Finite translation orbits on double families of abelian varieties (with an appendix by E. Amerik)

Paolo Dolce Francesco Tropeano

Abstract

We study two families of g-dimensional abelian varieties, induced by distinct rational maps defined on a common variety $\overline{\mathcal{A}}$ and mapping to two bases $\overline{S}_1$ and $\overline{S}_2$. Two non-torsion sections induce birational fiberwise translations on $\overline{\mathcal{A}}$. We consider the action of a specific subset of the group generated by these translations. Under the assumption that $\dim \overline{S}_1 (= \dim \overline{S}_2) \leq g$, we prove that the points with finite orbit are contained in a proper Zariski closed subset. This subset is explicitly described to a certain extent. Our results generalize a theorem of Corvaja, Tsimermann, and Zannier to higher dimensions.

0 Introduction

In the context of algebraic dynamics, it is natural to study the distribution of *special points* under the action of the automorphism group of an algebraic variety. Cantat and Dujardin, in [13, Theorem B], establish that if X is a smooth projective surface defined over a number field and $\Gamma \subset \mathrm{Aut}(X)$ is a subgroup satisfying certain properties, then the points of $X(\mathbb{C})$ with finite Γ-orbit are contained in a proper Zariski-closed subset of X. In [15, Theorem 1.1], Corvaja, Tsimerman, and Zannier improve upon this result in the special case of a projective surface endowed with a double elliptic fibration. They demonstrate that if Γ is the group generated by the two translations induced by the elliptic fibrations, then the points with finite orbit under the action of a specific small subset of Γ lie on the union of finitely many fibers of one of the two fibrations. Their proof employs tools from the theory of unlikely intersections, particularly leveraging the properties of the so-called *Betti map*. In this paper we generalize [15, Theorem 1.1] to the case of projective varieties endowed with a double fibration in g-dimensional abelian varieties over bases of dimension at most g.

General notations. We assume that *all* algebraic varieties and morphisms are defined over $\overline{\mathbb{Q}}$. An algebraic point p of a variety X will be denoted simply as $p \in X$ (or, occasionally, more explicitly as $p \in X(\overline{\mathbb{Q}})$). We do not make use of schematic points in this work. Furthermore, we denote by $X(\mathbb{C})$ the analytification of X, which naturally carries the structure of a complex manifold. The dimension of X as a complex manifold is denoted by $\dim X$.

In several proofs, we work with numerous positive real constants, typically denoted by variables such as $C, c_0, c_1, \dots$ Our convention is that these variables are *local to the paper*, meaning their values and interpretations are valid only within the specific proof in which they appear, unless explicitly stated otherwise.

This paper employs concepts from transcendental Diophantine problems, including o-minimal structures, definable sets, and definable families. For the foundational definitions and properties, we refer the reader to the seminal works [39] and [38].

Additionally, when we write an inequality using the symbol $\gg$, we mean that the left-hand side (LHS) is greater than or equal to the right-hand side (RHS) multiplied by a constant that is independent of the variables involved in the inequality.

Definition 0.1. Let $\overline{S}$ be a non-singular, irreducible variety. A *family of g-dimensional abelian varieties* is a proper flat morphism of finite type $f : \overline{\mathcal{A}} \to \overline{S}$ with a section, where $\overline{\mathcal{A}}$ is a non-singular irreducible variety and the generic fiber is an abelian variety of dimension g over $\overline{\mathbb{Q}}(\overline{S})$ (with a rational point). After removing the singular fibers and their images we obtain a g-dimensional abelian scheme $f : \mathcal{A} \to S$ (the fiberwise group law extends uniquely to a global map that gives the structure of abelian scheme over S, see [35, Theorem 6.14]).

The set of N-torsion points of a family of g-dimensional abelian varieties $\mathcal{A}$ is denoted by $\mathcal{A}[N]$, and moreover we put $\mathcal{A}_{\mathrm{tor}} = \bigcup_{N \geq 1} \mathcal{A}[N]$. We assume the existence of a non-torsion section $\sigma\colon S \to \mathcal{A}$ of f (i.e. the image of σ is not contained in any $\mathcal{A}[N]$) and that $\mathbb{Z}\sigma$ is Zariski dense in $\mathcal{A}$. We define the following automorphism:

$$\begin{aligned} t_\sigma : \mathcal{A}(\mathbb{C}) &\to \mathcal{A}(\mathbb{C}) \\ p &\mapsto p + \sigma(f(p)). \end{aligned}$$

Let Γ_σ be the group generated by t_σ that acts naturally on $\mathcal{A}(\mathbb{C})$, for any $p \in \mathcal{A}(\mathbb{C})$ we are interested in the orbit

$$\Gamma_\sigma(p) := \{t_\sigma^r(p) \colon r \in \mathbb{N}\}.$$

Clearly each orbit is contained in a single fiber of f, but it is important to study whether the locus $\mathfrak{F}^{(1)}$ of points $p \in \mathcal{A}(\mathbb{C})$ such that $\Gamma_\sigma(p)$ is finite can be confined in a subset lying over a proper closed subset of the base. We recall that a torsion value of σ is an element of $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ and obviously $\Gamma_\sigma(p)$ is finite if and only if $f(p)$ is a torsion value. Therefore, such study of $\mathfrak{F}^{(1)}$ can be reduced to the study of the Zariski density of the torsion values of σ. But the last property depends on the values of $\dim S$ and g in the following way: if $\dim S \geq g$ then $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ is Zariski dense in S if and only if the rank of the Betti map β_σ is $2g$ (see [20, Theorem 1.3]). Note that [8, Proposition 2.1.1] shows that $\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma \geq 2g$ implies that $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ is dense in $S(\mathbb{C})$ with respect to the analytic topology. On the other hand if $\dim S < g$ then $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ is not Zariski dense in S. This is a special case of the relative Manin-Mumford conjecture that has been recently proved in [20, Theorem 1.1].

We examine a variation of the aforementioned setting.

Definition 0.2. A *double g-dimensional abelian rational fibration* is the datum of two dominant rational maps $f_1 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_1$ and $f_2 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_2$, such that $\overline{\mathcal{A}}$, $\overline{S}_1$ and $\overline{S}_2$ are non-singular and irreducible varieties, and moreover the induced morphisms on the (maximal) loci where f_1 and f_2 are well defined induce families of g-dimensional abelian varieties. In particular, for each of them the generic fiber is an abelian variety over $k_{\overline{S}_1} := \overline{\mathbb{Q}}(\overline{S}_1)$ and $k_{\overline{S}_2} := \overline{\mathbb{Q}}(\overline{S}_2)$ respectively.

Note that $\dim(\overline{S}_1) = \dim(\overline{S}_2)$. Additionally, we usually require that $\overline{\mathcal{A}}$, $\overline{S}_1$ and $\overline{S}_2$ are *projective* and we denote with $\mathrm{Fund}(f_i)$ the fundamental locus of f_i, i.e. the proper closed subset on which the rational map f_i cannot be extended.

Assumptions. *In addition we impose the following rather standard conditions on these families:*

1) *The two abelian families are "distinct", in the sense that their common fibers (if any) lie over a proper Zariski closed subset E either of $\overline{S}_1$ or of $\overline{S}_2$. Let's assume $E \subseteq \overline{S}_1$.*

2) *We consider $i, j \in \{1, 2\}$ with $i \neq j$. We assume that $\mathrm{Fund}(f_j)$ is not horizontal with respect to f_i*[1]*. Hence, the set $\mathrm{Fund}(f_j) \setminus (\mathrm{Fund}(f_1) \cap \mathrm{Fund}(f_2))$ is contained in a closed subset $f_i^{-1}(W)$ where W is a proper Zariski closed subset of $\overline{S}_i$ defined over $\overline{\mathbb{Q}}$. We fix a W as above and we call it Ind_i. As a consequence, after removing from $\overline{S}_i$ and $\overline{\mathcal{A}}$ some suitable closed subset defined over $\overline{\mathbb{Q}}$, the maps f_i induce two families of abelian varieties over a quasi-projective base (we still have bad reduction). Moreover, after removing the respective singular fibers and discriminant loci we obtain two abelian schemes $f_i : \mathcal{A}_i \to S_i$. We assume the existence of non-torsion sections $\sigma_i : S_i \to \mathcal{A}_i$ of f_i.*

3) *$\mathbb{Z}\sigma_i$ is Zariski dense in $\mathcal{A}_i$.*

4) *The abelian schemes $\mathcal{A}_i \to S_i$ have no fixed part, i.e. the respective generic fibers have trivial $\overline{k_{\overline{S}_i}}/\overline{\mathbb{Q}}$-trace.*

The fiber of a point $s \in S_i(\mathbb{C})$ with respect to the morphism f_i will be denoted by $\mathcal{A}_{i,s}$ and the discriminant locus of f_i is $\mathrm{Sing}_i = \overline{S}_i \setminus S_i$. Consider the two birational transformations t_i of $\overline{\mathcal{A}}(\mathbb{C})$ acting by translation along the general fiber of f_i and mapping the zero section to σ_i:

$$\begin{aligned} t_i : \overline{\mathcal{A}}(\mathbb{C}) &\dashrightarrow \overline{\mathcal{A}}(\mathbb{C}) \\ p &\mapsto p + \sigma_i(f_i(p)). \end{aligned}$$

We study the action of the subgroup $\Gamma_{\sigma_1,\sigma_2} := \langle t_1, t_2 \rangle$ generated by t_1 and t_2 in the group of birational transformations $\mathrm{Bir}(\overline{\mathcal{A}}(\mathbb{C}))$; in particular we want to confine the points with finite orbits. First of all, since t_1 and t_2 are not defined everywhere on $\overline{\mathcal{A}}(\mathbb{C})$ we have to be careful with the notion of orbit. For $p \in \overline{\mathcal{A}}(\mathbb{C})$ we put:

$$\Gamma_{\sigma_1,\sigma_2}(p) := \{\gamma(p) \colon \gamma \in \Gamma_{\sigma_1,\sigma_2} \text{ and } \gamma \text{ is well defined at } p\}.$$

In fact, we shall focus on a subset of the orbit showing that already the points with finite orbits under the action of a "small subset" of $\Gamma_{\sigma_1,\sigma_2}$ lie in a proper Zariski closed subset of $\overline{\mathcal{A}}(\mathbb{C})$. This small subset of $\Gamma_{\sigma_1,\sigma_2}$ will be precisely the following:

$$O = O_{\sigma_1,\sigma_2} := \{t_1^{r_1} \circ t_2^{r_2} : r_1, r_2 \in \mathbb{N}\}.$$

For any $p \in \overline{\mathcal{A}}(\mathbb{C})$ we clearly have $O(p) \subseteq \Gamma_{\sigma_1,\sigma_2}(p)$ and moreover we define

$$\mathfrak{F} = \mathfrak{F}^{(2)} := \{p \in \overline{\mathcal{A}}(\mathbb{C}) : O(p) \text{ is finite}\}.$$

Remark 0.3. Note that if $p \in \mathfrak{F}$ then both $f_1(p)$ and $f_2(p)$ are torsion values for the relative sections, and in particular $p \in \overline{\mathcal{A}}(\overline{\mathbb{Q}})$. In other words $\mathfrak{F}$ is contained in the intersection between the f_1-fibers and the f_2-fibers of the torsion values.

The case $g = 1$ has been already treated in [15, Theorem 1.1] where it is shown that $\mathfrak{F}$ lies over finitely many fibers of f_2. The following theorem is our main result:

Theorem 0.4. *Let $f_1 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_1$ and $f_2 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_2$ be a double g-dimensional abelian rational fibration with $\overline{\mathcal{A}}$, $\overline{S}_1$ and $\overline{S}_2$ projective varieties. Moreover, assume that f_1 and f_2 satisfy the assumptions $1) - 4)$ above. If $\dim \overline{S}_1 = \dim \overline{S}_2 \leq g$, then there exist two proper Zariski closed subsets $Z_1 \subset \overline{S}_1$ and $Z_2 \subset \overline{S}_2$ such that*

$$\mathfrak{F} \setminus (\mathrm{Fund}(f_1) \cap \mathrm{Fund}(f_2)) \subseteq f_1^{-1}(Z_1) \cup f_2^{-1}(Z_2). \tag{1}$$

Our result can be seen as a generalization of the relative Manin-Mumford conjecture for sections in the following way: in the case of a single family of abelian varieties [20, Theorem 1.1] says that the relative locus $\mathfrak{F}^{(1)}$ is not Zariski dense for $\dim S \leq g - 1$. On the other hand, in the case of two families of abelian varieties with same base S, Theorem 0.4 implies that $\mathfrak{F}^{(2)}$ is not Zariski dense for $\dim S \leq g$.

[1] A subset $W \subset \overline{\mathcal{A}}$ is said *horizontal with respect to f_i* if $f_i(W)$ is Zariski-dense in $\overline{S}_i$ for $i = 1, 2$.

