)$ acts via linear transformations on $(\mathcal{P}^k_S)_s$: this gives rise to a monodromy representation. We are going to give a more detailed description of the monodromy representation but the rough idea is the following: pick a loop $\gamma(t)$ at s and use a trivialization of the bundle along the loop to move vectors in $(\mathcal{P}^k_S)_s$ along $(\mathcal{P}^k_S)_{\gamma(t)}$, back to $(\mathcal{P}^k_S)_s$. Actually this is a more general construction regarding monodromy representations associated to local systems, but in our case the monodromy action can be induced by homeomorphisms of the fiber. In order to be more precise, consider $\gamma \in \pi_1(S(\mathbb{C}), s)$ represented by a loop $\gamma : [0,1] \to S(\mathbb{C})$ based at s. Consider the fibration $\mathcal{A}_\gamma$ defined as the fiber product

$$\begin{array}{ccc} \mathcal{A}_\gamma & \longrightarrow & \mathcal{A}(\mathbb{C}) \\ {\scriptstyle \phi_\gamma}\downarrow & & \downarrow{\scriptstyle \phi} \\ [0,1] & \xrightarrow{\ \gamma\ } & S(\mathbb{C}). \end{array}$$

By the compactness of $[0,1]$, there exist real numbers $0 \leq \epsilon_i < \epsilon_{i+1}, 1 \leq i \leq N$, such that $\epsilon_1 = 0, \epsilon_N = 1$, and ϕ_γ trivialises on $[\epsilon_i, \epsilon_{i+1}]$. By gluing local trivialisations above segments $[\epsilon_i, \epsilon_{i+1}]$, we can trivialise the fibration $\phi_\gamma : \mathcal{A}_\gamma \to [0,1]$. From such a trivialization

$$\mathcal{A}_\gamma \cong \phi_\gamma^{-1}(0) \times [0,1],$$

we deduce a homeomorphism $\psi : \phi_\gamma^{-1}(1) \cong \phi_\gamma^{-1}(0)$. These two spaces are canonically homeomorphic to the fibre $\mathcal{A}_s(\mathbb{C})$.

The homeomorphism ψ induces a group automorphism ψ^k of $H^k(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$, for any $k \geq 0$. Therefore, we obtain the monodromy representations

$$\rho^k : \pi_1(S(\mathbb{C}), s) \to \operatorname{Aut} H^k(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \tag{6}$$

defined by

$$\rho^k(\gamma)(\eta) := \psi^k\eta, \qquad \eta \in H^k(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}).$$

Thus, thanks to the fact the local systems we are considering is obtained using a fibration, the monodromy representation is in fact induced by homeomorphisms of the fibre $\mathcal{A}_s(\mathbb{C})$ onto itself. In particular, it follows that the monodromy representation on the cohomology of the fibre $H^\bullet(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ is compatible with the cup-product on $H^\bullet(\mathbb{A}_s(\mathbb{C}),\mathbb{Z})$, in the sense that each $\rho(\gamma) \in \operatorname{Aut} H^\bullet(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ is a ring automorphism for the ring structure given by the cup-product.

Recall that by Poincaré duality we have isomorphisms $H^k(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \cong H_{2g-k}(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$. Moreover, by the Künneth formula the groups $H_k(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ and $H^k(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ are free abelian groups of finite rank $\binom{2g}{k}$ for all $k = 1,\dots,2g$. Therefore, we can look at $\mathcal{P}_S^{2g-1}$ as a local system with constant stalk $\mathbb{Z}^{2g}$ to be identified with singular homology; we will simply write $\mathcal{P}$ instead of $\mathcal{P}_S^{2g-1}$. In other words, from now on we fix $k = 2g-1$ and we only deal with the monodromy action induced on the first homology group identifying $\operatorname{Aut} H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \cong \mathrm{GL}_{2g}(\mathbb{Z})$:

$$\rho : \pi_1(S(\mathbb{C}), s) \to \mathrm{GL}_{2g}(\mathbb{Z}).$$

Furthermore, we have a non-degenerate intersection form

$$\langle \cdot,\cdot \rangle : H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \times H_{2g-1}(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \to \mathbb{Z}.$$

Any functional $H_{2g-1}(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \to \mathbb{Z}$ is an intersection index of some homology class from $H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ and, if a class $A \in H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ is such that $\langle A, B\rangle = 0$ for any $B \in H_{2g-1}(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$, then A vanishes as element of $H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$. The abelian variety $\mathcal{A}_s(\mathbb{C})$ has a polarization inducing an alternating form E which takes integer values on the lattice Λ_s. Identifying $\Lambda_s = H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ and fixing a symplectic basis $\omega_1,\dots,\omega_{2g}$ of $H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ for E we obtain functionals

$$\varphi_i := E(\cdot,\omega_i) : H_1(\mathcal{A}_s(\mathbb{C}),\mathbb{Z}) \to \mathbb{Z}, \qquad \text{for } i = 1,\dots,2g.$$

For any functional φ_i there exists a $(2g-1)$-cycle $\eta_i \in H_{2g-1}(\mathcal{A}_s(\mathbb{C}),\mathbb{Z})$ such that

$$\varphi_i(\omega_j) = \langle \eta_i, \omega_j\rangle.$$

Since the intersection form is dual to the cup-product and since each $\rho(\gamma) \in \mathrm{GL}_{2g}(\mathbb{Z})$ is a ring automorphism for the ring structure given by the cup-product, then $\rho(\gamma)$ has to preserve the intersection matrix induced by E, i.e.

$$P = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

In other words the image of the representation ρ is contained in the following group

$$\mathrm{Sp}_{2g}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2g}(\mathbb{Z}) \mid M^\top P M = P\}.$$

Thus, monodromy representation of periods introduced in Equation (6) can be actually though as a map:

$$\rho : \pi_1(S(\mathbb{C}), s) \to \mathrm{Sp}_{2g}(\mathbb{Z}). \tag{7}$$

Definition 2.3. We define the *monodromy group* of $\mathcal{A} \to S$ at s as $\mathrm{Mon}(\mathcal{A}, s) := \rho(\pi_1(S(\mathbb{C}), s))$.

Since $S(\mathbb{C})$ is path connected, all the groups $\mathrm{Mon}(\mathcal{A}, s)$ are conjugate when we vary the base point $s \in S(\mathbb{C})$: in other words, the monodromy group is defined up to an inner automorphism of the group $\mathrm{Sp}_{2g}(\mathbb{Z})$. Thus, fix once and for all a base point $s \in S(\mathbb{C})$ and denote the group $\mathrm{Mon}(\mathcal{A}, s)$ simply by $\mathrm{Mon}(\mathcal{A})$, without writing any dependencies on the base point.

Since we can identify each first homology group of the fibers with the corresponding lattice Λ_s, then the sheaf $\mathcal{P}$ can be identified with the sheaf Λ_S defined in Equation (1); it will be called *period sheaf* and will be denoted simply by Λ. In more concrete terms, notice that by construction studying the monodromy of the period sheaf corresponds to studying the analytic continuation of period functions (i.e. the coordinates of the period map defined in Equation (3)) along loops in $S(\mathbb{C})$ representing classes in the fundamental group based at s.

Periods from universal family. In the case of a universal family, a more explicit description of the monodromy action on periods is described in [16, Section 3]. Keeping the same notations as above, let us denote by $\pi : \mathfrak{A}_g \to \mathbb{A}_g$ the universal family of principally polarized g-dimensional abelian varieties with level-ℓ-structure for some $g \geq 1$ and $\ell \geq 3$. Let $\mathbb{H}_g$ denote Siegel's upper half space, i.e., the symmetric matrices in $\mathrm{Mat}_{g\times g}(\mathbb{C})$ with positive definite imaginary part. We have holomorphic uniformizing maps

$$u_B : \mathbb{H}_g \to \mathbb{A}_g(\mathbb{C}) \qquad \text{and} \qquad u : \mathbb{C}_g \times \mathbb{H}_g \to \mathfrak{A}_g(\mathbb{C}).$$

Recall that $\mathrm{Sp}_{2g}(\mathbb{R})$, the group of real points of the symplectic group, acts on $\mathbb{H}_g$. Let $x \in \mathbb{A}_g(\mathbb{C})$ and fix $\tau \in \mathbb{H}_g$ such that $x = u_B(\tau)$. If 1_g denotes the $g \times g$ unit matrix, then the columns of $(\tau, 1_g)$ are an $\mathbb{R}$-basis of $\mathbb{C}^g$ and we have $\mathfrak{A}_{g,x}(\mathbb{C}) \cong \mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$. The period lattice basis $(\tau, 1_g)$ allows us to identify $H_1(\mathfrak{A}_{g,x}(\mathbb{C}), \mathbb{Z})$ with $\mathbb{Z}^{2g}$. Let us consider a loop γ in $\mathbb{A}_g(\mathbb{C})$ based at x representing a class of $\pi_1(\mathbb{A}_g(\mathbb{C}), x)$. Then a lift $\widetilde{\gamma}$ of γ to $\mathbb{H}_g$ starting at τ ends at $M\tau \in \mathbb{H}_g$ for some

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Then $M\tau$ is the period matrix of the abelian variety $\mathbb{C}^g/(M\tau\mathbb{Z}^g + \mathbb{Z}^g)$ which is isomorphic to $\mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$: more precisely, the isomorphism $\mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g) \to \mathbb{C}^g/(M\tau\mathbb{Z}^g + \mathbb{Z}^g)$ is induced by the map

$$\tau u + v \mapsto ((c\tau + d)^\top)^{-1}(\tau u + v) = (M\tau, 1_g)(M^\top)^{-1}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $u, v \in \mathbb{R}^g$ are column vectors; in what follows sometimes we will interpret u, v as row vectors and we will consider the transposition of the previous relation. Therefore, the monodromy representation expressed in these coordinates is given by

$$\begin{aligned} \rho : \pi_1(\mathbb{A}_g(\mathbb{C})), x) &\to \mathrm{Sp}_{2g}(\mathbb{Z}) \\ [\gamma] &\mapsto (M^\top)^{-1}. \end{aligned} \tag{8}$$

Notice that the monodromy action is a right action on period functions, it can be obviously interpreted as a left action by putting $h \cdot (M\tau, 1_g) := (M\tau, 1_g)\rho(h)$.

The previous considerations about periods of universal families allow us to obtain some results on periods of the abelian scheme $\phi : \mathcal{A} \to S$: we are able to construct period functions of $\phi : \mathcal{A} \to S$ by means of periods of the universal family $\pi : \mathfrak{A}_g \to \mathbb{A}_g$. Let $V \subseteq \mathbb{A}_g(\mathbb{C})$ be a simply connected open set of an open covering of $\mathbb{A}_g$ where holomorphic period functions do exist for the universal abelian scheme, see Equation (2); denote by $\omega^{\mathfrak{A}_g}_{V,1}, \dots, \omega^{\mathfrak{A}_g}_{V,2g}$ such local (holomorphic) period functions. Fix a base point $s \in S(\mathbb{C})$, not a ramification point for $p : S \to \mathbb{A}_g$, and let $x = p(s) \in \mathbb{A}_g(\mathbb{C})$. In a connected and simply connected neighborhood U of s in $S(\mathbb{C})$ such that $p(U) = V$, we can holomorphically define a basis $\omega_{U,1}, \dots, \omega_{U,2g}$ of the period lattices by the equations

$$\omega_{U,i} = \omega^{\mathfrak{A}_g}_{V,i} \circ p. \tag{9}$$

Locally on suitable open subsets $U \subset S(\mathbb{C})$, this gives a basis for Λ_s made up of holomorphic functions $\omega_{U,1}, \dots, \omega_{U,2g} : U \to \mathbb{C}^g$.

Remark 2.4. Recall that in our situation we have a finite map $p : S \to \mathbb{A}_g$ and that the monodromy group of the universal family $\mathrm{Mon}(\mathfrak{A}_g)$ is a finite index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Therefore, the monodromy group $\mathrm{Mon}(\mathcal{A})$ of $\mathcal{A} \to S$ is still a finite index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ thanks to Equation (9); in particular, $\mathrm{Mon}(\mathcal{A})$ is Zariski-dense in $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Lemma 2.5. *The natural action of the group $Sp_{2g}(\mathbb{Z})$ on $\mathbb{Q}^{2g}$ is irreducible.*

Proof. Assume that there exists a non-trivial subvector space W of $\mathbb{Q}^{2g}$ which is fixed by the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Since the symplectic group Sp_{2g} is a simple algebraic Lie group defined over $\mathbb{Q}$, by a theorem of Borel and Harish-Chandra [6, Theorem 7.8] the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is a lattice in $\mathrm{Sp}_{2g}(\mathbb{R})$. By Borel density theorem (see [5]) we have that $\mathrm{Sp}_{2g}(\mathbb{Z})$ is Zariski dense in $\mathrm{Sp}_{2g}(\mathbb{R})$, and so it is Zariski dense in $\mathrm{Sp}_{2g}(\mathbb{Q})$. This implies that W is fixed by the action of $\mathrm{Sp}_{2g}(\mathbb{Q})$. Since the action of $\mathrm{Sp}_{2g}(\mathbb{Q})$ on $\mathbb{Q}^{2g}$ is irreducible (see [19, Proposition 3.2]), we get $W = \mathbb{Q}^{2g}$ which concludes the proof. □

Proposition 2.6. *The action of the monodromy group $Mon(\mathcal{A})$ on the lattice of periods is irreducible.*

Proof. This is an easy consequence of Remark 2.4 and Lemma 2.5. □

Remark 2.7. The previous considerations yield some conclusion on the global definition of periods on the base $S(\mathbb{C})$. We have seen that period functions $\omega_1, \dots, \omega_{2g}$ can be locally defined on simply connected open sets $U \subseteq S(\mathbb{C})$ via Equation (9) for any abelian scheme $\mathcal{A} \to S$. These functions may be analytically continued through the whole of $S(\mathbb{C})$, but it's impossible to globally define them: they turn out to be multi-valued functions, i.e. they have quite nontrivial monodromy when traveling along closed paths.

To be more precise, since the group $\mathrm{Mon}(\mathcal{A})$ is non-trivial then the functions $\omega_1, \dots, \omega_{2g}$ cannot be all defined continuously on the whole of $S(\mathbb{C})$. Moreover, since the action of $\mathrm{Mon}(\mathcal{A})$ on the lattice of periods is irreducible, neither one of $\omega_1, \dots, \omega_{2g}$ nor any single non-zero element of the period lattice can be defined on the whole of $S(\mathbb{C})$. This last thing is also proved in [15, Lemma 5.6] for any abelian scheme without using irriducibility properties of the monodromy action.

2.4 Relative monodromy of abelian logarithms

Let us consider the exponential map $\exp : \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}$ defined in Section 2.2. Notice that it is a topological cover. Thus, it induces a monodromy action of the fundamental group $\pi_1(\mathcal{A}(\mathbb{C}), p)$ on the fiber $\exp^{-1}(p)$, where we assume $\phi(p) = s$. Now, let $\sigma : S \to \mathcal{A}$ be a section of the abelian scheme. By restricting the exponential map to $\sigma(S)$ and identifying the fundamental groups $\pi_1(\sigma(S(\mathbb{C})), p)$ and $\pi_1(S(\mathbb{C}), s)$, we obtain a monodromy action of $\pi_1(S(\mathbb{C}), s)$ on the fiber $\exp^{-1}(\sigma(s))$.

Recalling the definition of logarithm given in Equation (4), the set $\exp^{-1}(\sigma(s))$ is the set of all possible determinations of the abelian logarithm of σ at the point s. Every two such determinations differ each other by an element of the lattice Λ_s, thus we can identify the set $\exp^{-1}(\sigma(s))$ with $\mathbb{Z}^{2g}$. Therefore, we get the monodromy action

$$c : \pi_1(S(\mathbb{C}), s) \to \mathbb{Z}^{2g} \tag{10}$$

which is determined by the analytic continuation of a fixed branch of $\log_\sigma(s)$ along loops representing classes in $\pi_1(S(\mathbb{C}), s)$.

To be more explicit, fix a determination $\log_\sigma(s)$ of the logarithm of σ at s. The analytic continuation of $\log_\sigma(s)$ along a loop representing some $h \in \pi_1(S(\mathbb{C}), s)$ is of the form

$$\log_\sigma + u_1^h \omega_1 + \cdots + u_{2g}^h \omega_{2g}, \tag{11}$$

where $(u_1^h, \dots, u_{2g}^h)^\top = c(h)$ is the column vector obtained by means of Equation (10).

Now, let us look at the simultaneous monodromy action of $G := \pi_1(S(\mathbb{C}), s)$ on periods and logarithm. By Equation (7) and Equation (10), we can provide a new representation

$$\theta_\sigma : G \to \mathrm{SL}_{2g+1}(\mathbb{Z}),$$

where every matrix $\theta_\sigma(h)$ is of the form

$$\theta_\sigma(h) = \begin{pmatrix} \rho(h) & c(h) \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where $\rho(h) \in \mathrm{Sp}_{2g}(\mathbb{Z}), c(h) \in \mathbb{Z}^{2g}$. Note that the matrix $\rho(h)$ acts on the periods and does not depend on σ. Moreover, the vector $c(h)$ encodes the action of h on determinations of the logarithm $\log_\sigma$. Define the monodromy group of the section σ as $M_\sigma := \theta_\sigma(G)$.

Let's continue considering an abelian scheme $\mathcal{A} \to S$ and a rational non-zero section $\sigma : S \to \mathcal{A}$. Generally neither a basis of periods nor a logarithm $\log_\sigma$ can be globally defined on the whole of $S(\mathbb{C})$ (see Remark 2.7 and Remark 2.10 below), but obviously they can be globally defined on the universal cover of S. Studying the related monodromy problems corresponds to finding out the minimal (unramified) cover of S on which both a basis of the periods and a logarithm of the section can be defined. With this in mind, we first call $S^* \to S$ the minimal (unramified) cover of S on which a basis for the period lattice can be globally defined and we set $S_\sigma \to S^*$ to be the minimal cover of S^* where one can define the logarithm of σ. The tower of covers is represented in the diagram:

$$S_\sigma \to S^* \to S. \tag{13}$$

In particular, the group $\mathrm{Mon}(\mathcal{A})$ corresponds to the Galois group of the covering map $S^* \to S$, while the group M_σ corresponds to the Galois group of the covering map $S_\sigma \to S$. Our interest is in studying the relative monodromy of the logarithm of a section with respect to the monodromy of periods, i.e. studying the covering map $S_\sigma \to S^*$. Topologically, this is the same as looking at the variation of logarithm via analytic continuation along loops on $S(\mathbb{C})$ which leave periods unchanged. Moreover, in terms of Equation (7) and Equation (12) this corresponds to studying the group $M_\sigma^{\mathrm{rel}} := \theta_\sigma(\ker\rho)$, which we define as *relative monodromy group of* σ.

The group M_σ^{rel} gives information on the "pure monodromy" of the Betti map, and this is a strong knowledge for obtaining transcendence results which for example allow to count torsion points in issues with "unlikely intersection" flavour. Note that M_σ^{rel} is clearly a subgroup of $\mathbb{Z}^{2g}$; it is useful for applications knowing when this subgroup is trivial and how large it is.

Example 2.8. Let us illustrate what happens in a simple case, i.e. when the section is torsion. Thus, let $\sigma : S \to \mathcal{A}$ be a torsion section. By the properties of Betti map, any logarithm of σ is a rational constant combination of periods; i.e.

$$\log_\sigma = q_1\omega_1 + \cdots + q_{2g}\omega_{2g},$$

where $q_1, \ldots, q_{2g} \in \mathbb{Q}$. Therefore, a loop which leaves unchanged periods via analytic continuation, leaves also unchanged the logarithm of such a section. In other words, the cover $S_\sigma \to S^*$ is trivial in this case and then $M_\sigma^{\mathrm{rel}} \cong \{0\}$.