Remark 0.5. If any of the sets $\sigma_i^{-1}(\mathcal{A}_{i,\mathrm{tor}})$ is not Zariski dense then the theorem is obviously true thanks to Remark 0.3. Therefore if $\dim S_1 = \dim S_2 < g$ then Theorem 0.4 follows directly from [20, Theorem 1.1]. For the same reason, thanks to [20, Theorem 1.3] we can restrict ourselves to prove just the case:

$$2 \dim S_1 = 2 \dim S_2 = 2g = \mathrm{rk}_{\mathbb{R}}\, \beta_1 = \mathrm{rk}_{\mathbb{R}}\, \beta_2 \,, \tag{2}$$

where β_i is the Betti map attached to the section σ_i. Observe that Equation (2) is crucial for the application of the so called "height inequality" of [17, Theorem 1.6] that relates the projective height of the base to the fiberwise Neron-Tate height. In our proof this result appears several times, and on different abelian schemes, to ensure that the height of "most of" the torsion values can be uniformly bounded. On the other hand, it is known that the height inequality fails in general without assumptions on the rank of the Betti map. See also [48, Theorem 5.3.5] for a generalization of height inequality which nevertheless requires the same hypotheses in the case of abelian schemes.

Remark 0.6. At first glance it might seem that in the case $1 = \dim S_1 = \dim S_2 = g$, Theorem 0.4 is slightly weaker than [15, Theorem 1.1] where the claim is just $\mathfrak{F} \setminus \mathrm{Fund}(f_2) \subseteq f_2^{-1}(Z)$ for a proper closed subset Z. However, Proposition 2.6 shows that the two statements are actually equivalent.

Remark 0.7. Let Z be a subset of $\overline{\mathcal{A}}$ which is not horizontal with respect to either f_1 or f_2. If Theorem 0.4 holds replacing $\mathfrak{F}$ by $\mathfrak{F} \cap (\overline{\mathcal{A}} \setminus Z)$, it also holds for $\mathfrak{F}$.

Our proof follows the general strategy employed in the low-dimensional setting of [15], which is a variation of the Pila-Zannier method originally introduced in [40]. After some preliminary considerations, we are ultimately reduced to showing that the points of the form $\sigma_2(b)$ for $b \in f_2(\mathfrak{F})$ have uniformly bounded torsion order. Denoting this order by $m := m(b)$, we use the properties of the Betti map to interpret a collection of conjugates of certain torsion values as rational points within a definable family in $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ in the sense of [39].

By analyzing the relationships between Weil heights, torsion orders, and conjugates of algebraic points, we establish a lower bound on the number of such rational points and an upper bound on their height. Crucially, these bounds depend on m. On the other hand, the result of Pila and Wilkie [39, Theorem 1.9] provides an upper bound on the number of rational points of bounded height in the transcendental part of such a definable family. Using the independence result [7, Theorem 3] of André, we prove that the definable family has an empty algebraic part. This allows us to compare the aforementioned bounds on the number of rational points and deduce a uniform upper bound for m.

However, our higher-dimensional setting introduces several subtle complications that were not present in [15]. Below, we outline the new technical ingredients developed in this paper:

(i) The height inequality of Dimitrov, Gao, and Habegger, established in [17], provides a uniform height bound only for torsion values contained in an open dense subset (see Corollary 1.4). Note that when the base is a curve, this poses no issue, as a uniform bound on a Zariski open dense subset is equivalent to a uniform bound for all torsion values. Consequently, in each step of our proof, we must carefully track the closed subset excluded by the height inequality. Additionally, we apply the height inequality to an abelian scheme with a f_2-fiber as its base, meaning the open dense subset with uniformly bounded height is not closed under addition (with respect to the base).

(ii) We require an upper bound on the torsion order of (the image of) torsion values that depends solely on the heights and degrees of the points. To this end, we prove the following:

Proposition 0.8 (See Proposition 1.7)**.** *Let $f : \mathcal{A} \to S$ be a g-dimensional abelian scheme (induced by a morphism of varieties) admitting a non-torsion section $\sigma : S \to \mathcal{A}$. Let K be the field of definition of S, let s be a torsion value for σ, and set $d(s) := [K(s) : \mathbb{Q}]$. Let $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a height on the base. Then, there exist real constants $c = c(g)$ and $C = C(g)$ (independent of the point s) and a Zariski open dense subset $U \subseteq S$ such that*

$$\mathrm{ord}(\sigma(s)) \le \left((14g)^{64g^2} d(s) \max\left(1,\, c \cdot h(s) + C,\, \log d(s)\right)^2 \right)^{\frac{35840 g^3}{16}} \qquad \forall s \in U(\overline{\mathbb{Q}}).$$

The proof combines a similar result for abelian varieties due to Rémond in [43][2] with certain modular properties of the Faltings height. Furthermore, when applying this result to f_1, we require it to be "compatible" with the height bound for torsion points with respect to f_2. To achieve this, we make careful choices of the heights.

[2]We note that Masser and Zannier also obtained a similar, though less sharp, bound in [31].

(iii) We prove the following result which is essential in several steps of the proof of Theorem 0.4:

Proposition 0.9 (See Proposition 1.10)**.** *Let's fix the following data: X is a projective variety; B is a closed subvariety of X; K is a number field containing the fields of definition of X and B. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{3}{4}[K(\alpha) : K]$ different K-embeddings $\tau \colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C_δ.*

Roughly speaking, this result states that, for a fixed uniform constant C and a subvariety B, there is a lower bound on the number of Galois conjugates of a point $\alpha \notin B$ with height at most C that do not lie "near" B. Importantly, this bound depends only on the degree of α. This generalizes [15, Lemma 2.8], which treats the case where B is a finite union of hypersurfaces. This tool is particularly useful for proving results of Zilber-Pink type, as it allows one to move torsion points into a "comfort zone" of the variety, where many arguments can be carried out with sufficient uniformity.

(iv) In the proof of Theorem 0.4, it is necessary to remove a Zariski closed subset from each fiber of f_2. However, we must ensure that this removal can be done "without harm". Specifically, in [15], it is shown that for a point $p \in \mathfrak{F}$ with $f_2(p) = b$ and $m = \operatorname{ord}(\sigma_2(b))$, one of the following two conditions holds: either "many" $k(b)$-conjugates of p lie outside the bad locus of $\mathcal{A}_{2,b}(\mathbb{C})$, or "many" translates of p satisfy the same property. Here, the term "many" refers to a quantity that depends solely on the order m in a uniform manner.

In [15], the case $g = 1$ is considered, where the bad locus on the fibers is always a finite set of points. This allows one to encircle each point with an arbitrarily small euclidean disk and prove the desired statements. However, in the higher-dimensional case, controlling the number of translates that lie in the bad locus becomes problematic. Consequently, we must modify the construction of the definable family in the Pila-Zannier method. In particular, we avoid working with translates altogether and rely solely on conjugates. It turns out that it is not enough to work on one fixed fiber $\mathcal{A}_{2,b}(\mathbb{C})$. Hence, we carry out this estimation on the fibers of the conjugates of b (over the fixed field of definition). Our arguments rely on an application of Proposition 1.7 (see Section 2.1.7).

Remark 0.10. Let us now explain where the assumptions 1)–4) are used in our proof. Assumptions 1) and 2) ensure that the geometric construction is well-defined and meaningful. Assumption 3) is required to guarantee the validity of the height inequality, while assumption 4) is necessary for the application of André's transcendence results.

Finally, we highlight that the present work raises several natural questions. First, it is meaningful to inquire whether our result is *sharp* with respect to the choice of $O \subset \Gamma_{\sigma_1,\sigma_2}$. Specifically:

Question 0.11. Can we find subsets $G \subset O$ that are as small as possible such that the points with finite G-orbits are confined to a proper Zariski-closed subset?

In this direction, Amerik and Cantat demonstrate in [1, Section 6.2] that the points with finite G-orbit become Zariski dense when G is sufficiently small. Furthermore, the following problem is also quite natural:

Question 0.12. What is the generalization of Theorem 0.4 in the case of $n > 2$ abelian rational fibrations $f_i : \overline{\mathcal{A}} \dashrightarrow \overline{S}_i$ for $i = 1, \ldots, n$? In particular, what is the optimal relationship between the dimensions of the bases and g in this setting?

The outline of the paper is the following: in Section 1 we collect the preliminary results. The proof of Theorem 0.4 is carried out in Section 2.1 and Section 2.2. Additionally, in Section 2.3, we make some comments on the shape of the Zariski closed subsets Z_1 and Z_2 that confine the fibers containing the points with finite orbit. Finally, Appendix A by E. Amerik provides explicit constructions of double abelian fibrations. It is worth noting that a well-known example of such fibrations is given in [46] for the case $g = 1$. While examples in higher dimensions can be obtained by considering products of distinct elliptic fibrations on a surface, the appendix presents new constructions for $g > 1$ that are not products.

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1 Auxiliary results

In this section we present all the tools needed for the proof of Theorem 0.4. We describe the results in the most general setting.

1.1 Betti map

Let S be a non-singular, irreducible quasi-projective variety and let $f : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$ with "a zero section" σ_0. Moreover we assume that $\sigma : S \to \mathcal{A}$ is a non-torsion section. Each fiber $\mathcal{A}_s(\mathbb{C})$ is analytically isomorphic to a complex torus $\mathbb{C}^g/\Lambda_s$ and for any subset $T \subseteq S(\mathbb{C})$ we denote $\Lambda_T := \bigsqcup_{s\in T} \Lambda_s$. The space $\mathrm{Lie}(\mathcal{A}) := \bigsqcup_{s\in S(\mathbb{C})} \mathrm{Lie}(\mathcal{A}_s(\mathbb{C}))$ has a natural structure of g-dimensional holomorphic vector bundle $\pi\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ (it is actually a complex Lie algebra bundle). By using the fiberwise exponential maps one can define a global map $\exp\colon \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}(\mathbb{C})$. Let $\Sigma_0 \subset \mathcal{A}(\mathbb{C})$ be the image of the zero section of the abelian scheme, then obviously $\exp^{-1}(\Sigma_0) = \Lambda_{S(\mathbb{C})}$. Clearly $S(\mathbb{C})$ can be covered by finitely many open simply connected subsets where the holomorphic vector bundle $\pi\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ trivializes. Let $U \subseteq S(\mathbb{C})$ be any of such subsets and consider the induced holomorphic map $\pi\colon \Lambda_U \to U$; it is actually a fiber bundle with structure group $\mathrm{GL}(n,\mathbb{Z})$. Since U is simply connected, by [16, Lemma 4.7] we conclude that $\pi\colon \Lambda_U \to U$ is a topologically trivial fiber bundle. Thus we can find $2g$ continuous sections of π:

$$\omega_i : U \to \Lambda_U\,, \quad i = 1, \dots 2g \tag{3}$$

such that $\{\omega_1(s), \dots, \omega_{2g}(s)\}$ is a set of periods for Λ_s for any $s \in U$. Since $\Lambda_U \subseteq \mathrm{Lie}(\mathcal{A})_{|U}$, we can put periods into the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Lie}(\mathcal{A})_{|U} \\ & \overset{\omega_i}{\nearrow} & \big\downarrow \exp_{|U} \\ S(\mathbb{C}) \supset U & \xrightarrow{\sigma_0|U} & \mathcal{A}_{|U}, \end{array}$$

where σ_0 is the zero section. Since σ_0 is holomorphic and exp is a local biolomorphism, then the period functions defined in Equation (3) are holomorphic. The map $\mathcal{P} = (\omega_1, \dots, \omega_{2g})$ is called a *period map*; roughly speaking it selects a $\mathbb{Z}$-basis for Λ_s which varies holomorphically for $s \in U$. The set $U \subseteq S(\mathbb{C})$ is simply connected therefore we can choose a holomorphic lifting $\ell_\sigma : U \to \mathrm{Lie}(\mathcal{A})$ of the restriction $\sigma_{|U}$; ℓ_σ is often called an *abelian logarithm*. Thus for any $s \in U$ we can write uniquely

$$\ell_\sigma(s) = \beta_1(s)\omega_1(s) + \dots + \beta_{2g}\omega_{2g}(s) \tag{4}$$

where $\beta_i : U \to \mathbb{R}$ is a real analytic function for $i = 1, \dots, 2g$. The map $\beta_\sigma : U \to \mathbb{R}^{2g}$ defined as $\beta_\sigma = (\beta_1, \dots, \beta_{2g})$ is called the *Betti map associated to the section* σ, whereas the β_i's are the *Betti coordinates*. Observe that the Betti map depends both on the choice of period map $\mathcal{P}$ and on the abelian logarithm ℓ_σ, but this is irrelevant for our applications. The main feature of the Betti map is that $\beta_\sigma(s) \in \mathbb{Q}^{2g}$ if and only if s is a torsion value of σ, so it allows us to treat the study of the torsion values of an abelian scheme as a transcendental Diophantine problem. Note that we need a non-torsion section σ otherwise β_σ would be obviously constant and equal to a rational point. Viceversa, we recall that as a consequence of Manin's "theorem of the kernel" (see [27] or [11]) if β_σ is locally constant then σ is torsion. Moreover, the fibers of β_σ are complex submanifolds of $S(\mathbb{C})$ (see [14, Proposition 2.1] or [8, Section 4.2]).

Remark 1.1. There exists a compact subset $D \subseteq U$ such that the Betti map β_σ restricted to D is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ (using the real charts). This follows for instance by using [37, Fact 4.3] and the fact that for $i = 1, \dots, 2g$ we have $\beta_i = \pi_i \circ \ell_\sigma$, where π_i is the projection on the i-th coordinate with respect to the period map.

The rank, in the sense of real differential geometry, of the Betti map at a point s is denoted by $\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma(s)$. It can be shown that it depends only on the point s (see for instance [8, Section 4.2.1] or [19, Section 4]). Moreover we define

$$\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma = \max_{s\in S(\mathbb{C})} \mathrm{rk}_{\mathbb{R}}\, \beta_\sigma(s) \tag{5}$$

and note that it obviously holds that $\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma \leq 2\min(g, \dim S)$. We call a section $\sigma\colon S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ *non-degenerate* if $\mathrm{rk}\, \beta_\sigma = 2\dim S$. The following crucial proposition allows us to have a uniform control on the fibers of the Betti map, under certain conditions.