Remark 2.9. Note that the Zariski-closures of the discrete groups $\mathrm{Mon}(\mathcal{A})$ and M_σ as far as the kernel of the projection $\ker(\overline{M_\sigma} \to \overline{\mathrm{Mon}(\mathcal{A})})$ are the differential Galois groups of some Picard-Vessiot extensions of $\mathbb{C}(S)$ obtained with $\omega_1, \ldots, \omega_{2g}, \log_\sigma$ and their derivatives (see [10] for further details about differential Galois theory and Picard-Vessiot extensions). In these terms, the differential Galois group $\ker(\overline{M_\sigma} \to \overline{\mathrm{Mon}(\mathcal{A})})$ was just studied in [1], [3], [4]; anyway, those results give no information on the relative monodromy of the logarithm over S^* because they involve the Zariski closures of M_σ and $\mathrm{Mon}(\mathcal{A})$: in fact, in [9] the authors pointed out that it may happen that the group $\ker(\overline{M_\sigma} \to \overline{\mathrm{Mon}(\mathcal{A})})$ contains quite limited information on M_σ^{rel} since the former may be larger than expected in comparison with the latter. Thus, in order to obtain information on M_σ^{rel}, we have to introduce considerations of different nature with respect to Bertrand's and André's theorems.

Similarly to period functions, once we have locally defined the logarithm of a section as in Equation (4) we can think about its analytic continuation through the whole of $S(\mathbb{C})$. As a first example, let us focus on the zero-section σ_0 which associates to each $s \in S(\mathbb{C})$ the origin O_s of the corresponding fiber $\mathcal{A}_s$. A logarithm of σ_0 is given by the zero function

$$\log_{\sigma_0} : S(\mathbb{C}) \to \mathbb{C}^g, \qquad \log_{\sigma_0}(s) = 0 \text{ for each } s \in S(\mathbb{C}).$$

Thus, in this case we can find a globally defined logarithm on the whole of $S(\mathbb{C})$: in fact it has no monodromy. For algebraic sections, this is the only case in which such a global logarithm exists. For the sake of completeness, we briefly resume a proof of this fact.

Remark 2.10. Let $\mathcal{A} \to S$ be an abelian scheme. We use Lang-Néron theorem [21, Thorem 1] to show that a non-zero algebraic section $\sigma : S \to \mathcal{A}$ which is not contained in the fixed part cannot admit a globally defined logarithm on $S(\mathbb{C})$. In fact, assume by contradiction that $\log_\sigma$ exists globally on $S(\mathbb{C})$, then $\frac{\sigma}{n} := \exp \circ \left(\frac{1}{n}\log_\sigma\right)$ for any $n \in \mathbb{Z} \setminus \{0\}$ is an analytic section such that $n \cdot \frac{\sigma}{n} = \sigma$. This means that σ is infinitely divisible in $\Sigma(S)$ contradicting Lang-Néron theorem.

By Equation (11), fixed a determination of $\log_\sigma$ on an open set U any element $h \in G := \pi_1(S(\mathbb{C}), s)$ acts by analytic continuation on it in the following way:

$$h \cdot \log_\sigma = \log_\sigma + u_1^h \omega_1 + \cdots + u_{2g}^h \omega_{2g},$$

where $u_1^h, \ldots, u_{2g}^h \in \mathbb{Z}$. Choose $h_1, h_2 \in G$; by looking at the action of h_1h_2 and recalling Equation (8), we have

$$\log_\sigma \xrightarrow{h_2} \log_\sigma + (u_1^{h_2}, \ldots, u_{2g}^{h_2}) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix} \xrightarrow{h_1} \log_\sigma + (u_1^{h_1}, \ldots, u_{2g}^{h_1}) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix} + (u_1^{h_2}, \ldots, u_{2g}^{h_2})\rho(h_1)^\top \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix}.$$

Therefore, we obtain the relation

$$(u_1^{h_1h_2}, \ldots, u_{2g}^{h_1h_2}) = (u_1^{h_1}, \ldots, u_{2g}^{h_1}) + (u_1^{h_2}, \ldots, u_{2g}^{h_2})\rho(h_1)^\top.$$

In other words the map $h \mapsto (u_1^h, \dots, u_{2g}^h)$ *is a cocycle for the described action of* G *on* $\mathbb{Z}^{2g}$. Moreover, in the next proposition we prove that a section admitting a globally defined logarithm is characterized by the fact that the map $h \mapsto (u_1^h, \dots, u_{2g}^h)$ is a coboundary for the aforementioned action, i.e. there exists a fixed vector $(u_1, \dots, u_{2g}) \in \mathbb{Z}^{2g}$ such that $(u_1^h, \dots, u_{2g}^h) = (u_1, \dots, u_{2g})(\rho(h)^\top - 1_{2g})$ for all $h \in G$. Thus, the just defined cocycle determines an element in the cohomology group $H^1(G, \mathbb{Z}^{2g})$ that describes the obstruction for a(n analytic) section to have a globally defined logarithm.

Proposition 2.11. *Let* $\sigma : S \to \mathcal{A}$ *be an analytic section and* $\log_\sigma$ *a determination of its logarithm over an open set* $U \subset S(\mathbb{C})$*. The section admits a globally defined logarithm on* $S(\mathbb{C})$ *if and only if the associated cocycle* $h \mapsto (u_1^h, \dots, u_{2g}^h)$ *is a coboundary.*

Proof. Suppose that σ admits a globally defined logarithm $\ell : S(\mathbb{C}) \to \mathbb{C}^g$. Then the two determinations ℓ and $\log_\sigma$ over U differ by a period $\omega := n_1\omega_1 + \cdots + n_{2g}\omega_{2g}$, i.e.

$$\log_\sigma = \ell + n_1\omega_1 + \cdots + n_{2g}\omega_{2g}.$$

For $h \in G$, we have

$$\begin{aligned} h \cdot \log_\sigma = h \cdot (\ell + n_1\omega_1 + \cdots + n_{2g}\omega_{2g}) &= \ell + (n_1, \dots, n_{2g})\rho(h)^\top \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix} = \\ &= \log_\sigma + \left[(n_1, \dots, n_{2g})\rho(h)^\top - (n_1, \dots, n_{2g})\right] \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix}. \end{aligned}$$

Thus the corresponding cocycle is given by

$$h \mapsto (n_1, \dots, n_{2g})\rho(h)^\top - (n_1, \dots, n_{2g})$$

for $h \in G$ and a fixed pair $(n_1, \dots, n_{2g}) \in \mathbb{Z}^{2g}$, hence it is a coboundary.

Viceversa, let us suppose to have a logarithm $\log_\sigma$ over U and that there exists a fixed $2g$-tuple $(n_1, \dots, n_{2g}) \in \mathbb{Z}^{2g}$ such that

$$h \cdot \log_\sigma = \log_\sigma + \left[(n_1, \dots, n_{2g})\rho(h)^\top - (n_1, \dots, n_{2g})\right] \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix}$$

for each $h \in G$. Let us define the function

$$\ell := \log_\sigma - n_1\omega_1 - \cdots - n_{2g}\omega_{2g},$$

which is another determination of the logarithm of σ. Looking at the action of G we obtain

$$\begin{aligned} h \cdot \ell &= h \cdot (\log_\sigma - n_1\omega_1 - \cdots - n_{2g}\omega_{2g}) = \\ &= \log_\sigma + \left[(n_1, \dots, n_{2g})\rho(h)^\top - (n_1, \dots, n_{2g})\right] \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix} - (n_1, \dots, n_{2g})\rho(h)^\top \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix} = \\ &= \log_\sigma - n_1\omega_1 - \cdots - n_{2g}\omega_{2g} = \ell. \end{aligned}$$

Therefore, ℓ is a globally defined logarithm of σ on $S(\mathbb{C})$.

□

3 Proofs of the main results

3.1 Invariance results

In this section we want to provide some preliminary results about the relative monodromy group M_σ^{rel} in order to prove that our main theorems are insensitive to some natural operations. First of all, let us notice that Theorem 1.1 and Theorem 1.2 are invariant under isogenies: this was proven in [33, Theorem 3.6]. In what follows we describe other invariance results.

Lemma 3.1. *Theorem 1.1 (resp. Theorem 1.2) is invariant by finite base change: in other words, if* $\varphi : \widetilde{S} \to S$ *is a finite morphism and* $\widetilde{\mathcal{A}} \to \widetilde{S}$ *is the pullback of* $\mathcal{A} \to S$ *via* φ*, then Theorem 1.1 (resp. Theorem 1.2) for* $\widetilde{\mathcal{A}} \to \widetilde{S}$ *implies Theorem 1.1 (resp. Theorem 1.2) for* $\mathcal{A} \to S$.

Proof. Let us denote by $\omega_1, \ldots, \omega_{2g}$ a basis of periods of $\mathcal{A} \to S$. We can define $\widetilde{\omega_1}, \ldots, \widetilde{\omega_{2g}}$, a basis of periods of $\widetilde{\mathcal{A}} \to \widetilde{S}$, by the equations

$$\widetilde{\omega_1} = \omega_1 \circ \varphi, \quad \ldots, \quad \widetilde{\omega_{2g}} = \omega_{2g} \circ \varphi. \tag{14}$$

Fix two base points $\widetilde{s} \in \widetilde{S}(\mathbb{C})$ and $s \in S(\mathbb{C})$ such that $\varphi(\widetilde{s}) = s$. We have the associated monodromy representations:

$$\rho : \pi_1(S(\mathbb{C}), s) \to \mathrm{Mon}(\mathcal{A}), \qquad \widetilde{\rho} : \pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s}) \to \mathrm{Mon}(\widetilde{\mathcal{A}}).$$

By Equation (14), we have $\widetilde{\rho}(h) = \rho(\varphi_*(h))$ for each $h \in \pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s})$ where $\varphi_* : \pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s}) \to \pi_1(S(\mathbb{C}), s)$ denotes the induced homomorphism between fundamental groups. In particular, we obtain

$$\varphi_*(\ker \widetilde{\rho}) = \ker \rho \cap \varphi_*(\pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s})). \tag{15}$$

Let $\sigma : S \to \mathcal{A}$ be a non-torsion section of $\mathcal{A} \to S$. Since $\widetilde{\mathcal{A}} \to \widetilde{S}$ is obtained as pullback of $\mathcal{A} \to S$, then the abelian varieties $\widetilde{\mathcal{A}}_{\widetilde{s}}$ and $\mathcal{A}_s$ are canonically identified; therefore the pullback $\varphi^*(\sigma) := \sigma \circ \varphi : \widetilde{S} \to \widetilde{\mathcal{A}}$ is a non-torsion section of $\widetilde{\mathcal{A}} \to \widetilde{S}$. We have the associated monodromy representations

$$\theta_\sigma : \pi_1(S(\mathbb{C}), s) \to \mathrm{SL}_{2g+1}(\mathbb{Z}), \qquad \theta_{\varphi^*(\sigma)} : \pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s}) \to \mathrm{SL}_{2g+1}(\mathbb{Z}).$$