Proposition 1.2. *Let $2\dim S = 2g = \mathrm{rk}_{\mathbb{R}}\,\beta_\sigma$. There exist a non-empty Zariski open set U of $S(\mathbb{C})$ such that: for any $x \in U$ there is a compact subanalytic set $D \subseteq S(\mathbb{C})$ containing x and a constant $c = c(D)$ such that the Betti map $\beta_\sigma \colon D \to \mathbb{R}^{2g}$ has finite fibers of cardinality at most c.*

Proof. From the condition on the rank of the Betti map it follows immediately that there exists a non-empty Zariski open set $U \subseteq S(\mathbb{C})$ on which β_σ is a submersion. Pick any compact subanalytic D inside U and contained in a chart. Restrict the Betti map on D and identify the latter with an euclidean compact in $\mathbb{R}^{2g}$. Since β_σ is now a submersion, the fibers must have real codimension equal to $2g$ (see for instance [25, Corollary 5.13]), which means that the fibers are discrete, and hence finite (D is compact). It remains to prove the uniform bound on the cardinality. So consider the subanalytic set

$$Z := \{(z, \beta_\sigma(z)) \colon z \in D\} \subset \mathbb{R}^{2g} \times \mathbb{R}^{2g}.$$

Let $\pi_2 : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}^{2g}$ the projection on the second factor, then for any $p \in \mathbb{R}^{2g}$ we obviously have

$$Z \cap \pi_2^{-1}(p) = \beta_\sigma^{-1}(p).$$

By Gabrielov's theorem (see [49, Theorem A.4] or [12, Theorem 3.14]) $Z \cap \pi_2^{-1}(p)$ has at most c connected components, hence $\beta_\sigma^{-1}(p)$ has cardinality at most c. ☐

1.2 Height bounds

In this short subsection we use the same notation of Section 1.1. Let $\mathcal{M}$ be a relative f-ample and symmetric line bundle on $\mathcal{A}$, then we define $\hat{h} : \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ to be the fiberwise Néron-Tate height i.e.

$$\hat{h}(p) = \hat{h}_{\mathcal{M}}(p) := \lim_{n\to\infty} \frac{1}{4^n} h_{\mathcal{M}}\left(2^n p\right).$$

Note that $\hat{h}(p) = \hat{h}_{\mathcal{M}_s}(p)$ with $s = f(p)$. Moreover we consider a height function $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ on the base. The following height inequality proved in [17, Theorem B.1] (see also [48, Theorem 5.3.5] for a more general approach) is a crucial result that relates the values of $\hat{h}$ and h:

Theorem 1.3 (Height inequality for abelian schemes)**.** *Let X be an irreducible and non-degenerate*[3] *subvariety of $\mathcal{A}$ that dominates S. Then there exist two constants $c_1 > 0$ and $c_2 \geq 0$ and a Zariski non-empty open subset $V \subseteq X$ with*

$$\hat{h}(p) \geq c_1 h(f(p)) - c_2 \quad \textit{for all } p \in V\left(\overline{\mathbb{Q}}\right).$$

Proof. See [17, Theorem B.1]. ☐

Corollary 1.4. *Assume that $f \colon \mathcal{A} \to S$ is endowed with a non-degenerate section $\sigma : S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$. Then there exists a constant $C \geq 0$ and a non-empty Zariski open subset $V \subseteq S$ such that*

$$h(s) \leq C \quad \textit{for all } s \in V(\overline{\mathbb{Q}}) \cap \sigma^{-1}(\mathcal{A}_{\mathrm{tor}}). \tag{6}$$

Remark 1.5. Note that if the abelian scheme $\mathcal{A} \to S$ and the section σ are defined over $\overline{\mathbb{Q}}$ then $S \setminus V$ is a Zariski closed subset defined over $\overline{\mathbb{Q}}$ by [19, Theorem 1.8].

1.3 Torsion bounds

Let's quickly recall the definition of the stable Faltings height. Let A be a g-dimensional abelian variety over a number field K. Consider a finite extension $L \supseteq K$ such that $A \otimes L$ is semistable; moreover let $\mathcal{A} \to S := \operatorname{Spec} O_L$ be the connected component of the Neron model of $A \otimes L$ and denote with $\epsilon \colon S \to \mathcal{A}$ be the zero section. The sheaf of relative differentials $\Omega^g_{\mathcal{A}/S}$ pulls back on the base S through ϵ and we put $\omega_{\mathcal{A}/S} := \epsilon^* \Omega^g_{\mathcal{A}/S}$. The *stable Faltings height of A* is defined as:

$$h_F(A) := \frac{1}{[L : \mathbb{Q}]} \widehat{\deg}\left(\omega_{\mathcal{A}/S}\right)$$

[3] The references [17] and [20] use a slightly different (but equivalent) definition of Betti map and they have a notion of non-degenerate subvariety. A section σ is non-degenerate in our sense if and only if the subvariety $\sigma(S(\mathbb{C}))$ of $\mathcal{A}$ is non-degenerate in the sense of Dimitrov, Gao, Habbegger.

where $\widehat{\deg}$ is the Arakelov degree calculated on $\omega_{\mathcal{A}/S}$ seen as hermitian line bundle on the base. It can be shown that h_F doesn't depend on the field extension (for details check [18]).

Let's recall an important property of the stable Faltings height. If $\phi\colon A \to A'$ is a K-isogeny between abelian varieties over K, then [41, Corollary 2.1.4] says that the stable Faltings heights of A and A' are related in the following way:

$$|h_F(A) - h_F(A')| \leq \frac{1}{2}\log\deg(\phi) \tag{7}$$

Moreover the stable Faltings height can be used to bound the exponent and the cardinality of the group of rational torsion points. The result is due to Rémond:

Proposition 1.6. *Let A be an abelian variety of dimension g defined over a number field K. The finite group $A(K)_{\mathrm{tor}}$ has exponent at most $\kappa(A)^{\frac{35}{16}}$ and cardinality at most $\kappa(A)^{4g+1}$, where $d = [K\colon \mathbb{Q}]$ and $\kappa(A) = \left((14g)^{64g^2} d\max(1, h_F(A), \log d)^2\right)^{1024g^3}$.*

Proof. See [43, Proposition 2.9]. □

For a slightly weaker result involving principally polarized abelian varieties and the semistable Faltings height see [29, Proposition 7.1]. Let $\mathfrak{A}_g$ be the coarse moduli space over $\mathbb{C}$ of g-dimensional principally polarized abelian schemes. It is known that $\mathfrak{A}_g$ is a quasi-projective variety defined over $\mathbb{Q}$ and moreover there is a canonical projective embedding which induces a height function[4] $h_{\mathrm{mod}}\colon \mathfrak{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$ (see for instance [18, §3]). There is a close relationship between h_{mod} and the stable Faltings height h_F, in fact if $x \in \mathfrak{A}_g(K)$ is the point corresponding to a semistable abelian variety A over a number field K, then there exists a constant C independent from A and K such that:

$$|h_{\mathrm{mod}}(x) - rh_F(A)| \leq C \tag{8}$$

where r is a certain positive integer. For the proof of this deep result see [18, Theorem 3.1].

Proposition 1.7. *Let $f : \mathcal{A} \to S$ be a g-dimensional abelian scheme (induced by a morphism of varieties) admitting a non-torsion section $\sigma : S \to \mathcal{A}$. Let K be the field of definition of S, let s be a torsion value for σ and put $d(s) := [K(s) : \mathbb{Q}]$. Let $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a height on the base corresponding to an ample line bundle, there exist real constants $c = c(g), C = C(g)$ (so independent from the point s) and a Zariski open dense subset $U \subseteq S$ such that*

$$\mathrm{ord}(\sigma(s)) \leq \left((14g)^{64g^2} d(s)\max\left(1,\, c\cdot h(s) + C,\, \log d(s)\right)^2\right)^{\frac{35840g^3}{16}} \qquad \forall s \in U(\overline{\mathbb{Q}})\,.$$

Proof. Recall that $\mathcal{A}_s$ is an abelian variety over the number field $K(s) \supseteq K$. The first step consists in reducing to the principally polarized case. The explicit construction is explained in [17, Proof of Theorem B.1 (Fourth devissage)], here we just recall the result: there is a quasi-finite dominant étale morphism $\rho : S' \to S$ with S' irreducible and a principally polarized abelian scheme $g\colon \mathcal{A}' \to S'$ such that there exists a S'-isogeny

$$\phi : \mathcal{A}' \to \mathcal{A}'' := \mathcal{A} \times_S S'\,.$$

Note that if $s' \in S'$ is a point lying above $s \in S$, then $\mathcal{A}''_{s'} = \mathcal{A}_s \otimes K(s')$, thus $h_F(\mathcal{A}_s) = h_F(\mathcal{A}''_{s'})$. By Equation (7) we have that $h_F(\mathcal{A}''_{s'}) \leq h_F(\mathcal{A}'_{s'}) + \deg(\phi_{s'})$, but notice that $\deg(\phi_{s'})$ doesn't depend on s', therefore we can just write:

$$h_F(\mathcal{A}_s) \leq h_F(\mathcal{A}'_{s'}) + C_1\,. \tag{9}$$

Consider the induced morphism

$$\begin{aligned} m_g : S' &\to \mathfrak{A}_g \\ s' &\mapsto [\mathcal{A}'_{s'}] =: x_{s'}\,. \end{aligned}$$

The stable Faltings height of $\mathcal{A}'_{s'}$ is calculated over a finite extension $L \supseteq K(s')$ such that $\mathcal{A}'_{s'} \otimes L$ is semistable, in other words $h_F(\mathcal{A}'_{s'}) = h_F(\mathcal{A}'_{s'} \otimes L)$. From this fact and Equation (8) we obtain

$$h_F(\mathcal{A}'_{s'}) < C_2 + h_{\mathrm{mod}}(x_{s'})\,. \tag{10}$$

[4]There is no general agreement on the notation of this height function on $\mathfrak{A}_g$. Some authors for instance denote it as h_{geo} and use h_{mod} for the Faltings height instead.

On the other hand, by fixing a height function $h' : S'(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated to the pull-back of the line bundle inducing h_{mod} and by the usual functorial properties of the Weil height we have

$$|h'(s') - h_{\mathrm{mod}}(x_{s'})| < C_3 \tag{11}$$

for a constant C_3. Since any line bundle can be written as the difference between two very ample line bundles, we can consider a height h'' on S' corresponding to an ample line bundle such that $h' \leq h''$. From [44, Theorem 1] applied the morphism $\rho : S' \to S$ it follows that the following relation holds on an open Zariski dense subset of S':

$$h'(s') \leq h''(s') \leq C_4 h(\rho(s')) + C_5\,. \tag{12}$$

Since ρ is an open map, the claim follows after putting together Equations (9) to (12) and Proposition 1.6 applied to $\mathcal{A}_s$. ☐

1.4 Control on conjugate points

Let's fix an affine variety $Y(\mathbb{C}) \subseteq \mathbb{A}^N(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C})$ defined over a number field K. For any point $p \in Y(\mathbb{C})$ we denote by $K(p)$ the field generated by the coordinates of p; this is the same as the residue field of p when the latter is seen as an abstract point of Y. With the letter h we denote both the absolute height on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and $\mathbb{A}^1(\overline{\mathbb{Q}})$, since the formal meaning is clear from the argument of h. Further, we denote by $\|\cdot\|$ the euclidean norm in $\mathbb{A}^N(\mathbb{C})$. We fix a closed subvariety B' of Y and we define

$$W'_\delta := \{x \in Y(\mathbb{C})\colon d(x, B'(\mathbb{C})) < \delta\}, \quad \text{for } \delta \in \mathbb{R}_{>0}$$

where

$$d(x, B'(\mathbb{C})) := \inf_{b \in B'(\mathbb{C})} \|x - b\|\,.$$

Moreover let's consider the set $C'_\delta := Y(\mathbb{C}) \setminus W'_\delta$.

Lemma 1.8. *Let H be a subset of $Y(\mathbb{C})$ and let C be a compact subset of H. Fixed $p \in Y(\mathbb{C}) \setminus H$, there exists a constant c (uniform with respect to $b \in C$) such that*

$$d(p, H) \geq c \cdot \|p - b\| \quad \textit{for each } b \in C.$$

Proof. For each $b \in C$, let us consider a constant a_b which satisfies $0 < a_b < \frac{d(p,H)}{\|p-b\|}$ (note that it exists since $p \notin H$). Observe that a_b is a constant which depends on b and such that

$$d(p, H) - a_b \cdot \|p - b\| > 0.$$

Then there exists an open (analytic) neighbourhood N_b of b such that

$$d(p, H) - a_b \cdot \|p - b'\| > 0 \qquad \text{for each } b' \in N_b.$$

The family $\{N_b : b \in H\}$ is an open covering of the compact set C. Thus there exists a finite subcovering $\{N_{b_i} : i = 1, \ldots, n\}$. The constant $c := \min_{1 \leq i \leq n}(a_{b_i})$ works uniformly on C. In fact for each $b \in C$ we have

$$c \cdot \|p - b\| \leq a_b \cdot \|p - b\| < d(p, H).$$

☐

Proposition 1.9. *Let K be a number field which contains the field of definition of the subvariety B'. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in Y(\overline{\mathbb{Q}}) \setminus B'(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{3}{4}[K(\alpha) : K]$ different K-embeddings $\tau\colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C'_δ.*

Proof. Fix $\beta = (\beta_1, \ldots, \beta_N) \in B'(\overline{\mathbb{Q}})$ such that there exists an index i with $\beta_i \in K(\alpha)$ (observe that such a β always exists); and write $\alpha := (\alpha_1, \ldots, \alpha_N)$. Clearly $h(\alpha) \geq h(\alpha_i)$ and $h(\beta) \geq h(\beta_i)$. This implies

$$h(\alpha_i - \beta_i) \leq h(\alpha_i) + h(\beta_i) + \log(2) \leq h(\alpha) + h(\beta) + \log(2). \tag{13}$$

Fix $\delta > 0$. We define

$$\Sigma := \{\tau : K(\alpha) \hookrightarrow \mathbb{C}\colon \ \mathrm{id} = \tau_{|K} \text{ and } \alpha^\tau \notin C'_\delta\}$$

and denote by k the cardinality of Σ. Since τ is a K-embedding we have $\beta^\tau \in B'(\overline{\mathbb{Q}})$. Moreover observe that, given $\tau \in \Sigma$, we have $\alpha^\tau \notin B'(\mathbb{C})$. Thus, by Lemma 1.8 for $p = \alpha^\tau, H = B'(\mathbb{C})$ and $C = \{\beta^\tau : \tau \in \Sigma\}$, and since $\alpha^\tau \notin C'_\delta$ (by definition of Σ) there exists a constant c_τ such that

$$\frac{1}{|\alpha_i^\tau - \beta_i^\tau|} \geq \frac{1}{\|\alpha^\tau - \beta^\tau\|} \geq \frac{c_\tau}{d(\alpha^\tau, B(\mathbb{C}))} > \frac{c_\tau}{\delta}.$$

Considering $c := \min_{\tau\in\Sigma}(c_\tau)$ we obtain a constant c such that:

$$\frac{1}{|\alpha_i^\tau - \beta_i^\tau|} \geq \frac{c}{\delta} \quad \text{for fixed } i \text{ and for all } \tau \in \Sigma.$$

Then for δ small enough we obtain

$$\begin{aligned} h(\alpha_i - \beta_i) &\geq \frac{1}{[K(\alpha):\mathbb{Q}]} \sum_\nu \log\max\left(1, \left|\frac{1}{\alpha_i - \beta_i}\right|_\nu\right) \geq \\ &\geq \frac{1}{[K(\alpha):\mathbb{Q}]} \sum_{\tau\in\Sigma} \log\max\left(1, \left|\frac{1}{\alpha_i^\tau - \beta_i^\tau}\right|\right) \geq \frac{k}{[K(\alpha):\mathbb{Q}]} \log\left(\frac{c}{\delta}\right). \end{aligned} \tag{14}$$

By (13), (14) and the fact that α has bounded height we obtain

$$k \leq \frac{(a + h(\beta) + \log(2)) \cdot [K(\alpha):\mathbb{Q}]}{\log(c/\delta)}.$$

For δ small enough we have

$$\frac{a + h(\beta) + \log(2)}{\log(c/\delta)} \leq \frac{1}{4[K:\mathbb{Q}]}.$$

Therefore

$$k \leq \frac{1}{4}[K(\alpha):K].$$

□

Now let's fix a projective variety X defined over K and a closed subvariety B of X. For any point $p = (x_0 : \ldots : x_N) \in X(\mathbb{C})$ pick any $x_i \neq 0$ and then put $K(p) := K\left(\frac{x_j}{x_i} : j = 0, \ldots, N\right)$. Note that $K(p)$ doesn't depend on the choice of x_i (i.e. the standard affine chart) and moreover $K(p)$ is the residue field of p when the latter is seen as an abstract point of X. We prove a higher dimensional generalization of a quite useful result already appeared for the projective line in [29, 30, 31, Lemma 8.2] and for hypersurfaces in [15, Lemma 2.8]. Roughly speaking the result claims the following: K is the field of definition of B, $a \in \mathbb{R}$ and $\alpha \in X(\overline{\mathbb{Q}})$ is any point not contained in $B(\mathbb{C})$ with height at most a; then we can give an explicit lower bound, depending only on $[K(\alpha):K]$, on the number of $K(\alpha)$ conjugates of α that lie in a "big enough" compact not intersecting $B(\mathbb{C})$.

We first construct the compact subset. Denote by $U_0, \ldots, U_N$ the standard affine charts of the projective space. Let's define

$$W_{i,\delta} := \{x \in X(\mathbb{C}) \cap U_i : d(x, B(\mathbb{C}) \cap U_i) < \delta\} \qquad \text{for fixed } \delta \in \mathbb{R}_{>0} \text{ and } i = 1, \ldots, N. \tag{15}$$

Then we put $W_\delta := \bigcup_{i=0}^N W_{i,\delta}$ and note that it is an open subset of $X(\mathbb{C})$ containing $B(\mathbb{C})$. Therefore $C_\delta := X(\mathbb{C}) \setminus W_\delta$ is a compact set not intersecting $B(\mathbb{C})$.

Proposition 1.10. *Let K be a number field which contains the field of definition of the subvariety B. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{3}{4}[K(\alpha):K]$ different K-embeddings $\tau : K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C_δ.*

Proof. Fix $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$ and fix a chart U_i such that $\alpha \in U_i$. Since the chart is invariant under the action of each τ, we can apply Proposition 1.9 for $Y(\mathbb{C}) = X(\mathbb{C}) \cap U_i$, $B'(\mathbb{C}) = Y(\mathbb{C}) \cap B(\mathbb{C})$ and $C'_\delta = C_\delta \cap U_i$. Therefore, we obtain a real number δ_i which only depends on K, a and U_i and which satisfies the statement for $\alpha \in U_i$. We can repeat the argument for any standard chart and after defining $\delta := \min_{0\leq i\leq N}(\delta_i)$, we can conclude. □

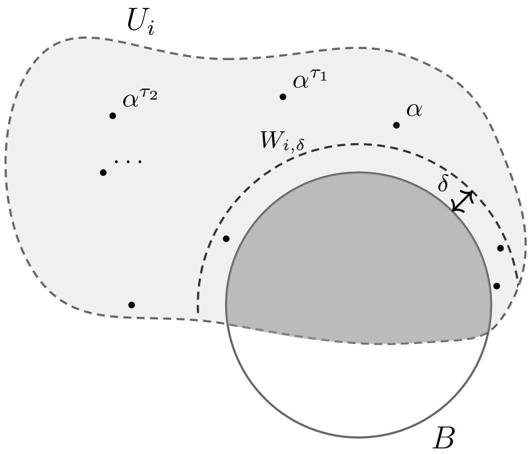

Figure 1: A representation of the portion of conjugates of α that stay away from a euclidean open set $W_{i,\delta}$ that tightly encircles a Zariski closed set B. The set U_i is a selected affine chart.

Remark 1.11. Observe that the the intersection of C_δ with each standard chart U_i is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. In fact, first of all let's identify $U_i \cap X(\mathbb{C})$ with $\mathbb{R}^{2N}$, then the map $\mathbb{R}^{2N} \ni p \mapsto d(p, B(\mathbb{C}) \cap U_i)$ is a globally subanalytic function (see for instance [10, Example 2.10]). At this point we apply [47, §1 Lemma 2.3] to conclude that the set $W_{i,\delta} = U_i \cap W_\delta$ is globally subanalytic for any $\delta > 0$. Finally, note that the intersection $C_\delta \cap U_i$ is the complement set $(U_i \cap X(\mathbb{C})) \setminus (U_i \cap W_{i,\delta})$, so it is also globally subanalytic.

2 The main theorem

In this section we prove Theorem 0.4. The proof is rather long and technical; it will be eventually split in two cases after a common setup. We use the same notations fixed in the introduction.

2.1 Setup of the proof

Our proof necessitates a considerably intricate preparation, which we delineate as follows.

2.1.1 Construction of the heights

We first construct a specific ample line bundle on S_1 such that the pullback through f_1 is ample on $\mathcal{A}_1$. These two line bundles will give two (Weil) heights respectively on S_1 and $\mathcal{A}_1$ that will be fixed for the rest of the proof. We need such setup for two reasons: firstly we want to induce "quasi Néron-Tate heights" on the fibres $\mathcal{A}_{2,b}$. Then we want these heights to be functorially related to the height on the base S_1 (they come from a pullback of a line bundle on the base) in order to apply the height machine.

By [17, Section 3] there exists a relative f_1-ample line bundle $\mathcal{M}'$ on $\mathcal{A}_1$ such that $\mathcal{M}' = f_1^*(\mathcal{N})$ for a line bundle $\mathcal{N}$ on S_1. We write $\mathcal{N} = \mathcal{D}_1 \otimes \mathcal{D}_2^{-1}$ where $\mathcal{D}_1$ and $\mathcal{D}_2$ are ample line bundles. By [45, TAG 0892] the line bundle $\mathcal{M}' \otimes f_1^*(\mathcal{D}_2^k) = f_1^*(\mathcal{N} \otimes \mathcal{D}_2^k)$ is ample for $k \in \mathbb{N}$ big enough. We put $\mathcal{L} := \mathcal{N} \otimes \mathcal{D}_2^k$ and $\mathcal{M} := f_1^*(\mathcal{L})$. Note that $\mathcal{L} := N \otimes \mathcal{D}_2^k = \mathcal{D}_1 \otimes \mathcal{D}_2^{k-1}$ is also ample.

We fix two heights $h_\mathcal{L}$ and $h_\mathcal{M}$ on S_1 and $\mathcal{A}_1$ respectively. Let's consider the abelian scheme $f_2 : \mathcal{A}_2 \to S_2$ with the morphism $[-1] : \mathcal{A}_2 \to \mathcal{A}_2$ and restrict $\mathcal{M}$ on $\mathcal{A}_2$ (keeping the same name for it). The restriction induces a height $h_\mathcal{M}$ on the fibers of $\mathcal{A}_2$ which don't intersect the fundamental locus of f_1. Define the line bundles $\mathcal{M}_1 := \mathcal{M} \otimes [-1]^*\mathcal{M}^{-1}$ and $\mathcal{M}_2 := \mathcal{M} \otimes [-1]^*\mathcal{M}$. Observe that $\mathcal{M}_1$ is ample and skew-symmetric, while $\mathcal{M}_2$ is ample and symmetric. We get two canonical heights on $\mathcal{A}_2$:

$$\hat{h}_{\mathcal{M}_i}(p) := \lim_{n\to\infty} \frac{1}{2^{in}} h_{\mathcal{M}_i}(2^n p) ,$$

and define

$$\hat{h}_\mathcal{M} := \frac{1}{2}\hat{h}_{\mathcal{M}_1} + \frac{1}{2}\hat{h}_{\mathcal{M}_2} .$$

The height $\hat{h}_\mathcal{M}$ has three relevant properties for our aims:

(i) If $x \in \mathcal{A}_{2,\mathrm{tor}}$, then $\hat{h}_{\mathcal{M}}(x) = 0$.

(ii) $\hat{h}_{\mathcal{M}}(x+y) + \hat{h}_{\mathcal{M}}(x-y) = 2\hat{h}_{\mathcal{M}}(x) + \hat{h}_{\mathcal{M}}(y) + \hat{h}_{\mathcal{M}}(-y)$ for any x, y such that $f_2(x) = f_2(y)$.

(iii) $\hat{h}_{\mathcal{M}} - h \circ f_1 = O(1)$.

We then fix the heights $h_b := \hat{h}_{\mathcal{M}}|_{\mathcal{A}_{2,b}}$ on the fibers $\mathcal{A}_{2,b}$: every time we refer to a height on a fiber $\mathcal{A}_{2,b}$ we mean h_b.

2.1.2 Removing Zariski closed subsets

By Remark 0.7 it's enough to prove Theorem 0.4 for $\mathfrak{F} \cap \mathcal{A}'$, where $\mathcal{A}'$ is obtained from $\mathcal{A}$ after removing some non-horizontal Zariski closed subsets with respect to f_1 or f_2. Let's describe precisely how to obtain $\mathcal{A}'$.

Let R_1 be the the Zariski closed subset of $\overline{S}_1$ defined as the union of the following proper Zariski closed subsets:

- The locus Sing_1 of singular fibers of the abelian scheme $f_1 : \mathcal{A}_1 \to S_1$.
- The locus E where two fibers $\mathcal{A}_{1,s_1}$ and $\mathcal{A}_{2,s_2}$ are equal.
- The locus Ind_1 containing the f_1-images of points where the rational map f_2 is not defined (see Assumption 2)).
- The locus $\mathcal{C}(\beta_1)$ of critical points of the Betti map β_1, where the Betti map β_1 is not a submersion. This is the locus where Proposition 1.2 fails.
- The locus $\mathcal{C}_{\text{Rém},1}$ where the inequality in Proposition 1.7 does not hold.
- The locus $\mathcal{C}_{\text{height},1}$ where the height bound in Corollary 1.4 does not hold.

Let R_2 be the the Zariski closed subset of $\overline{S}_2$ defined as the union of the following proper Zariski closed subsets:

- The locus Sing_2 of singular fibers of the abelian scheme $f_2 : \mathcal{A}_2 \to S_2$.
- The locus Ind_2 containing the f_2-images of points where the rational map f_1 is not defined (see Assumption 2)).
- The locus $\mathcal{C}_{\text{Rém},2}$ where the inequality in Proposition 1.7 does not hold.
- The locus $\mathcal{C}_{\text{height},2}$ where the height bound in Corollary 1.4 does not hold.