Using again the fact that the abelian varieties $\widetilde{\mathcal{A}}_{\widetilde{s}}$ and $\mathcal{A}_s$ are canonically identified, we obtain that a determination of the logarithm of $\varphi^*(\sigma)$ at $\widetilde{s}$ can be defined by the equation

$$\log_{\varphi^*(\sigma)}(\widetilde{s}) := \log_\sigma(s). \tag{16}$$

Hence, we have $\theta_{\varphi^*(\sigma)}(h) = \theta_\sigma(\varphi_*(h))$ for each $h \in \pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s})$. Now, let us consider the relative monodromy group

$$M^{\mathrm{rel}}_{\varphi^*(\sigma)} = \theta_{\varphi^*(\sigma)}(\ker \widetilde{\rho}).$$

By Equation (15) and Equation (16), we obtain

$$M^{\mathrm{rel}}_{\varphi^*(\sigma)} = \theta_\sigma(\varphi_*(\ker \widetilde{\rho})) = \theta_\sigma(\ker \rho \cap \varphi_*(\pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s}))).$$

Since $\ker \rho \cap \varphi_*(\pi_1(\widetilde{S}(\mathbb{C}), \widetilde{s})) \subset \ker \rho$, we get $M^{\mathrm{rel}}_{\varphi^*(\sigma)} \subseteq M^{\mathrm{rel}}_\sigma$. The conclusion follows.

□

Let us consider a non-torsion section $\sigma : S \to \mathcal{A}$. Note that the representation θ_σ is defined in terms of Equation (11), thus it depends on the branch of logarithm we fix. Here, we want to prove that M^{rel}_σ is independent of the choice of branch of $\log_\sigma$ and remains unchanged under some operations on the section.

Proposition 3.2. *Let* $\sigma : S \to \mathcal{A}$ *be a non-torsion section. Then*

(i) the group M^{rel}_σ *does not depend on the choice of branch of* $\log_\sigma$*;*

(ii) the groups M^{rel}_σ *and* $M^{\mathrm{rel}}_{n\sigma}$ *are isomorphic.*

Proof. (i): Choose two branches of $\log_\sigma$, say ℓ^1_σ and ℓ^2_σ and denote by $M^{\mathrm{rel}}_{\sigma,1}$ and $M^{\mathrm{rel}}_{\sigma,2}$ the corresponding relative monodromy groups. Since the two branches ℓ^1_σ and ℓ^2_σ have to differ by a period, the conclusion follows: in fact, since any loop α in $S(\mathbb{C})$ whose homotopy class lies in $\ker \rho$ leaves periods unchanged, then ℓ^1_σ and ℓ^2_σ have the same variation by analytic continuation along α; thus we get $M^{\mathrm{rel}}_{\sigma,1} = M^{\mathrm{rel}}_{\sigma,2}$.

(ii): Fixed a branch ℓ_σ of $\log_\sigma$, a determination $\ell_{n\sigma}$ of the logarithm of $n\sigma$ can be defined by the equation

$$\ell_{n\sigma} = n\ell_\sigma.$$

For any loop α representing a homotopy class $h \in \pi_1(S(\mathbb{C}), s)$ we have the corresponding variations:

$$h \cdot \ell_{n\sigma} = \ell_{n\sigma} + \omega^{n\sigma}_\alpha, \qquad h \cdot \ell_\sigma = \ell_\sigma + \omega^\sigma_\alpha,$$

where $\omega^{n\sigma}_\alpha = n\omega^\sigma_\alpha$. Thus M^{rel}_σ and $M^{\mathrm{rel}}_{n\sigma}$ are isomorphic.

□

3.2 Non triviality of the relative monodromy

In this section we prove Theorem 1.1. We are going to combine some topological techniques and cohomology theory in order to get some control on the relative monodromy of sections.

Let's keep the same notation as above. Thanks to Lemma 3.1, we can assume that the abelian scheme $\phi : \mathcal{A} \to S$ is endowed with a finite surjective map $p : S \to \mathbb{A}_g$, where $\mathfrak{A}_g \to \mathbb{A}_g$ denotes some universal family of g-dimensional abelian varieties with some fixed level-ℓ-structure and without locally constant parts. Let $\sigma : S \to \mathcal{A}$ be a non-torsion section.

One of the main ideas of the proof is to construct a Kuga family associated to our abelian scheme, so that we can employ the result of Mok and To [25, Main Theorem] to obtain some constraints on the logarithm of sections. To this end, let's consider the classifying map $p : S \to \mathbb{A}_g$ and look at the induced homomorphism on fundamental groups, i.e. $p_* : \pi_1(S(\mathbb{C}), s) \to \pi_1(\mathbb{A}_g(\mathbb{C}), x)$ where $s \in S(\mathbb{C})$ and $x \in \mathbb{A}_g(\mathbb{C})$ are two base points such that $x = p(s)$. The inclusion $p_*\pi_1(S(\mathbb{C}), s) \subseteq \pi_1(\mathbb{A}_g(\mathbb{C}), x)$ gives rise to a finite unramified morphism $q' : S'(\mathbb{C}) \to \mathbb{A}_g(\mathbb{C})$ that by [22, Corollary 12.19] satisfies

$$q'_*\pi_1(S'(\mathbb{C}), s') = p_*\pi_1(S(\mathbb{C}), s), \tag{17}$$

where s' is a fixed base point with $q'(s') = x$. By Equation (17), we can lift p to a morphism $q : S(\mathbb{C}) \to S'(\mathbb{C})$ such that $q(s) = s'$. Considering the corresponding abelian schemes we obtain the diagram

$$\begin{array}{ccccc} \mathcal{A}(\mathbb{C}) & \longrightarrow & \mathcal{A}'(\mathbb{C}) & \longrightarrow & \mathfrak{A}_g(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ S(\mathbb{C}) & \xrightarrow{q} & S'(\mathbb{C}) & \xrightarrow{q'} & \mathbb{A}_g(\mathbb{C}). \end{array} \quad (p = q' \circ q) \tag{18}$$

Notice that the abelian scheme $\mathcal{A}' \to S'$ is a Kuga family by construction, since $q' : S'(\mathbb{C}) \to \mathbb{A}_g(\mathbb{C})$ is an unramified morphism.

Remark 3.3. We want to point out some algebraic and topological properties of the above construction. From the algebraic point of view, observe that the morphisms q and q' are both finite. In fact since p is finite, then q and q' are quasi-finite. Since q' is a finite-sheeted (unramified) topological cover, then it is proper in the sense that the inverse image of every compact set is compact. This implies that q' is a proper morphism in the sense of scheme theory, and thus q' is finite. In particular, q' is also a separated morphism. Since $p = q' \circ q$ is finite and q' is separated, then q is proper. Since q is proper and quasi-finite, then q is finite.

From the topological point of view, observe that the induced homomorphism $q_* : \pi_1(S(\mathbb{C}), s) \to \pi_1(S'(\mathbb{C}), s')$ is surjective. This follows from the diagram

$$\pi_1(S(\mathbb{C}), s) \xrightarrow{q_*} \pi_1(S'(\mathbb{C}), s') \xrightarrow{q'_*} p_*\pi_1(S(\mathbb{C}), s), \quad (p_* = q'_* \circ q_*) \tag{19}$$

where $q'_* : \pi_1(S'(\mathbb{C}), s') \to p_*\pi_1(S(\mathbb{C}), s)$ is an isomorphism.

Let's fix some notations. We denote by exp and exp$'$ the abelian exponential maps of the schemes $\mathcal{A} \to S$ and $\mathcal{A}' \to S'$, respectively. Moreover, recall that Σ^{an} is the sheaf of holomorphic sections of the scheme $\mathcal{A} \to S$, while $\Sigma(S)$ denotes its algebraic sections. Analogously, we denote by $\Sigma'^{,\mathrm{an}}$ and $\Sigma'(S')$ the sheaf of holomorphic sections and the algebraic sections of the abelian scheme $\mathcal{A}' \to S'$, respectively. Now, we want to define a way of mapping sections of $\mathcal{A} \to S$ to sections of $\mathcal{A}' \to S'$, and viceversa.

First of all we define the the pull-back operator: in fact notice that for any $\tau' \in \Sigma'^{,\mathrm{an}}(S')$ the composition $\tau' \circ q$ defines a holomorphic map $S(\mathbb{C}) \to \mathcal{A}'(\mathbb{C})$; after canonically identifying the fibers of the type $\mathcal{A}_y$ and $\mathcal{A}'_{q(y)}$, we obtain an element $q^*(\tau') \in \Sigma^{\mathrm{an}}(S)$. In other words, we obtain a morphism of abelian groups

$$q^* : \Sigma'^{,\mathrm{an}}(S') \to \Sigma^{\mathrm{an}}(S).$$

In addition if τ' is algebraic then $q^*(\tau')$ is still algebraic, i.e. $q^*(\Sigma'(S')) \subseteq \Sigma(S)$.