We fix a number field K containing all the fields of definitions of $\overline{\mathcal{A}}, \overline{S}_1, \overline{S}_2, f_1, f_2, \sigma_1, \sigma_2$ and all the proper Zariski closed subset listed above. Let's define

$$\mathcal{A}' := \mathcal{A}_2 \setminus \left(f_1^{-1}(R_1) \cup f_2^{-1}(R_2)\right) . \tag{16}$$

For any f_2-fiber $\mathcal{A}_{2,b} := f_2^{-1}(b)$, we define the Zariski open subset

$$F_b := \mathcal{A}_{2,b} \cap \mathcal{A}'. \tag{17}$$

The restriction to F_b allows to get rid of the 'problematic' Zariski closed subset $\mathcal{A}_{2,b} \setminus \mathcal{A}'$. To be more precise, in the whole proof we need to remove the following subsets:

- The Zariski closed subset $\overline{S}_2 \setminus f_2(\mathcal{A}')$ on the base $\overline{S}_2$.
- The Zariski closed subset $\overline{S}_1 \setminus f_1(\mathcal{A}')$ on the base $\overline{S}_1$.
- The Zariski closed subset $\mathcal{A}_{2,b} \setminus F_b$ on each fiber $\mathcal{A}_{2,b}$.

2.1.3 Uniform bounds

Recall that all the σ_1-torsion values in $f_1(\mathcal{A}')$ and all the σ_2-torsion values in $f_2(\mathcal{A}')$ have uniformly bounded height and call μ_1 this constant. Moreover, let us denote with μ_2 the constant defined by property (iii) in Section 2.1.1. Define the constant

$$C_{\text{height}} := 2\mu_1 + 3\mu_2. \tag{18}$$

If $p \in \mathcal{A}'$ and $b = f_2(p)$ we clearly have that $K(b) \subseteq K(p)$. We define the set of complex K-embeddings of the field $K(p)$:

$$\Sigma_p := \{\tau \colon K(p) \hookrightarrow \mathbb{C} \mid \tau_{|K} = id\}. \tag{19}$$

Given $\tau \in \Sigma_p$ we get $f_2(p^\tau) = b^\tau$, but observe that two conjugates of b might coincide. Each element of Σ_p induces by restriction a complex K-embedding of $K(b)$ in a surjective way.

Let us consider the Zariski open subset $f_1(\mathcal{A}')$ of $\overline{S}_1$. Since we have the uniform bound Equation (18) for the height of the σ_1-torsion values in $f_1(\mathcal{A}')$ and since we removed the Zariski closed subset $\mathcal{C}_{\text{Rém},1}$ from the base, we can apply Proposition 1.7 and we obtain two constants $\eta = \eta(g)$ and $\eta' = \eta'(g)$ depending only on g such that

$$\operatorname{ord}(\sigma_1(s)) \le C'_{\text{Rém}} \cdot [K(s) : K]^{C_{\text{Rém}}} \qquad \text{for any } s \in f_1(\mathcal{A}'), \tag{20}$$

where

$$C_{\text{Rém}} = C_{\text{Rém}}(g) := 3 \cdot \frac{35840g^3}{16}, \qquad C'_{\text{Rém}} = C'_{\text{Rém}}(g, K) := (14g)^{64g^2}(\eta' \cdot C_{\text{height}} + \eta) \cdot [K : \mathbb{Q}]^{C_{\text{Rém}}}. \tag{21}$$

Analogously, we can consider the Zariski open subset $f_2(\mathcal{A}')$ of $\overline{S}_2$. Since we have removed the Zariski closed subset $\mathcal{C}_{\text{Rém},2}$ from the base, by using again the uniform bound Equation (18) for the σ_2-torsion values in $f_2(\mathcal{A}')$ and using again Proposition 1.7 we obtain

$$\operatorname{ord}(\sigma_2(b)) \le C'_{\text{Rém}} \cdot [K(b) : K]^{C_{\text{Rém}}} \qquad \text{for any } b \in f_2(\mathcal{A}'), \tag{22}$$

with the same constants defined in Equation (21).

2.1.4 Removing euclidean open subsets

During the proof we need to apply our arguments with enough uniformity after removing the aforementioned Zariski closed subsets on the bases $\overline{S}_1, \overline{S}_2$ and on each fiber $\mathcal{A}_{2,b}$. We want to cut out small euclidean open subsets which encircle the Zariski closed subsets, so that we can work on compact analytic subsets containing enough conjugates of the points that we want to study.

Firstly, we consider the Zariski closed subset $\overline{S}_2 \setminus f_2(\mathcal{A}')$ on the base $\overline{S}_2$. By applying Proposition 1.10 with respect to the height bound C_{height}, we get an analytic compact set

$$\Delta \subseteq f_2(\mathcal{A}') \tag{23}$$

(in the above notation we have $\Delta = C_\delta$ for some $\delta > 0$ small enough) such that for any $b \in f_2(\mathcal{A}')$ with $h(b) \le C_{\text{height}}$ there are at least $\frac{3}{4}[K(b) : K]$ different K-embeddings $\tau : K(b) \hookrightarrow \mathbb{C}$ satisfying $b^\tau \in \Delta$. By Remark 1.11 the compact set Δ has the property that the intersection $\Delta \cap U_i$ with each standard chart is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.

Analogously, we want to cut out small euclidean open subsets of each f_2-fiber and of the base $\overline{S}_1$ which encircle the sets $\mathcal{A}_{2,b} \setminus F_b$ and $\overline{S}_1 \setminus f_1(\mathcal{A}')$ respectively, so that we can work on a compact subsets of each fiber and of the base. We follow the same construction as in Equation (15). Since this construction does not depend on the shape of the Zariski closed subset removed in Equation (16), we explain it for general closed subsets.

Let's embed the fiber $\mathcal{A}_{2,b}(\mathbb{C})$ inside some $\mathbb{P}^N(\mathbb{C})$ and let $U'_0, \dots, U'_N \subseteq \mathbb{P}^N(\mathbb{C})$ be the standard charts. Let us consider a Zariski closed subset $Y \subseteq \overline{S}_1$ and define

$$X_b = \mathcal{A}_{2,b}(\mathbb{C}) \cap f_1^{-1}(Y(\mathbb{C})). \tag{24}$$

After identifying $\mathcal{A}_{2,b}(\mathbb{C}) \cap U'_i$ with $\mathbb{R}^{2N}$, we can consider the globally subanalytic sets

$$V_{i,\delta} := \{z \in \mathcal{A}_{2,b}(\mathbb{C}) \cap U'_i \colon d(z, X_b \cap U'_i) < \delta\}$$

for any $\delta > 0$ small enough and define

$$V_{b,\delta} := \bigcup_{i=0}^{N} V_{i,\delta}. \tag{25}$$

This shows that the Zariski closed subset X_b is contained in a small enough euclidean open subset $V_{b,\delta} \subseteq \mathcal{A}_{2,b}(\mathbb{C})$ whose intersection $V_{b,\delta} \cap U_i'$ with each standard chart of $\mathbb{P}^N(\mathbb{C})$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.

Denote by $U_0, \ldots, U_M$ the standard affine charts on $\overline{S}_1(\mathbb{C})$. Analogously, we can encircle Y with a small enough open set of which we can control the size (chart-by-chart), so let us consider the sets

$$W_{i,\delta} := \{z \in S_1(\mathbb{C}) \cap U_i \colon d(z, Y \cap U_i) < \delta\}$$

for any $\delta > 0$ small enough, and define

$$W_\delta := \bigcup_{i=0}^{M} W_{i,\delta}. \tag{26}$$

We can carry out the construction of $V_{b,\delta}$ and W_δ such that $f_1(V_{b,\delta}) \subseteq W_\delta$, so that their size is controlled via the same δ.

We apply this construction to the Zariski closed sets $\mathcal{A}_{2,b} \setminus F_b$ and $\overline{S}_1 \setminus f_1(\mathcal{A}')$. Therefore, in the rest of the proof we denote by $V_{b,\delta} \subset \mathcal{A}_{2,b}(\mathbb{C})$ a euclidean open subset which contains the locus $\mathcal{A}_{2,b} \setminus F_b$ and by W_δ a euclidean open subset which contains the locus $\overline{S}_1 \setminus f_1(\mathcal{A}')$ with the property $f_1(V_{b,\delta}) \subseteq W_\delta$. We choose $\delta > 0$ small enough to ensure that Proposition 1.10 can be applied on the compact sets $\mathcal{A}_{2,b} \setminus V_{b,\delta}$ and $\overline{S}_1 \setminus W_\delta$ with respect to the height bound C_{height}. Notice that the intersections $V_{b,\delta} \cap U_i'$ and $W_\delta \cap U_i$ with each standard chart of $\mathbb{P}^N(\mathbb{C})$ and $\mathbb{P}^M(\mathbb{C})$ respectively is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. Define

$$T_{b,\delta} := \mathcal{A}_{2,b}(\mathbb{C}) \setminus V_{b,\delta}, \qquad \Delta' := \overline{S}_1 \setminus W_\delta\,. \tag{27}$$

2.1.5 Auxiliary families of abelian schemes

We need to construct an auxiliary abelian scheme for any $b \in \Delta$ that will play a crucial role in the whole proof. Let us consider the variety F_b introduced in Equation (17) and define an abelian scheme

$$\mathcal{X} := \mathcal{A}_1 \times_{S_1} F_b \to F_b \qquad \text{for any } b \in \Delta, \tag{28}$$

so that by abuse of notation we can identify the fiber $\mathcal{X}_z = \mathcal{A}_{1,f_1(z)}$. Note that $\mathcal{X}$ depends on the choice of b, but for simplicity of notations we don't write such dependence. Clearly, such fibers are all non-singular since we have removed the discriminant locus of f_1. In addition, this abelian scheme is endowed with a non-torsion section $s_{\mathcal{X}} := \sigma_1 \circ f_1$.

The restriction to F_b allows to get rid of the 'problematic' Zariski closed subset $\mathcal{A}_{2,b} \setminus \mathcal{A}'$. Consequently, the $s_{\mathcal{X}}$-torsion values lying in $\mathcal{A}'$ inherit the height bound Equation (18) and the following bound on their order:

$$\operatorname{ord}(s_{\mathcal{X}}(z)) \le C'_{\text{Rém}} \cdot [K(z) : K]^{C_{\text{Rém}}} \qquad \text{for any } z \in F_b. \tag{29}$$

Moreover, when we need we can further restrict to the compact analytic subset $T_{b,\delta}$ constructed in Equation (27), ensuring that each point $z \in T_{b,\delta}$ with height at most C_{height} has enough conjugates in $T_{b,\delta}$.

2.1.6 Reduction steps

Let us consider $b \in f_2(\mathcal{A}')$. If b is a σ_2-torsion value it has height bounded by C_{height}, so we can ensure that it has enough conjugates in the compact set Δ constructed in Equation (23). Since the order of $\sigma_2(b)$ and the set $f_2(\mathcal{A}')$ are invariant under the action of any K-embedding $\tau \colon K(b) \hookrightarrow \mathbb{C}$, in our proof we can always replace b by b^τ and consequently assume $b \in \Delta$. Roughly speaking we have just explained that we can assume that b lies in a "big enough" compact set of $\overline{S}_2(\mathbb{C})$ that avoids the bad locus of f_2.

Fix $b \in \Delta$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$. Since $p \in \mathfrak{F}$, then $f_1(p)$ is a σ_1-torsion value and $f_2(p)$ is a σ_2-torsion value. We denote $m = m(b) := \operatorname{ord}(\sigma_2(b))$ and define

$$\mathfrak{O} := \{\operatorname{ord}(\sigma_2(b)) \colon b \in f_2(\mathfrak{F}) \cap \Delta)\} \subseteq \mathbb{N}, \tag{30}$$

where clearly the order is intended in $\mathcal{A}_{2,b}$. Moreover, for any $r = 0, 1, \ldots, m-1$ we define

$$p_r := t_2^r(p) = p + r\sigma_2(b) \qquad \text{and} \qquad n_r := \operatorname{ord}\sigma_1(f_1(p_r))\,. \tag{31}$$

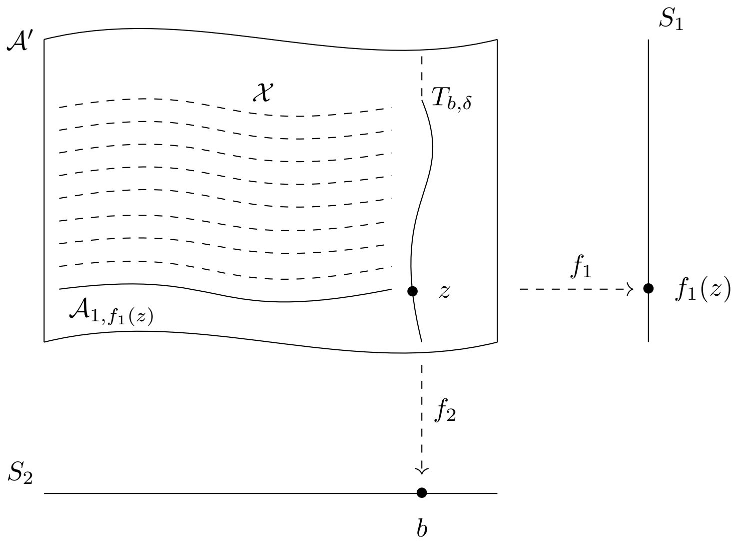

Figure 2: A schematization of the family $\mathcal{X} \to T_{b,\delta}$.