In the opposite direction, we define the trace operator

$$\mathrm{Tr} : \Sigma^{\mathrm{an}}(S) \to \Sigma'^{,\mathrm{an}}(S')$$

as follows: we already noticed that for each $y \in S(\mathbb{C})$ the fibers $\mathcal{A}_y$ and $\mathcal{A}'_{q(y)}$ are canonically identified, so for $\tau \in \Sigma^{\mathrm{an}}(S)$ we define $\mathrm{Tr}(\tau) : S'(\mathbb{C}) \to \mathcal{A}'(\mathbb{C})$ by setting

$$\mathrm{Tr}(\tau)(y') := \sum_{y \in q^{-1}(y')} m_y \tau(y) \in \mathcal{A}'_{y'},$$

where m_y is the ramification index of q at y. We have $\mathrm{Tr}(\Sigma(S)) \subseteq \Sigma'(S')$; in addition, since $\mathcal{A}' \to S'$ is a Kuga family we have $\mathrm{Tr}(\Sigma(S)) \subseteq \Sigma'(S')_{\mathrm{tor}}$ by Theorem 2.1, where $\Sigma'(S')_{\mathrm{tor}}$ is the group of torsion sections. Moreover, we point out that the following property holds for each holomorphic section $\tau' : S'(\mathbb{C}) \to \mathcal{A}'(\mathbb{C})$:

$$\mathrm{Tr} \circ q^*(\tau') = \deg q \cdot \tau'. \tag{20}$$

It is clear that the trace operator and the pullback operator can be defined also for the sections of the vector bundles $(\mathcal{O}_S^{\mathrm{an}})^{\oplus g}$ and $(\mathcal{O}_{S'}^{\mathrm{an}})^{\oplus g}$, i.e. for vectors of holomorphic functions on S and S'. They commute with the exponentials, in the sense that for all holomorphic functions $f : S(\mathbb{C}) \to \mathbb{C}^g$ and $f' : S'(\mathbb{C}) \to \mathbb{C}^g$ we get

$$\mathrm{Tr}(\exp(f)) = \exp'(\mathrm{Tr}(f)) \qquad \text{and} \qquad \exp(q'^*(f')) = q'^*(\exp'(f')). \tag{21}$$

Now we want to consider some cohomology groups related to the monodromy of logarithms. In our situation the bases of abelian schemes are too large to be able of having control on their cohomology. If we consider an algebraic curve $C \subset S(\mathbb{C})$, we can think of restricting the diagram of Equation (18) to C: more precisely, if we denote $C' := q(C)$ and $D := q'(C')$ we get a new diagram:

$$\begin{array}{ccccc} \mathcal{A}(\mathbb{C})|_C & \longrightarrow & \mathcal{A}'(\mathbb{C})|_{C'} & \longrightarrow & \mathfrak{A}_g(\mathbb{C})|_D \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{q_C} & C' & \xrightarrow{q'_{C'}} & D, \end{array} \qquad C \xrightarrow{p_C} D \tag{22}$$

where $q_C, q'_{C'}$ and p_C denote the restrictions of the corresponding maps. In this situation, assuming for the moment that C' and D are still of dimension 1, we can clearly consider the analogous notions of trace operator and pullback operator as well as the restriction of a section $\sigma : S \to \mathcal{A}$ to C, which will be denoted by σ_C. We also have analogous notions of monodromy representations and monodromy groups, e.g. we will write $\rho_C, \theta_{\sigma_C}, \mathrm{Mon}(\mathcal{A}|_C)$ and $M^{\mathrm{rel}}_{\sigma_C}$ for the obvious objects associated to the scheme $\mathcal{A}|_C \to C$ and to the section $\sigma_C : C \to \mathcal{A}|_C$; we do the same for C'. Moreover, if we denote by $i : C \hookrightarrow S(\mathbb{C})$ and $i' : C' \hookrightarrow S'(\mathbb{C})$ the natural embeddings and choose base points $s \in C$ and $s' \in C'$, from a topological point of view we obtain the following diagram involving the induced homomorphisms between fundamental groups (for the surjectivity of the upper horizontal arrow see Remark 3.3):

$$\begin{array}{ccc} \pi_1(S(\mathbb{C}), s) & \xrightarrow{q_*} & \pi_1(S'(\mathbb{C}), s') \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i'_*} \\ \pi_1(C, s) & \xrightarrow{q_{C*}} & \pi_1(C', s'). \end{array} \tag{23}$$

Anyway, this kind of restriction causes the loss of much information from the starting schemes. With the next result we want to overcome the problem of having large bases, by showing that we can restrict the abelian schemes to some suitable curves lying into the bases so that the topological and monodromy properties are preserved.

Lemma 3.4. *There exists a curve $C \subseteq S(\mathbb{C})$ with the following properties:*

(i) *the morphism $q_C : C \to C'$ appearing in Equation (22) is finite, the curves C and C' are affine and the section $\sigma_C : C \to \mathcal{A}|_C$ is non-torsion and non-isoconstant;*

(ii) $\mathrm{Tr}(\sigma_C) = \mathrm{Tr}(\sigma)|_{C'}$*;*

(iii) *the homomorphisms i_* and i'_* appearing in Equation (23) are surjective;*

(iv) *if we denote by ρ' the monodromy representation of the abelian scheme $\mathcal{A}' \to S'$, then we have $\rho_C = \rho \circ i_*, \rho_{C'} = \rho' \circ i'_*$ and Mon$(\mathcal{A})$ = Mon$(\mathcal{A}|_C)$, Mon$(\mathcal{A}')$ = Mon$(\mathcal{A}'|_{C'})$;*

(*v*) $\ker q_{C*} \subseteq \ker \rho_C$;

(*vi*) $\theta_{\sigma_C} = \theta_\sigma \circ i_*$ *and* $\theta_\sigma(\pi_1(S(\mathbb{C}), s)) = \theta_{\sigma_C}(\pi_1(C, s))$;

(*vii*) *if* $M_\sigma^{rel} = \{0\}$, *then* $M_{\sigma_C}^{rel} = \{0\}$.

Proof. (i) In order to construct the curve C let us consider the general hyperplane section $Y = H \cap S(\mathbb{C})$ and choose a base point $s \in Y$. By Lefschetz theorem on quasi-projective varieties [18, 2.2], the natural map $\pi_1(Y, s) \to \pi_1(S(\mathbb{C}), s)$ between fundamental groups is surjective. After iterating the process of taking such generic hyperplane sections we obtain an irreducible curve C_1 with the property that the natural map $\pi_1(C_1, s) \to \pi_1(S(\mathbb{C}), s)$ is surjective for a base point $s \in C_1$. The restriction of q to C_1 defines a morphism q_{C_1} which is finite: in fact, since every closed embedding is a finite morphism and $q : S \to S'$ is a finite morphism (see Remark 3.3), then the restriction q_{C_1} is finite. We define $C' := q_{C_1}(C_1)$ and $C := q^{-1}(C')$. Notice that C' is an affine curve since q_{C_1} is finite and surjective; moreover, C could be no longer irreducible but it is still affine because q is an affine morphism (since it is finite). Furthermore, since $q : S \to S'$ is finite then it is closed. We have a closed embedding $i : C \hookrightarrow S(\mathbb{C})$, and we can conclude that the restriction $q_C : C \to C'$ is a finite morphism. Up to remove a finite number of points from C' with the corresponding preimages, we can suppose that both C and C' are affine curves. In addition, by the genericity of the hyperplane sections we can assume that the section $\sigma_C : C \to \mathcal{A}|_C$ is non-torsion and non-isoconstant.

(ii) This property is true by construction, since $C = q^{-1}(C')$.

(iii) By (*i*) we have a surjective homomorphism $\pi_1(C_1, s) \to \pi_1(S(\mathbb{C}), s)$. Since C_1 is an irreducible component of C and since removing a finite number of points preserves the fact of having a surjective map on fundamental groups, then the morphism $i_* : \pi_1(C, s) \to \pi_1(S(\mathbb{C}), s)$ is surjective. Then, the surjectivity of $i'_* : \pi_1(C', s') \to \pi_1(S'(\mathbb{C}), s')$ follows by Equation (23).

(iv) Just recall that for abelian schemes obtained as base change we define periods as in Equation (9), so that we have $\rho_C = \rho \circ i_*$ and $\rho_{C'} = \rho' \circ i'_*$. Therefore, the claim follows from (*iii*).

(v) Let $h \in \ker q_{C*}$. Since we can define periods of $\mathcal{A}|_C \to C$ as in Equation (9) and $q_{C*}(h)$ has a trivial monodromy action on the periods of $\mathcal{A}'|_{C'} \to C'$, then we get $h \in \ker \rho_C$; hence the claim.

(*vi*) Since we have the embedding $i : C \hookrightarrow S(\mathbb{C})$, we clearly have $\theta_{\sigma_C}(\pi_1(C), s) \subseteq \theta_\sigma(\pi_1(S(\mathbb{C}), s))$. In the other direction, by (*iv*) it is enough to show that each variation of a logarithm of σ induced by an element of $\pi_1(S(\mathbb{C}), s)$ can be obtained as a variation of a logarithm of σ_C induced by the monodromy action of some element of $\pi_1(C, s)$; and this is true thanks to the surjectivity of i_* which was proven in part (*iii*). This also proves the equality $\theta_{\sigma_C} = \theta_\sigma \circ i_*$.

(*vii*) Let us assume $M_\sigma^{\text{rel}} = \{0\}$ and consider $h \in \ker \rho_C$. By part (*iv*) we have $\rho_C = \rho \circ i_*$, and this implies that $i_*(h) \in \ker \rho$. The assumption $M_\sigma^{\text{rel}} = \{0\}$ implies that $\theta_\sigma(i_*(h)) = 0$. By part (*vi*) we have $\theta_{\sigma_C} = \theta_\sigma \circ i_*$ which implies $\theta_{\sigma_C}(h) = \theta_\sigma(i_*(h)) = 0$, i.e. $M_{\sigma_C}^{\text{rel}} = \{0\}$.

□

Remark 3.5. In Lemma 3.4 the fact that σ_C is non-torsion follows by the genericity of hyperplane sections, but notice that actually it is also a consequence of part (*iii*) of Lemma 3.4.