Let Σ_p be the set defined in Equation (19). For any $\tau \in \Sigma_p$ we fix the following notation to denote the 'transaltes' of p^τ:

$$a_r = a_r^{(b,p,\tau)} := f_1(p^\tau + r\sigma_2(b^\tau)) \qquad \text{for } r = 0, \dots, m-1. \tag{32}$$

Further, we can decompose the compact set Δ as a finite union of small definable compact sets Ξ_i. We work in one of those compact sets that contains b and we call it Ξ, in symbols we have

$$\Delta \subseteq \bigcup \Xi_i, \qquad b \in \Xi. \tag{33}$$

Analogously, we can decompose the compact set Δ' on $\overline{S}_1$ (see Equation (27)) as a finite union of small definable compact sets Ξ_i' where the Betti map of the section σ_1 is defined. We work in one of those compact sets that contains $f_1(p)$ and we call it Ξ', in symbols we have

$$\Delta' \subseteq \bigcup \Xi_i', \qquad f_1(p) \in \Xi'. \tag{34}$$

When we want to control the conjugates of p with respect to Ξ and/or Ξ' we will use the following subsets of Σ_p:

$$\Sigma_{p,\Xi} := \{\tau \in \Sigma_p : b^\tau \in \Xi\}, \qquad \Sigma_{p,\Xi,\Xi'} := \{\tau \in \Sigma_p : b^\tau \in \Xi,\ f_1(p)^\tau \in \Xi'\}. \tag{35}$$

Up to replace b, p with b^τ, p^τ and up to change Ξ and Ξ', since the number of Ξ_i's and Ξ_i''s is fixed and by construction of Δ and Δ', we can apply Proposition 1.10 to b and $f_1(p)$ and conclude the following:

$$\#\Sigma_{p,\Xi} \gg [K(p) : K] \qquad \text{and} \qquad \#\Sigma_{p,\Xi,\Xi'} \gg [K(p) : K]\,, \tag{36}$$

where the implicit constants are independent from p and b.

2.1.7 Consequences of the height bounds

Let $p \in \mathfrak{F} \cap \mathcal{A}'$ and $b = f_2(p)$. Using the construction in Section 2.1.2, we ensure that such points satisfy a uniform height bound as well as certain inequalities involving torsion orders and degrees. However, we are particularly interested in studying translates of p and their conjugates. Since Zariski closed subsets are not preserved under translation, the behavior of points defined in Equation (31) and Equation (32) could, in principle, be irregular. Nevertheless, we prove in Proposition 2.1 that a uniform bound for the heights of such points can be established. A crucial aspect of our approach is the use of height functions on $\mathcal{A}_1$, S_1, and the fibers $\mathcal{A}_{2,b}$, as defined in Section 2.1.1. Indeed, the result fails if the chosen height functions are not appropriately related. As a consequence, we show in Proposition 2.2 that it is possible to control the distribution of conjugates of p and their images on the two bases S_1 and S_2. Specifically, as explained in Section 2.1.4 we generally work with a subset of the base $S_1(\mathbb{C})$ as defined in Equation (27) and we must ensure that a "good portion" of conjugates is stable with respect to the euclidean coverings defined in Equation (33) and Equation (34).

Proposition 2.1. *Let $h = h_{\mathcal{L}} : S_1(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ and $h_b : \mathcal{A}_{2,b}(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ be the height functions defined in Section 2.1.1. Given $b \in \Delta$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$ we have*

$$h_b(p + r\sigma_2(b)) \leq C_{\text{height}}, \quad h(f_1(p + r\sigma_2(b))) \leq C_{\text{height}} \qquad \text{for each } r = 0, \dots, m-1,$$

where $C_{height} > 0$ is the constant introduced in Equation (18) and is independent from m, b and p.

Proof. Since $p \in \mathfrak{F}$ then it is a $s_{\mathcal{X}}$-torsion value. Since all the σ_1-torsion values in $f_1(\mathcal{A}')$ have uniformly bounded height by a constant μ_1, denoting with μ_2 the constant defined by property (iii) of the height $\hat{h}_{\mathcal{M}}$ we obtain the following uniform bound on the height of p:

$$h_b(p) \leq \mu_1 + \mu_2.$$

Notice that $r\sigma_2(b)$ is a torsion point of $\mathcal{A}_{2,b}$, so by property (i) of the height $\hat{h}_{\mathcal{M}}$ we have $h_b(r\sigma_2(b)) = 0$. Thus, by property (ii) of the height $\hat{h}_{\mathcal{M}}$, for any $r = 0, \dots, m-1$ we obtain

$$h_b(p + r\sigma_2(b)) \leq h_b(p + r\sigma_2(b)) + h_b(p - r\sigma_2(b)) = 2h_b(p) \leq 2(\mu_1 + \mu_2).$$

In other words, each point of the type $p + r\sigma_2(b)$ has uniformly bounded height. The full claim then follows using again property (iii) of the height $\hat{h}_{\mathcal{M}}$ and Equation (18).

□

We use the notations introduced in Equations (32) to (35). Fix $m \in \mathfrak{O}, b \in \Delta$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$ and $\operatorname{ord}(\sigma_2(b)) = m$. Since $K(b) \subseteq K(p)$, by Equation (22) we obtain

$$m = \operatorname{ord}(\sigma_2(b)) \leq C'_{\text{Rém}}[K(p) : K]^{C_{\text{Rém}}} \qquad \text{for any } b \in f_2(\mathcal{A}'). \tag{37}$$

By Proposition 2.1, the element $f_1(p)$ has height bounded by C_{height} uniformly. Let us consider conjugation with respect to the set Σ_p defined in Equation (19). As explained before Equation (27) and after Equation (23), we choose $\delta > 0$ small enough such that[5]

$$\#\{a_0^{(b,p,\tau)} : \tau \in \Sigma_p\} \cap \Delta' \geq \frac{3}{4}[K(p) : K] \qquad \text{and} \qquad \#\{b^\tau : \tau \in \Sigma_p\} \cap \Delta \geq \frac{3}{4}[K(p) : K]\,.$$

Therefore, we obtain

$$\#\{a_0^{(b,p,\tau)} : \tau \in \Sigma_p \text{ and } b^\tau \in \Delta\} \cap \Delta' \geq \frac{1}{2}[K(p) : K]\,.$$

We define

$$\mathcal{J}_m^{(b,p)} := \{a_0^{(b,p,\tau)} : \tau \in \Sigma_{p,\Xi,\Xi'}\} \cap \Delta'\,. \tag{38}$$

Since the number of the sets Ξ_i and Ξ'_i is fixed, up to replace b, p with Σ_p-conjugates b^τ, p^τ, we can always choose compact sets Ξ among the Ξ_i and Ξ' among the Ξ'_i such that

$$b \in \Xi,\ f_1(p) \in \Xi' \qquad \text{and} \qquad \#\mathcal{J}_m^{(b,p)} \gg [K(p) : K]\,. \tag{39}$$

Proposition 2.2. *Assume that $\mathfrak{O}$ is infinite. Let us consider $m \in \mathfrak{O}$ and $b \in \Delta$ such that $\operatorname{ord}(\sigma_2(b)) = m$. Let $p \in \mathfrak{F} \cap \mathcal{A}'$ be such that $f_2(p) = b$. Assume $b \in \Xi$ and $f_1(p) \in \Xi'$ such that Equation (39) holds. For any $m \gg 1$ we have*

$$\#\mathcal{J}_m^{(b,p)} \gg m^{\frac{1}{C_{\text{Rém}}}}\,, \tag{40}$$

where the implicit constant is independent from m, b and p.

Proof. We proceed by contradiction: after choosing a sequence contained in $\mathfrak{O}$, for any m there exist $b \in \Xi$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ with $f_1(p) \in \Xi'$ such that

$$\frac{\#\mathcal{J}_m^{(b,p)}}{m^{\frac{1}{C_{\text{Rém}}}}} \xrightarrow[m\to\infty]{} 0\,. \tag{41}$$

By Equation (37) and Equation (39) we obtain

$$\#\mathcal{J}_m^{(b,p)} \gg [K(p) : K] \gg m^{\frac{1}{C_{\text{Rém}}}}\,.$$

Finally we get

$$\frac{\#\mathcal{J}_m^{(b,p)}}{m^{\frac{1}{C_{\text{Rém}}}}} \gg 1,$$

which is a contradiction with Equation (41). □

[5]We are taking conjugates of the field $K(p)$, which may be larger than $K(b)$ and $K(f_1(p))$: some of these conjugates may coincide but their distribution is preserved.

2.1.8 Strategy of the proof

It is enough to prove that

the set $\mathfrak{O}$ defined in Equation (30) is bounded, i.e. the orders $m \in \mathfrak{O}$ are uniformly bounded.

In fact, if $\mathfrak{O}$ is bounded by a uniform constant C, then

$$\{f_2(p) : p \in \mathfrak{F} \cap \mathcal{A}'\} \subseteq \{b \in f_2(\mathcal{A}') \colon \operatorname{ord}(\sigma_2(b)) \leq C\} \subseteq \sigma_2^{-1}\left(\bigcup_{N \leq C} \mathcal{A}_2[N]\right). \tag{42}$$

Theorem 0.4 follows, since σ_2 is non-torsion. We will partition $\mathfrak{O}$ in two subsets $\mathfrak{O}'$ and $\mathfrak{O}''$ and show that each of them contains a finite number of elements.

2.2 Proof

All the notations introduced in Equations (16) to (36) will be fixed in the rest of the paper.

2.2.1 First case

For any $m \in \mathfrak{O}$ we consider $b \in \Delta$ such that $\operatorname{ord}(\sigma_2(b)) = m$. Let F_b be the Zariski open subset of the fiber $\mathcal{A}_{2,b}$ introduced in Equation (17) and let $T_{b,\delta}$ be the euclidean compact set defined in Equation (27). Given a point $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$ we use the notation Equation (31) to denote the σ_2-translates of p and their orders with respect to the f_1-group law. Let $C_{\text{Rém}}$ be the constant introduced in Equation (21) and let's define

$$\mathfrak{O}' := \left\{ m \in \mathfrak{O} \colon \exists b \in \Delta \text{ and } \exists p_r \in F_b \text{ such that } n_r > m^{g(2C_{\text{Rém}}+1)} \right\}.$$

We will prove that the set $\mathfrak{O}'$ is finite, giving a uniform upper bound for $m \in \mathfrak{O}'$. We fix

$$m \in \mathfrak{O}',\ b \in \Delta \text{ with } \operatorname{ord}(\sigma_2(b)) = m,\ p \in \mathfrak{F} \cap \mathcal{A}' \text{ with } f_2(p) = b,$$

and a point

$$\zeta := p_r = p + r\sigma_2(b) \in F_b \qquad \text{such that} \qquad n := n_r > m^{g(2C_{\text{Rém}}+1)}, \tag{43}$$

for some $r \in \{0, \dots, m-1\}$. Up to choose $\delta > 0$ small enough, we have $\zeta \in T_{b,\delta}$.

Consider the abelian scheme $\mathcal{X} \to F_b$ defined in Equation (28) and fix $z \in F_b(\mathbb{C})$. As explained in Equation (3), there exists a simply connected open set $U_z' \subseteq F_b(\mathbb{C})$ in the complex topology containing z where a period map is defined:

$$\mathcal{P}_{\mathcal{X}}^{(b)} = \left(\omega_{1,\mathcal{X}}^{(b)}, \dots, \omega_{2g,\mathcal{X}}^{(b)}\right).$$

In other words we have holomorphic functions $\omega_{i,\mathcal{X}}^{(b)} : U_z' \to \mathbb{C}^g$ for $i = 1, \dots, 2g$ which fix a basis of the corresponding lattice $\Lambda_{z'}$ for each $z' \in U_z'$. Thus, the family of open simply connected sets $\{U_z' : z \in T_{b,\delta}\}$ is a covering of $T_{b,\delta}$. Fixing a standard chart U_i' which contains z, we can consider a simply connected open definable subset $U_z \subseteq U_z' \cap U_i'$ which contains z and whose analytic closure D_z is contained in $U_z' \cap U_i'$. In other words, we can consider an open covering $\{U_z \colon z \in T_{b,\delta}\}$, where each U_z is a simply connected open set with the following properties: its analytic closure D_z in the fixed chart of F_b is a definable compact set in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ and all the period functions $\omega_{i,\mathcal{X}}^{(b)}$ with $i = 1, \dots, 2g$ are defined as holomorphic functions on D_z. Since $T_{b,\delta}$ is compact, it can be covered with finitely many small compact simply-connected sets of the type D_z.

Since $U_z' \subseteq F_b(\mathbb{C})$ is simply connected, we obtain notions of abelian logarithm $\ell_{\mathcal{X}}^{(b)}$ and Betti map $\beta_{\mathcal{X}}^{(b)} = \left(\beta_{1,\mathcal{X}}^{(b)}, \dots, \beta_{2g,\mathcal{X}}^{(b)}\right)$ of the section $s_{\mathcal{X}}$ on each U_z' as explained in Equation (4). Note that the abelian logarithm is a holomorphic function on each compact set D_z and the Betti map is described by the equation

$$\ell_{\mathcal{X}}^{(b)}(z) = \beta_{1,\mathcal{X}}^{(b)}(z)\omega_{1,\mathcal{X}}^{(b)}(z) + \ldots + \beta_{2g,\mathcal{X}}^{(b)}(z)\omega_{2g,\mathcal{X}}^{(b)}(z),$$

where the Betti coordinates $\beta_{i,\mathcal{X}}^{(b)}$ are real-analytic functions on each compact set D_z. In addition note that $\beta_{\mathcal{X}}^{(b)}$ doesn't have any critical points on $T_{b,\delta}$ by construction (we have expressly removed them).