Now, let us consider an algebraic curve $C \subset S(\mathbb{C})$ satisfying Lemma 3.4. We want to consider the restriction of the exact sequence in Equation (5) to the curve C. Since C is a Stein space, by [14, Theorem 5.3.1] the restriction $\mathscr{Lie}(\mathcal{A})|_C$ is isomorphic to $(\mathcal{O}_C^{\text{an}})^{\oplus g}$, where $\mathcal{O}_C^{\text{an}}$ denotes the sheaf of holomorphic functions on C. Therefore, we obtain the exact sequence

$$0 \to \Lambda_C \to (\mathcal{O}_C^{\text{an}})^{\oplus g} \to \Sigma^{\text{an}}|_C \to 0. \tag{24}$$

Let us study the exact sequence in cohomology groups induced by Equation (24). By Remark 2.7 and by part (*iv*) of Lemma 3.4 we can conclude that no non-zero period can be globally defined on C. In other words, we get $H^0(C, \Lambda_C) = 0$. In addition, since C is a Stein space we have

$$H^1(C, (\mathcal{O}_C^{\text{an}})^{\oplus g}) = \bigoplus_{i=1}^{g} H^1(C, (\mathcal{O}_C^{\text{an}})) = 0.$$

Thus, we obtain the exact sequence of cohomology groups

$$0 \to H^0(C, (\mathcal{O}_C^{\mathrm{an}})^{\oplus g}) \to H^0(C, \Sigma^{\mathrm{an}}|_C) \to H^1(C, \Lambda_C) \to 0. \tag{25}$$

By Proposition 2.11, we know that the obstruction to define a global logarithm of a section is given by some Galois cohomology class. We want to explain the interplay between the relevant Galois cohomology group and the exact sequence in Equation (25). Let's fix the notation $G := \pi_1(C, s)$ for a base point $s \in C$ and denote by $\rho_C : G \to \mathrm{Mon}(\mathcal{A}|_C) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z})$ the monodromy representation associated to the abelian scheme $\mathcal{A}|_C \to C$; we use the notation $\overline{h} := \rho_C(h)$. With the aim of proving how the Galois cohomology interacts with the sheaf cohomology in our situation, we put the following remark in order to interpret the sections of the sheaf of periods as continuous maps on the universal cover of C which are well-behaved with respect to the action of G.

Remark 3.6. Notice that we can view Λ_C geometrically also as a covering space of the base C with fibers isomorphic to $\mathbb{Z}^{2g}$. Denote by $u_C : \widetilde{C} \to C$ the universal cover of the curve C and note that the pull-back of Λ_C to the universal covering space $\widetilde{C}$ is the constant sheaf $\mathbb{Z}^{2g}_{\widetilde{C}}$. We can obtain the map $\Lambda_C \to C$ via the diagram

$$\begin{array}{ccc} \widetilde{C} \times \mathbb{Z}^{2g} & \longrightarrow & \Lambda_C \\ \downarrow & & \downarrow \\ \widetilde{C} & \xrightarrow{u_C} & C. \end{array}$$

Observe that the group G acts on $\widetilde{C}$ in the usual way and on $\mathbb{Z}^{2g}$ via its monodromy representation ρ_C; the space C is then the orbit space of the diagonal action of G on $\widetilde{C} \times \mathbb{Z}^{2g}$.

For every open set $V \subset C$, we can consider the open set $u_C^{-1}(V)$ which is invariant by the action of G on $\widetilde{C}$. Thus, the group of sections $\Gamma(V, \Lambda_C)$ corresponds to the group of continuous (i.e. locally constant) maps $w : u_C^{-1}(V) \to \mathbb{Z}^{2g}$ satisfying $\overline{h}w = w \circ h$, for all $h \in G$, i.e. $\overline{h} \cdot (w \circ h^{-1}) = w$.

We now give a description of the group $H^1(C, \Lambda_C)$ in terms of the Galois cohomology with respect to the action of G on $\mathbb{Z}^{2g}$ induced by the projection ρ_C. We have the following

Proposition 3.7. *With the above notation the Čech cohomology group $H^1(C, \Lambda_C)$ is canonically isomorphic to the group $H^1(G, \mathbb{Z}^{2g})$, where the action of G on $\mathbb{Z}^{2g}$ is induced by the projection $\rho_C : G \to Mon(\mathcal{A}|_C)$.*

Proof. Let $\mathcal{V} = (V_i)_{i \in I}$ be an open covering of C such that each open set V_i and each non-empty intersection $V_{i,j} = V_i \cap V_j$ is contractible, so that the first cohomology space is determined by the cohomology classes of the cocycles associated to this covering. Let us define a homomorphism $H^1(\mathcal{V}, \Lambda_C) = H^1(C, \Lambda_C) \to H^1(G, \mathbb{Z}^{2g})$ and prove it is an isomorphism.

Let $(\eta_{i,j})_{i,j}$ be a cocycle in $H^1(\mathcal{V}, \Lambda_C)$, where $\eta_{i,j} \in \Gamma(V_{i,j}, \Lambda_C)$. We refer to the diagram of Remark 3.6 and denote by $u_C : \widetilde{C} \to C$ the universal cover of C; moreover, let us define $U_i = u_C^{-1}(V_i)$ and $U_{i,j} = u_C^{-1}(V_{i,j}) = U_i \cap U_j$. By considering the pull-backs $u_C^*(\eta_{i,j})$, we obtain a cocycle $w_{i,j}$ with values in $\mathbb{Z}^{2g}$ for the covering $u_C^*(\mathcal{V}) = (\pi^{-1}(V_i))_{i \in I}$. By the above remark, the elements $w_{i,j}$ can be viewed as continuous (i.e. locally constant) functions $U_{i,j} \to \mathbb{Z}^{2g}$, satisfying

$$w_{i,j} = \overline{h} \cdot (w_{i,j} \circ h^{-1}) \qquad \text{for each } h \in G. \tag{26}$$

Since $\widetilde{C}$ is simply connected, we can write

$$w_{i,j} = w_i - w_j$$

for suitable continuous functions $w_i : U_i \to \mathbb{Z}^{2g}$, for $i \in I$. Let us define, for every $h \in G$ and $i \in I$, the continuous function $\gamma_{h,i} : U_i \to \mathbb{Z}^{2g}$ by

$$\gamma_{h,i} = w_i - \overline{h} \cdot (w_i \circ h^{-1}).$$

From (26) it follows that on $U_{i,j} = U_i \cap U_j$ the two functions $\gamma_{h,i}, \gamma_{h,j}$ coincide. Then we obtain, by gluing the $\gamma_{h,i}$ for $i \in I$, a well-defined continuous function $\gamma_h : \widetilde{C} \to \mathbb{Z}^{2g}$, which necessarily is a constant vector. By construction, the map $h \mapsto \gamma_h$ is a cocycle and defines a class $[h \mapsto \gamma_h]$ in $H^1(G, \mathbb{Z}^{2g})$. Also, its class only depends on the class $[(\eta_{i,j})_{i,j}]$ in $H^1(C, \Lambda_C)$. Therefore, we obtain a group homomorphism, defined as follows:

$$H^1(\mathcal{V}, \Lambda_C) \to H^1(G, \mathbb{Z}^{2g}), \qquad [(\eta_{i,j})_{i,j}] \mapsto [h \mapsto \gamma_h].$$

Let us verify that this homomorphism is injective. Take then a cocycle $(\eta_{i,j})_{i,j}$ giving rise, via the above procedure, to a coboundary $h \mapsto \gamma_h$. Then we can write $\gamma_h = \gamma - \overline{h} \cdot \gamma$ for a fixed vector $\gamma \in \mathbb{Z}^{2g}$; now, setting $w_i' = w_i - \gamma$ we obtain that

$$w_{i,j} = w_i' - w_j',$$

where the w_i' satisfy the invariance condition

$$\overline{h} \cdot (w_i' \circ h^{-1}) = w_i', \qquad \text{for each } h \in G$$

similar to the relation holding for the $w_{i,j}$. By Remark 3.6, this fact means that the w_i' are of the form $w_i' = u_C^*(\eta_i)$ for suitable sections $\eta_i \in \Gamma(V_i, \Lambda_C)$, so that $\eta_{i,j} = \eta_i - \eta_j$ is a coboundary.

Let us now verify that the homomorphism $H^1(\mathcal{V}, \Lambda_C) \to H^1(G, \mathbb{Z}^{2g})$ is also surjective. Let then $h \mapsto \gamma_h$ be a cocycle with values in $\mathbb{Z}^{2g}$, so that it satisfies

$$\gamma_{h_1 h_2} = \overline{h}_1 \cdot \gamma_{h_2} + \gamma_{h_1}, \qquad \text{for each } h_1, h_2 \in G.$$

We want to define functions $w_i : U_i \to \mathbb{Z}^2$ satisfying

$$w_i - \overline{h} \cdot (w_i \circ h^{-1}) = \gamma_h \qquad \text{for each } h \in G. \tag{27}$$

Recall that each open set $U_i = p^{-1}(V_i)$ is a disjoint union of connected open sets of the form $h(U_i^0)$, for $h \in G$ and for some component U_i^0 of U_i (actually any choice of a component would work). We can then choose the function $w_i : U_i \to \mathbb{Z}^{2g}$ so that it vanishes on U_i^0 and, on the component $h^{-1}(U_i^0)$ its value is $-\overline{h}^{-1} \cdot \gamma_h$. Then the cocycle condition satisfied by γ implies the relations Equation (27) on each component $h(U_i^0)$, so on the whole U_i. At this point we have that the functions $w_i - w_j =: w_{i,j}$ on $U_i \cap U_j$ are invariant, in the sense that $\overline{h} \cdot (w_{i,j} \circ h^{-1}) = w_{i,j}$, so they are of the form $u_C^*(\eta_{i,j})$ for sections $\eta_{i,j} \in \Gamma(V_{i,j}, \Lambda_C)$.

□

Therefore, thanks to the exact sequence Equation (25) and Proposition 3.7 we obtain a surjective group homomorphism

$$\Sigma^{\mathrm{an}}|_C(C) \ni \tau \mapsto [\tau] \in H^1(C, \Lambda_C) \simeq H^1(G, \mathbb{Z}^{2g})$$

associating to every holomorphic section a cohomology class, which measures the obstruction of having global abelian logarithm.

Remark 3.8. Notice that by Lemma 3.4 also the curve C' is a Stein space. Moreover, by Equation (9) the group $\mathrm{Mon}(\mathcal{A}|_C)$ is a finite index subgroup of $\mathrm{Mon}(\mathcal{A}'|_{C'})$; since $\mathrm{Mon}(\mathcal{A})$ acts irreducibly on the lattice of periods (see Proposition 2.6), by part (iv) of Lemma 3.4 we obtain that the action of $\mathrm{Mon}(\mathcal{A}'|_{C'})$ on the corresponding lattice of periods is still irreducible. In other words we get $H^0(C', \Lambda_{C'}) = 0$. Therefore, the analogous results of Equation (25) and Proposition 3.7 hold if we replace the curve C by the curve C'. In particular, for some base point $s' \in C'$ we have the surjective group homomorphism

$$\Sigma'^{,\mathrm{an}}|_{C'}(C') \ni \tau' \mapsto [\tau'] \in H^1(C', \Lambda_{C'}) \simeq H^1(\pi_1(C', s'), \mathbb{Z}^{2g}),$$

where the action of $\pi_1(C', s')$ on $\mathbb{Z}^{2g}$ is induced by the monodromy representation $\rho_{C'} : \pi_1(C', s') \to \mathrm{Mon}(\mathcal{A}'|_{C'})$.

Remark 3.9. Let us point out the relationship between the cohomology groups $H^1(\pi_1(S'(\mathbb{C}), s'), \mathbb{Z}^{2g})$ and $H^1(\pi_1(C', s'), \mathbb{Z}^{2g})$. By part (iii) of Lemma 3.4 the homomorphism $i'_* : \pi_1(C', s') \to \pi_1(S', s')$ is surjective and this induces a homomorphism

$$f : H^1(\pi_1(S'(\mathbb{C}), s'), \mathbb{Z}^{2g}) \to H^1(\pi_1(C', s'), \mathbb{Z}^{2g});$$

more precisely, if $c' : \pi_1(S'(\mathbb{C}), s') \to \mathbb{Z}^{2g}$ is a cocycle representing some cohomology class $[c']_{S'}$ in $H^1(\pi_1(S'(\mathbb{C}), s'), \mathbb{Z}^{2g})$ we can define $f([c']_{S'})$ as the cohomology class in $H^1(\pi_1(C', s'), \mathbb{Z}^{2g})$ of the cocycle $c'' : \pi_1(C', s') \to \mathbb{Z}^{2g}$ defined as $c''(h) := c'(i'_*(h))$. The analogous relation holds for the cohomology groups $H^1(\pi_1(S(\mathbb{C}), s), \mathbb{Z}^{2g})$ and $H^1(\pi_1(C, s), \mathbb{Z}^{2g})$.

We are ready to prove the following theorem:

Theorem 3.10. *Let $\phi : \mathcal{A} \to S$ be an abelian scheme such that (up to a finite base change) the modular map $p : S \to \mathbb{A}_g$ is a finite surjective morphism. If $\sigma \in \Sigma(S)$ is a non-torsion section, then the relative monodromy group M_σ^{rel} is non-trivial.*

Proof. Thanks to Theorem 2.1 and Proposition 3.2 up to multiplying σ by a positive integer we can suppose $\mathrm{Tr}(\sigma) = 0$. Let C be a curve satisfying Lemma 3.4.

We work by contradiction by assuming $M_\sigma^{\mathrm{rel}} = \{0\}$. By part (vii) of Lemma 3.4 this implies $M_{\sigma_C}^{\mathrm{rel}} = \{0\}$, or equivalently $\ker \rho_C \subseteq \ker \theta_{\sigma_C}$. Let $q_{C*} : \pi_1(C, s) \to \pi_1(C', s')$ be the homomorphism appearing in Equation (23). By part (v) of Lemma 3.4 we have $\ker q_{C*} \subseteq \ker \rho_C$; thus, we have $\ker q_{C*} \subseteq \ker \theta_{\sigma_C}$. Introducing the notation $\overline{G} := q_{C*}\pi_1(C, s)$ we obtain that there exists a homomorphism $f : \overline{G} \to \mathrm{SL}_{2g+1}(\mathbb{Z})$ such that the diagram

$$\begin{array}{ccc} \pi_1(C,s) & \xrightarrow{\ q_{C*}\ } & \overline{G} \\ & \searrow^{\theta_{\sigma_C}} & \downarrow f \\ & & \mathrm{SL}_{2g+1}(\mathbb{Z}) \end{array} \tag{28}$$

commutes; in other words, θ_{σ_C} factors through q_{C*}. By using the same notation as in Equation (12), this means that for $h \in \pi_1(C, s)$ the vector $c(h)$ only depends on the element $\overline{h} := q_{C*}(h) \in \overline{G}$. Then we obtain a cocycle

$$\overline{G} \ni \overline{h} \mapsto c(h) \in \mathbb{Z}^{2g}, \tag{29}$$

representing a certain cohomology class in $H^1(\overline{G}, \mathbb{Z}^{2g})$ which we denote with $[\sigma_C]_{\overline{G}}$. In other words, we have obtained an injective homomorphism $H^1(\pi_1(C, s), \mathbb{Z}^{2g}) \hookrightarrow H^1(\overline{G}, \mathbb{Z}^{2g})$ which can be used to define a homomorphism

$$\xi : H^0(\Sigma^{\mathrm{an}}|_C, C) \to H^1(\overline{G}, \mathbb{Z}^{2g}),$$

which to a section τ over C associates a cohomology class which we denote by $[\tau]_{\overline{G}}$. We want to show that $[\sigma_C]_{\overline{G}}$ can be canonically identified with a cohomology class $[\sigma]_{C'}$ in $H^1(\pi_1(C', s'), \mathbb{Z}^{2g})$.

To this end, let us construct some homomorphisms between cohomology groups. Since we are assuming $M_\sigma^{\mathrm{rel}} = \{0\}$ and the map $q_* : \pi_1(S(\mathbb{C}), s) \to \pi_1(S'(\mathbb{C}), s')$ is surjective by Remark 3.3, with the same reasoning as in Equation (28) we obtain an injective homomorphism

$$\zeta_1 : H^1(\pi_1(S(\mathbb{C}), s), \mathbb{Z}^{2g}) \hookrightarrow H^1(\pi_1(S'(\mathbb{C}), s'), \mathbb{Z}^{2g})$$

which to a cohomology class $[\tau]_S$ associates a cohomology class denoted with $[\tau]_{S'}$. By Remark 3.9 we have a homomorphism

$$\zeta_2 := f : H^1(\pi_1(S'(\mathbb{C}), s'), \mathbb{Z}^{2g}) \to H^1(\pi_1(C', s'), \mathbb{Z}^{2g}).$$

By composition we obtain a map

$$\zeta_3 := \zeta_2 \circ \zeta_1 : H^1(\pi_1(S(\mathbb{C}), s), \mathbb{Z}^{2g}) \to H^1(\pi_1(C', s'), \mathbb{Z}^{2g})$$

which to a class $[\tau]_S$ associates a class denoted with $[\tau]_{C'}$. Since $\overline{G}$ is a subgroup of $\pi_1(C', s')$ the restriction homomorphism induces a homomorphism between cohomology groups

$$\eta : H^1(\pi_1(C', s'), \mathbb{Z}^{2g}) \to H^1(\overline{G}, \mathbb{Z}^{2g}).$$

Notice that we cannot identify the cohomology groups $H^1(\overline{G}, \mathbb{Z}^{2g})$ and $H^1(\pi_1(C', s'), \mathbb{Z}^{2g})$, but we want to prove that η becomes an isomorphism when restricted to cohomology classes coming from global sections in $\Sigma(S)$. Thus, let us denote by $r : H^0(S, \Sigma^{\mathrm{an}}) \to H^0(C, \Sigma^{\mathrm{an}}|_C)$ the restriction map which to a section τ over S associates the restriction τ_C. Define the homomorphism

$$\zeta : r(H^0(S, \Sigma^{\mathrm{an}})) \to H^1(\pi_1(C', s'), \mathbb{Z}^{2g}), \qquad \tau_C \mapsto [\tau]_{C'},$$

where $\tau \in \Sigma(S)$ is such that $r(\tau) = \tau_C$: notice that this map is well-defined by Lemma 3.4. In fact, if $\tau_1, \tau_2 \in \Sigma(S)$ are such that $\tau_{1,C} = \tau_{2,C}$, then $M_{\tau_1}^{\mathrm{rel}} = M_{\tau_2}^{\mathrm{rel}}$ by part (vi) of Lemma 3.4. This implies $[\tau_1]_S = [\tau_2]_S$ and so $[\tau_1]_{C'} = [\tau_2]_{C'}$.

Furthermore, by restriction of ξ we define the homomorphism

$$\xi|_{r(H^0(S,\Sigma^{\mathrm{an}}))} : r(H^0(S, \Sigma^{\mathrm{an}})) \to H^1(\overline{G}, \mathbb{Z}^{2g}), \qquad \tau_C \mapsto [\tau_C]_{\overline{G}}.$$

We claim that $\mathrm{Im}(\zeta) \cong \mathrm{Im}(\xi|_{r(H^0(S,\Sigma^{\mathrm{an}}))})$. The restriction of η to $\mathrm{Im}(\zeta)$ induces a homomorphism

$$\eta' := \eta|_{\mathrm{Im}(\zeta)} : \mathrm{Im}(\zeta) \to \mathrm{Im}(\xi|_{r(H^0(S,\Sigma^{\mathrm{an}}))}), \qquad [\tau]_{C'} \mapsto [\tau_C]_{\overline{G}},$$

where $r(\tau) = \tau_C$; let's prove that η' is an isomorphism. Notice that it is well-defined and surjective by construction, so it's enough to prove that it is injective. To this end, consider $\tau_1, \tau_2 \in \Sigma(S)$ such that $[\tau_{1,C}]_{\overline{G}} = [\tau_{2,C}]_{\overline{G}}$. This implies

$$[\tau_{1,C} - \tau_{2,C}]_{\overline{G}} = [\tau_{1,C}]_{\overline{G}} - [\tau_{2,C}]_{\overline{G}} = 0.$$

Thanks to the injective homomorphism obtained by Equation (29), we get $[\tau_{1,C} - \tau_{2,C}]_C = 0$. By part (vi) of Lemma 3.4, we also obtain $[\tau_1 - \tau_2]_S = 0$. By applying ζ_3 we get $[\tau_1 - \tau_2]_{C'} = 0$, which means that η' is injective.