Summarizing: we have obtained the existence of finitely many simply connected compact sets D_i with $i = 1, \dots, N_{\text{comp}}$ which are definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ and where the Betti map $\beta_{\mathcal{X}}^{(b)}$ is $\mathbb{R}_{\text{an,exp}}$-definable and a submersion.

Remark 2.3. Fix $z \in T_{b,\delta}$. Observe that period functions, logarithms and Betti maps of $\mathcal{X} \to F_b$ are uniform with respect to b, since each fiber $\mathcal{X}_z$ only depend on the image $f_1(z)$. Moreover, the number N_{comp} of compact sets D_i's just constructed can be supposed to be uniform, i.e. constant with respect to $b \in \Delta$: in fact the open covering of the $T_{b,\delta}$'s given by the open part of the D_i's can be assumed to be induced (after intersecting with f_2-fibers) by a global open covering of the compact set $f_2^{-1}(\Delta)$ with the same properties.

Fix one of the previous compact sets which contains ζ and call it D. By Equation (29) we have

$$n^{\frac{1}{C_{\text{Rém}}}} \ll [K(\zeta) : K], \tag{44}$$

where the implicit constant depends only on g and K, which are fixed. On the other hand, recalling that the degree of the isogeny induced by the multiplication by m is m^{2g}, by Equation (43) we deduce

$$[K(b) : K] = [K(\sigma_2(b)) : K] \leq m^{2g} < n^{\frac{2}{2C_{\text{Rém}}+1}}. \tag{45}$$

We are now going to define a series of positive constants $c_0, c_1, \ldots$ that we need keep until the end of this section. By Equation (44) and Equation (45) we obtain

$$d := [K(\zeta) : K(b)] = \frac{[K(\zeta) : K]}{[K(b) : K]} \gg \frac{n^{\frac{1}{C_{\text{Rém}}}}}{n^{\frac{2}{2C_{\text{Rém}}+1}}} = n^{\frac{1}{c_0}}, \qquad \text{where } \; c_0 := C_{\text{Rém}}(2C_{\text{Rém}} + 1).$$

Consider the conjugates of ζ over $K(b)$, and call them ζ_j where $j = 1, \ldots, d$; they are torsion values of $s_{\mathcal{X}}$, since the section $s_{\mathcal{X}}$ is defined over K. As explained after Equation (29), up to choose $\delta > 0$ small enough, we can assume that the number of these conjugates lying in a same compact set of the type D_i is $\gg d$, where the implicit constant depends only on the original data (it can be taken for instance equal to $1/(2N_{\text{comp}})$ by Remark 2.3). From now on, we will denote by $\Omega = \Omega_b \subseteq A_{2,b}(\mathbb{C})$ the compact set (among the D_i's) just described. Hence, we may assume

$$\#\{\zeta_j \in \Omega\} \gg n^{\frac{1}{c_0}}. \tag{46}$$

By Equation (33), we decompose the compact set Δ as a finite union of small definable compact sets Ξ_j and we choose a set Ξ among them containing b. We consider the Betti map

$$\beta(z) := \beta^{(b)}(z) := (\beta^{(b)}_{1,\mathcal{X}}(z), \ldots, \beta^{(b)}_{2g,\mathcal{X}}(z)). \tag{47}$$

The Betti coordinates $\beta^{(b)}_{i,\mathcal{X}}$ are real-analytic with respect to the variable $z \in \Omega_b$ and also with respect to $b \in \Xi$. We consider the $\mathbb{R}_{\text{an,exp}}$-definable family $Z := \Xi \times \mathbb{R}^{2g}$, where the fibers are the real-analytic varieties $Z_b = \{b\} \times \mathbb{R}^{2g}$. Notice that when b is a torsion value of σ_2, then

$$\{b\} \times \beta(\Omega_b) \subseteq Z_b. \tag{48}$$

We denote by Z_b^{alg} (resp. $\beta(\Omega_b)^{\text{alg}}$) the algebraic part of Z_b (resp. $\beta(\Omega_b)$). We now prove that $\beta(\Omega_b)^{\text{alg}}$ is empty. This follows a standard procedure, relying on the algebraic independence of the coordinates of the logarithm with respect to the periods (see, for instance, [31, Lemma 6.2]). For completeness, we outline the main steps below, keeping the following important clarification in mind.

Remark 2.4. We point out that the argument described below works only for $g \geq 2$ since we need at least two components of the abelian logarithm. Nevertheless, the case $g = 1$ can be treated with small modifications in the construction of the family Z: indeed it is enough to consider two auxiliary abelian schemes instead of $\mathcal{X}$ only. In this way we have two Betti maps and two logarithms (each of them with one component). Then we apply the same procedure described in this section on the new definable family Z that now lives in $\mathbb{R}^2 \times \mathbb{R}^4$. For the details of the case $g = 1$ the reader can check directly [15, Theorem 1.1] where, what we have just described in this remark, is exactly the technique carried out.

Assume by contradiction that the algebraic part of $\beta(\Omega_b)$ is non-empty, so there is a real-algebraic arc γ contained in $\beta(\Omega_b)^{\text{alg}}$. In what follows we omit the dependence on b and $\mathcal{X}$ to simplify the notation. Consider the real-analytic set $U := \beta^{-1}(\gamma) \subseteq \Omega$. Since γ is a real algebraic arc and the points $\beta(z)$ with $z \in U$ satisfy the real algebraic equations defining γ, then the Betti coordinates β_i are algebraically dependent over $\mathbb{C}(S)$ when restricted to U. Moreover, this also implies that the field generated by the $2g$ Betti coordinates (when restricted to U) over $\mathbb{C}(S)$ has transcendence degree at most 1; in other words, any two of the Betti coordinates verify an algebraic equation over $\mathbb{C}(S)$. Thus, we have two cases: either the $2g$ Betti coordinates restricted to U all depend algebraically on any of them which is not constant, or otherwise they are all constant.

In the first case: let's consider the coordinates of the period functions $\omega_i = (\omega_{i1}, \ldots, \omega_{ig})$ for $i = 1, \ldots, 2g$. Here, all the functions are intended to be restricted to U, unless otherwise specified. The field generated by ω_i, β_i over $\mathbb{C}(S)$ has transcendence degree 1 over $\mathbb{C}(S)(\{\omega_{ij}\})$ and contains the coordinates of the abelian logarithm ℓ. This implies that the coordinates of ℓ are algebraically dependent over $\mathbb{C}(S)(\{\omega_{ij}\})$. However all these functions are locally holomorphic, so the dependence would hold identically on their domain Ω, which violates the independence result [7, Theorem 3] of André (see also [31, Lemma 5.1]).

In the second case, i.e. when the Betti coordinates are all constant when restricted to U, they are constant on their domain Ω by the same principle as above. This implies that the corresponding sections are identically torsion, which is a contradiction. Therefore, we have

$$\beta(\Omega_b)^{\mathrm{alg}} = \emptyset \quad \text{and consequently} \quad \beta(\Omega_b) = \beta(\Omega_b) \setminus \beta(\Omega_b)^{\mathrm{alg}}. \tag{49}$$

For the properties of the Betti map, each point ζ_j in Equation (46) gives rise to a rational point $\beta(\zeta_j) \in Z_b$ with denominators at most n. Some of these rational points might coincide, but since $\zeta_j \in \mathcal{A}'$ we can apply Proposition 1.2 and conclude that

$$\#\{\beta(\zeta_j) : \zeta_j \in \Omega_b\} \gg n^{\frac{1}{c_0}}, \tag{50}$$

where the constant depends only on the involved compact sets, which are fixed. In order to apply the Pila-Wilkie counting theorem for rational points we need the following height function on $\mathbb{Q}^{2g}$:

$$H\left(\frac{x_1}{y_1}, \ldots, \frac{x_{2g}}{y_{2g}}\right) := \max_i\{\max |x_i|, |y_i|\}. \tag{51}$$

All the rational points in Equation (50) have height $\ll n$, say $\leq c_1 n$.

Remark 2.5. Let's explain more in detail why c_1 is uniform. Firstly, the denominators of $\beta(\zeta_j)$ are bounded. Moreover we can bound the numerators on each compact set D_z, since the Betti map attains a maximum on each of them. Since the number of compact sets was previously fixed, we can choose analytic continuation of the Betti map such that the numerators of $\beta(\zeta_j)$ are bounded uniformly.

For any subset $\Sigma \subseteq \mathbb{R}^{2g}$ we define

$$\Sigma(\mathbb{Q}, T) := \{q \in \Sigma(\mathbb{Q}) \mid H(q) \leq T\}, \qquad N(\Sigma, T) := \#\Sigma(\mathbb{Q}, T). \tag{52}$$

We have

$$N(\beta(\Omega_b), c_1 n) \geq c_2 n^{\frac{1}{c_0}}, \qquad \text{for some constant } c_2. \tag{53}$$

On the other hand by [39, Theorem 1.9], for any $\varepsilon > 0$ there exists a constant $c(Z, \varepsilon)$ such that

$$N(Z_b - Z_b^{\mathrm{alg}}, T) \leq c(Z, \varepsilon) T^{\varepsilon}, \tag{54}$$

where the constant is independent from $b \in \Xi$. By Equation (48) and Equation (49), taking $\varepsilon = 1/(2c_0)$ we obtain

$$c_2 n^{\frac{1}{c_0}} \leq N(\beta(\Omega_b), c_1 n) \leq c(Z)(c_1 n)^{\frac{1}{2c_0}}$$

where all constants $c(Z), c_0, c_1, c_2$ are uniform with respect to $b \in \Xi$. This implies $n^{\frac{1}{2c_0}} \leq c_3$, that is $n^{\frac{1}{2C_{\text{Rém}}+1}} \leq c_3^{2C_{\text{Rém}}}$. In particular, by Equation (43) this implies

$$m < n^{\frac{1}{g(2C_{\text{Rém}}+1)}} \leq c_3^{\frac{2C_{\text{Rém}}}{g}}.$$

This estimate holds uniformly with respect to $b \in \Xi$. Since we have a finite number of fixed compact sets Ξ_j which cover Δ, we obtain a uniform bound for $m \in \mathfrak{O}'$.

2.2.2 Second case

We keep the same notations used in Section 2.2.1. Define

$$\mathfrak{O}'' := \{m \in \mathfrak{O} : \forall b \in \Delta \text{ and } \forall p_r \in F_b \text{ we have } n_r \leq m^{g(2C_{\text{Rém}}+1)}\}.$$

We will prove that the set $\mathfrak{O}''$ is finite. Assume by contradiction that it is not finite. We fix

$$m \in \mathfrak{O}'',\ b \in \Delta \text{ with } \operatorname{ord}(\sigma_2(b)) = m,\ p \in \mathfrak{F} \cap \mathcal{A}' \text{ with } f_2(p) = b.$$

Therefore, for any $r \in \{0, \dots, m-1\}$ we have

$$p_r = p + r\sigma_2(b) \in F_b \quad \Longrightarrow \quad n_r \leq m^{g(2C_{\text{Rém}}+1)}. \tag{55}$$

We consider again the abelian scheme $\mathcal{X} \to F_b$ introduced in Equation (17) with the euclidean compact set in Equation (27). We decompose $T_{b,\delta}$ as a finite union of compact subsets $\{D_i\}$ where periods, abelian logarithm and Betti map are defined, as in Section 2.2.1. By Equation (33) we decompose Δ and Δ' as a finite union of definable compact sets and we choose compact sets Ξ and Ξ' among them containing b and $f_1(p)$, respectively. Since we are assuming that $\mathfrak{O}''$ is infinite, by Proposition 2.2 for any $m \gg 1$ we have

$$\#\mathcal{J}_m^{(b,p)} \gg m^{\frac{1}{C_{\text{Rém}}}}, \tag{56}$$

where the set $\mathcal{J}_m^{(b,p)}$ is introduced in Equation (38) and contains the f_1-images of all the $\Sigma_{p,\Xi,\Xi'}$-conjugates of p. The implicit constant is independent from m, b and p.

Denote by β_{σ_1} the Betti map of σ_1 on S_1. We consider the $\mathbb{R}_{\text{an,exp}}$-definable family $Z := \Xi \times \mathbb{R}^{2g}$ with fibers $Z_b = \{b\} \times \mathbb{R}^{2g}$. When b is a torsion value of σ_2 we have:

$$\{b\} \times \beta_{\sigma_1}(\mathcal{J}_m^{(b,p)}) \subseteq \{b\} \times \beta_{\sigma_1}(\Xi') \subseteq Z_b. \tag{57}$$

In the following we use same height of Equation (51) and the same notation of Equation (52). By reasoning exactly as in the previous case it is possible to show that $\beta_{\sigma_1}(\Xi')^{\text{alg}}$ is empty. Also here we have to appeal to Remark 2.4: the case $g = 1$ needs a slightly different approach with a definable family in $\mathbb{R}^2 \times \mathbb{R}^4$; again, all the details are in [15].