Thus, by what we have just proven and by Remark 3.8 the section σ_C determines a cohomology class $[\sigma_C]_{C'}$ in $H^1(\pi_1(C', s'), \mathbb{Z}^{2g}) \cong H^1(C', \Lambda_{C'})$. Again thanks to Remark 3.8, we have that there exists a (possibly transcendental) section $\sigma' : C' \to \mathcal{A}'|_{C'}$ whose cohomology class $[\sigma']$ in $H^1(\pi_1(C', s'), \mathbb{Z}^{2g})$ coincides with the class $[\sigma_C]_{C'}$ defined above, i.e. $[\sigma'] = [\sigma_C]_{C'}$. Therefore, by Proposition 2.11 we obtain a section of the form

$$\sigma_C - q_C^*(\sigma') \in \Sigma^{\text{an}}(C)$$

that admits a logarithm. Notice that $\sigma_C - q_C^*(\sigma')$ cannot be the zero-section, more precisely we can prove that σ' is a holomorphic section which is not algebraic. In fact, assume by contradiction that σ' is an algebraic section such that $q_C^*(\sigma') = \sigma_C$. By part (ii) of Lemma 3.4 and Equation (20) we have $\operatorname{Tr}(\sigma)|_{C'} = \operatorname{Tr}(\sigma_C) = \deg q \cdot \sigma'$. By Theorem 2.1 σ' would be torsion, which is a contradiction.

Let us write

$$\Sigma^{\text{an}}(C) \ni \sigma_C - q_C^*(\sigma') = \exp(f), \tag{30}$$

for some holomorphic function $f : C \to \mathbb{C}^g$.

Now, by using properties of the operators q_C^* and Tr, we will see that σ_C also has to admit a logarithm. Taking the trace of f, we can write

$$f = \left(f - \frac{1}{\deg q} q_C^* \operatorname{Tr}(f)\right) + \frac{1}{\deg q} q_C^* \operatorname{Tr}(f) = l - l',$$

where

$$l := f - \frac{1}{\deg q} q_C^* \operatorname{Tr}(f), \quad \text{and} \quad l' := -\frac{1}{\deg q} q_C^* \operatorname{Tr}(f).$$

Observe that $\operatorname{Tr}(l) = 0$. Substituting in equation (30), we obtain

$$\sigma_C - q_C^*(\sigma') = \exp(l) - \exp(l'). \tag{31}$$

Recalling that $\operatorname{Tr}(\sigma_C) = 0$ by part (ii) of Lemma 3.4 and Theorem 2.1, taking traces on both sides of the previous equation and rearranging we obtain

$$\begin{aligned} 0 = \operatorname{Tr}(q_C^*(\sigma') - \exp(l')) &= \deg q \cdot \sigma' - \operatorname{Tr}(\exp(l')) = \\ &= \deg q \cdot \sigma' - \exp(\operatorname{Tr}(l')). \end{aligned}$$

Thus, $\deg q \cdot \sigma'$ admits a logarithm, and the same must hold for σ' and for $q_C^*(\sigma')$: in particular, we have $q_C^*(\sigma') = \exp(l')$. From equation (31) we then deduce that σ_C admits a globally defined logarithm as well (necessarily $\sigma_C = \exp(l)$). By part (i) of Lemma 3.4 and Remark 2.10, this leads to a contradiction.

□

3.3 Rank of relative monodromy

In this section we prove Theorem 1.2. We keep the same notations as above, in particular recall that $\rho : \pi_1(S(\mathbb{C}), s) \to \operatorname{Mon}(\mathcal{A})$ denotes the monodromy representation associated to the abelian scheme $\phi : \mathcal{A} \to S$.

Theorem 3.11. *Let $\phi : \mathcal{A} \to S$ be an abelian scheme such that (up to a finite base change) the modular map $p : S \to \mathbb{A}_g$ is a finite surjective morphism. If $\sigma : S \to \mathcal{A}$ is a non-torsion section, then the relative monodromy group M_σ^{rel} is isomorphic to $\mathbb{Z}^{2g}$.*

Proof. Let us proceed by contradiction by assuming that $M_\sigma^{\rm rel}$ is a submodule of $\mathbb{Z}^{2g}$ of rank g' with $g' < 2g$; notice that by Theorem 1.1 we have $g' > 0$. In other words, this means that for every $h \in H := \ker \rho$ the logarithm $\log_\sigma$ is transformed by h as

$$\log_\sigma \overset{h}{\mapsto} \log_\sigma + u_1^h \mu_1 + \cdots + u_{g'}^h \mu_{g'},$$

where $\mu_1, \ldots, \mu_{g'}$ are fixed non-zero periods (depending on σ). Recall that, for $k \in \pi_1(S(\mathbb{C}), s)$ the logarithm $\log_\sigma$ will be sent by k to a new determination of the form

$$\log_\sigma + v_1\omega_1 + \cdots + v_{2g}\omega_{2g},$$

where $v_1, \ldots, v_{2g} \in \mathbb{Z}$ and $\omega_1, \ldots, \omega_{2g}$ denote period functions. By Proposition 2.6 there exists $k \in \pi_1(S(\mathbb{C}), s)$ such that $\rho(k) \cdot M_\sigma^{\rm rel} \nsubseteq M_\sigma^{\rm rel}$, i.e. the action of $\rho(k)$ does not preserve the relative monodromy group. Therefore, we can fix $k \in \pi_1(S(\mathbb{C}), s)$ and $h \in H$ such that $\rho(k^{-1}) \cdot (u_1^h\mu_1 + \cdots + u_{g'}^h\mu_{g'}) \notin M_\sigma^{\rm rel}$; define $h' := k^{-1}hk$ and observe that $h' \in H$ since $H \trianglelefteq G$. If we look at the monodromy action of h' on the abelian logarithm we obtain

$$\begin{aligned}\log_\sigma &\overset{k}{\mapsto} \log_\sigma + v_1\omega_1 + \cdots + v_{2g}\omega_{2g} \overset{h}{\mapsto} \log_\sigma + v_1\omega_1 + \cdots + v_{2g}\omega_{2g} + u_1^h\mu_1 + \cdots + u_{g'}^h\mu_{g'} \\ &\overset{k^{-1}}{\longmapsto} \log_\sigma + \rho(k^{-1}) \cdot (u_1^h\mu_1 + \cdots + u_{g'}^h\mu_{g'}).\end{aligned}$$

Since $h' \in H$, then we have $\rho(k^{-1}) \cdot (u_1^h\mu_1 + \cdots + u_{g'}^h\mu_{g'}) \in M_\sigma^{\rm rel}$, which contradicts the choice of k and h. This concludes the proof.

□

4 Some applications

Now, we give some useful applications of our results.

4.1 Manin's kernel theorem

A first application of the non-triviality of relative monodromy Theorem 1.1 is a strong version of Manin's kernel theorem for an abelian scheme $\mathcal{A} \to S$ admitting a finite surjective map on some universal family without locally constant parts. The same result in the case of elliptic schemes was obtained in [9]. We refer to the Betti coordinates defined in Section 2.2 (see Equation (4)).

Theorem 4.1. *Let $\mathcal{A} \to S$ be an abelian scheme with a section $\sigma \in \Sigma(S)$. Assume that it admits a finite base change with a finite map on some universal family of abelian varieties without locally constant parts. The Betti coordinates $\beta_{\sigma,i}$ with $i = 1, \ldots, 2g$ are globally defined functions on $S(\mathbb{C})$ if and only if σ is a torsion section. In that case, the Betti coordinates are rational constants.*

Proof. Clearly, if σ is a torsion section the claim follows by definition. Conversely, if the Betti coordinates are globally defined on $S(\mathbb{C})$ then the relative monodromy group $M_\sigma^{\rm rel}$ is trivial. By Theorem 1.1 the section must be torsion.

□

4.2 Algebraic independence of periods and logarithms

Our theorem about the rank of the relative monodromy group Theorem 1.2 gives a different proof of a result due to André [1, Theorem 3] about the algebraic independence of logarithms under our hypothesis on the abelian scheme $\mathcal{A} \to S$. Let's briefly describe the setting.

Recall that the monodromy action of the fundamental group $\pi_1(S(\mathbb{C}), s)$ on periods and the logarithm of a section σ induces a representation $\theta_\sigma : \pi_1(S(\mathbb{C}), s) \to \mathrm{SL}_{2g+1}(\mathbb{Z})$ described in Equation (12). Consider the projection $\theta_\sigma(\pi_1(S(\mathbb{C}), s)) \to \mathrm{Sp}_{2g}(\mathbb{Z})$, whose image is the monodromy group $\mathrm{Mon}(\mathcal{A})$. By general theory, the Zariski-closure $\overline{\theta_\sigma(\pi_1(S(\mathbb{C}), s))}$ in SL_{2g+1} is the differential Galois group of the Picard-Vessiot extension of $\mathbb{C}(S)$ generated by the coordinates of $\omega_1, \ldots, \omega_{2g}, \log_\sigma$ and their directional derivatives along a tangent vector field ∂ with respect to the Gauss-Manin connection. Clearly the homomorphism $\theta_\sigma(\pi_1(S(\mathbb{C}), s)) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ extends to an algebraic group homomorphism

$$\xi : \overline{\theta_\sigma(\pi_1(S(\mathbb{C}), s))} \to \mathrm{Sp}_{2g}$$

Here the aforementioned result [1, Theorem 3] in the case where the abelian scheme is endowed (up to a base change) with a finite surjective morphism on a suitable universal family.

Theorem 4.2. *Let $\mathcal{A} \to S$ be an abelian scheme admitting (up to a base change) a finite surjective morphism $S \to \mathbb{A}_g$, where $\mathfrak{A}_g \to \mathbb{A}_g$ is a universal family of abelian varieties without locally constant parts. If $\sigma \in \Sigma(S)$ is a non-torsion section, the kernel $\ker \xi$ is isomorphic to $\mathbb{G}_a^{2g}$. In particular, the coordinates of $\log_\sigma$ have transcendence degree g over $\mathbb{C}(S)(\omega_1, \dots, \omega_{2g})$, where $\mathbb{C}(S)(\omega_1, \dots, \omega_{2g})$ denotes the extension of $\mathbb{C}(S)$ generated by the coordinates of periods.*

Proof. By Theorem 1.2 the kernel of the morphism $\xi|_{\theta_\sigma(\pi_1(S(\mathbb{C}),s))}$ is isomorphic to $\mathbb{G}_a^{2g}$, which implies the first part of the statement. In particular, the transcendence degree of the extension generated by the coordinates of $\log_\sigma$ and their directional derivatives over $\mathbb{C}(S)(\omega_1, \dots, \omega_{2g}, \partial\omega_1, \dots, \partial\omega_{2g})$ is $2g$, where we are fixing a tangent vector field ∂. The conclusion follows.

□

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