By Equation (55), for the properties of the Betti map, the points $\beta_{\sigma_1}(\mathcal{J}_m^{(b,p)})$ are rational with denominators at most $m^{g(2C_{\text{Rém}}+1)}$. By Remark 2.5, the points of $\beta_{\sigma_1}(\mathcal{J}_m^{(b,p)})$ have height $\ll m^{g(2C_{\text{Rém}}+1)}$, say $\leq c_4 m^{g(2C_{\text{Rém}}+1)}$. By [39, Theorem 1.9], for any $\varepsilon > 0$ there exists a constant $c(Z, \varepsilon)$ such that

$$N(Z_b - Z_b^{\text{alg}}, c_4 m^{g(2C_{\text{Rém}}+1)}) \leq c(Z, \varepsilon)(c_4 m^{g(2C_{\text{Rém}}+1)})^{\varepsilon}, \tag{58}$$

where the constant is independent from $b \in \Xi$. On the other hand, since $p \in \mathcal{A}'$, by Proposition 1.2 and Equation (56) we conclude that

$$N(\beta_{\sigma_1}(\Xi'), c_4 m^{g(2C_{\text{Rém}}+1)}) \geq c_5 m^{\frac{1}{C_{\text{Rém}}}} \qquad \text{for some constant } c_5, \tag{59}$$

where the constant depends only on the involved compact sets, which are fixed. Therefore, by choosing $\varepsilon < \frac{1}{gC_{\text{Rém}}(2C_{\text{Rém}}+1)}$, from Equation (57) we finally obtain:

$$m \leq \left(\frac{c(Z)c_4^{\varepsilon}}{c_5}\right)^{\frac{C_{\text{Rém}}}{1-\varepsilon g C_{\text{Rém}}(2C_{\text{Rém}}+1)}}.$$

This bound holds uniformly on Ξ and Ξ'. Since $\{\Xi_j\}$ and $\{\Xi'_j\}$ are fixed finite covering of Δ and Δ' respectively, we get a uniform bound for $m \in \mathfrak{O}''$ concluding the proof.

2.3 Some comments on the shape of Z_1 and Z_2

At the beginning of the proof, we removed some proper Zariski closed subset from the total space $\overline{\mathcal{A}}$ (see Section 2.1.2). Consequently, those sets fall inside the Zariski closed sets Z_1 and Z_2 appearing in Theorem 0.4. Thanks to the previous considerations, we get explicit expressions of Z_1 and Z_2 as it follows:

$$Z_1 = \text{Sing}_1 \cup E \cup \text{Ind}_1 \cup \mathcal{C}(\beta_1) \cup \mathcal{C}_{\text{Rém},1} \cup \mathcal{C}_{\text{height},1},$$

$$Z_2 = \text{Sing}_2 \cup \text{Ind}_2 \cup C_{\text{Rém},2} \cup \mathcal{C}_{\text{height},2} \cup \sigma_2^{-1}\left(\bigcup_{N \leq C} \mathcal{A}_2[N]\right),$$

where C is the uniform bound on $\mathfrak{O}$ (see Equation (42)). Unfortunately the constant C is implicit.

When $\dim \overline{S}_1 = \dim \overline{S}_2 = g = 1$, we have $\overline{S}_1 = \overline{S}_2 = \mathbb{P}^1$. In this case, we denote both bases simply by S. Here, the subsets $\mathcal{C}_{\text{height},i}$ are empty for obvious reasons, and the locus $f_1^{-1}(E)$ can be equivalently described as a finite union of f_2-fibers. The loci $\mathcal{C}_{\text{Rém},i}$ are empty in this case since we don't need to use Faltings height. Furthermore, the closed set $\mathcal{C}(\beta_1)$ does not need to be removed: since the Betti map β_1 is non-constant and the base S is an irreducible curve, the fibers of β_1 are all finite (even in the presence of critical points), and Gabrielov's theorem holds everywhere.

Finally, the following proposition shows that in the case $1 = \dim S = g$ all the points of $(\mathfrak{F} \setminus \text{Fund}(f_2)) \cap f_1^{-1}(\text{Sing}_1)$ are contained in a set of the form $f_2^{-1}(Z)$, where Z is a proper Zariski closed subset of $\overline{S}_2$. In other words we recover the stronger result proved in [15], i.e. $\mathfrak{F} \setminus \text{Fund}(f_2)$ is contained in a finite number of f_2-fibers (see Remark 0.6).

Proposition 2.6. *Let $1 = \dim S = g$, then there exists a proper closed Zariski subset $Z \subset S(\mathbb{C})$ such that:*

$$(\mathfrak{F} \setminus \text{Fund}(f_2)) \cap f_1^{-1}(\text{Sing}_1) \subseteq f_2^{-1}(Z).$$

Proof. Assume that Sing_1 has cardinality n and denote by Z_1 and Z_2 the proper Zariski closed subsets of $\overline{S}_1$ and $\overline{S}_2$ arising from Theorem 0.4, respectively. By Bézout theorem we know that $\#(\mathcal{A}_{2,s}(\mathbb{C}) \cap f_1^{-1}(\text{Sing}_1)) \leq 9n$. Let's put $H = (\mathfrak{F} \setminus \text{Fund}(f_2)) \cap f_1^{-1}(\text{Sing}_1)$ and let's consider the following partition of H:

$$H_1 := \{p \in H : \#(O(p)) \leq 9n\}, \qquad H_2 := \{p \in H : \#(O(p)) > 9n\}.$$

The set $f_2(H_1)$ is finite, since the following containment holds:

$$f_2(H_1) \subseteq \sigma_2^{-1}\left(\bigcup_{N=1}^{9n} \mathcal{A}[N]\right).$$

Fix $p \in H_2$. Observe that there exists $r \in \mathbb{N}$ such that $t_2^r(p) \notin f_1^{-1}(\text{Sing}_1)$: if not, we would have a contradiction by the fact that $O(p) = \{t_2^r(p) : r \in \mathbb{N}\} \subseteq f_1^{-1}(\text{Sing}_1) \cap \mathcal{A}_{2,s}(\mathbb{C})$ and $\#(O(p)) > 9n$. Therefore, for such r we have $f_1(t_2^r(p)) \notin Z_1$. Hence, by Theorem 0.4, we get $f_2(t_2^r(p)) \in Z_2$. Since t_2 acts on the f_2-fibers, we conclude that $f_2(t_2^r(p)) = f_2(p) \in Z_2$. This proves that $f_2(H_2) \subseteq Z_2$. The claim follows if we put $Z = Z_2 \cup f_2(H_1)$.

□

A Construction of double abelian fibrations in the IHS case

by E. Amerik

The purpose of this appendix is to remark that examples of the situation studied in this paper exist in every even dimension, and to provide some explicit constructions, as well as indications how to prove abstract existence results in a case which has been extensively studied by geometers. The general framework is as follows. We consider an **irreducible holomorphically symplectic (IHS) manifold** X, that is, a simply-connected manifold X such that $H^0(X, \Omega^2_X)$ is one-dimensional and generated by a nowhere degenerate form σ. We can take X projective, or more generally compact Kähler (in the situation we are looking for, projectivity shall be automatic). A typical example of such a manifold is a K3 surface S, or, more generally, the n-th punctual Hilbert scheme $S^{[n]}$, parameterizing subschemes of S of finite length n. In all explicit examples, we shall be dealing with $S^{[n]}$, but the general results are valid in the general IHS context.

It is well-known that on the second cohomology $H^2(X,\mathbb{Z})$ there is an integral non-degenerate quadratic form q, called the Beauville-Bogomolov form, which can be seen as an analogue of the intersection form on a surface. If $X \to B$ is a fibration, then the inverse image of an ample line bundle on B is nef and q-isotropic. Conversely, a famous "Lagrangian", or "hyperkähler SYZ", conjecture, checked in all known examples, in particular for $S^{[n]}$, states that if L is a nef line bundle on X with $q(L) = 0$, then some power of L is base-point-free, so that its sections define a fibration $\phi = \phi_L : X \to B$. Matsushita [32] proved that a non-trivial fibration on an IHS manifold is equidimensional, and all smooth fibers are lagrangian tori. In particular, if ϕ has a section, one obtains a family of abelian varieties on an open subset of X, say $\phi^0 : X^0 \to B^0$.

Oguiso ([36]) proved that the Picard number of the generic fiber of such a fibration is always equal to one. In particular, the generic fiber is simple, so that the family does not have a fixed part as soon as it is not isotrivial. In fact it is easy to deduce from [9] or [6] that no finite base-change of ϕ^0 has a fixed part unless the family is isotrivial.

By the same reason, the multiples of any non-torsion section or multisection of a family of abelian varieties arising in this way must be Zariski-dense.

If f is an automorphism of X such that its action on $H^2(X,\mathbb{Z})$ preserves the class of L as above, then a power of f preserves the fibration $\phi_L : X \to B$ ([26]) and acts on the smooth fibers as a translation ([6]). There is a way to say whether an automorphism ψ of the Neron-Severi lattice $NS(X) \subset H^2(X,\mathbb{Z})$ preserving the class of L comes from an actual automorphism $f : X \to X$, see **"Hodge-theoretic Torelli theorem"** by Markman, [28]: it should belong to the (Hodge) monodromy group[6], and it should take some ample class to an ample class. The Hodge monodromy group is of finite index in the automorphism group of $(NS(X), q)$, so replacing any ψ by a power we may assume it is in there. The ample cone is governed by so-called MBM classes, a higher-dimensional analogue of (-2)-classes on K3 surfaces ([2], [3]). These are primitive classes in $H^2(X,\mathbb{Z})$ of bounded negative square ([4]). Inside the cone of classes of positive square in $NS(X) \otimes \mathbb{R}$, the ample cone is a connected component of the complement to the union of the orthogonal hyperplanes to the MBM classes of Hodge type $(1,1)$. On all known examples of IHS manifolds, in particular on $S^{[n]}$, these classes can be described explicitely. If no MBM class is orthogonal to L in $(NS(X), q)$, then, up to taking a power, an automorphism of the lattice which fixes L lifts to an automorphism of X: indeed the image of an ample class near L in $NS(X) \otimes \mathbb{Q}$ shall be ample, so this is a consequence of Hodge-theoretic Torelli. The automorphisms preserving L, up to a finite index, form a free abelian group of rank $\rho - 2$, where ρ is the Picard number of X (we assume here that $\rho \geq 3$, then the statement is obtained from hyperbolic geometry, see [6]). If there are such MBM classes but not too many, some automorphisms may lift, see e.g. [33]: one has to further subtract from $\rho - 2$ the dimension of the subspace they generate. Such automorphisms are sometimes called **parabolic**.

Let us start with the following explicit example. Let S be a smooth quartic surface in $\mathbb{P}^3$ (it is, of course, a K3 surface). It is well-known and easy to see that S can contain only finitely many (complex) lines, so if S is defined over a number field, then the lines are defined over a (possibly larger) number field too. Assume S contains a line l. Take all planes through l, it is a pencil of planes (they are parameterized by $\mathbb{P}^1$). For each such plane P_t, the intersection with S is $l \cup C_t$, where C_t is a plane cubic. This gives a fibration $\phi : S \to \mathbb{P}^1$ where the smooth fibers are curves of genus 1. The line l induces a multisection: indeed l intersects each C_t in three points. So it is a trisection.

[6]The monodromy group is the group of automorphisms of $H^2(X,\mathbb{Z})$ generated by all parallel transports in families, and the Hodge monodromy group is the image of its Hodge type-preserving subgroup in the group of automorphisms of the Neron-Severi lattice.

If S contains another line l', which does not intersect l (this is possible, e.g. on a Fermat surface, but also on others - in fact over a codimension-two subvariety of the parameter space for quartic surfaces), this gives a section of ϕ, indeed each P_t and hence each C_t intersects l' at one point. In its turn, taking the pencil of planes P'_t through l', we obtain another fibration of S, $\phi' : S \to \mathbb{P}^1$, with genus one fibers C'_t residual to l' in the intersection of S and P'_t, a section induced by l, and a trisection induced by l' itself.

On the resulting abelian schemes, these trisections are non-torsion, see e.g. [21] where it is explained that a torsion multisection of an elliptic fibration of a K3 surface cannot be a rational curve. One can also choose S in such a way that it contains an additional line m skew to both l and l': it shall induce an additional section of both fibrations. Keeping in mind the general theory of automorphisms of IHS manifolds and MBM classes, one may also produce non-torsion sections on S as follows.

Proposition A.1. *If S is general with the above properties, then S admits an automorphism h of infinite order preserving ϕ and acting as a translation along its fibers.*

Proof. For such an S, the lattice $NS(X)$ is of rank 3, generated by the classes H (the hyperplane section class), l and l', and the class L of C_t is $H - l$. The orthogonal complement to L is generated by L itself and $H - 3l'$, which has square -20. Hence there are no MBM classes in the orthogonal complement to L: indeed these have square -2. So the result follows from Hodge-theoretic Torelli. ☐

We derive in particular that S also has a non-torsion section $h(l')$ of ϕ. The same applies to ϕ' (with $L' = H - l'$) and gives a non-torsion section $h'(l)$.

Consider now the k-th punctual Hilbert scheme $S^{[k]}$ of a K3 surface S: it parameterizes subschemes of S of length k, e. g. k-ples of distinct points, or of not necessarily distinct points with some extra structure. It is often viewed as a resolution of singularities of the k-th symmetric power of S. Any fibration $g : S \to \mathbb{P}^1$ naturally induces the fibration $g^{[k]} : S^{[k]} \to \mathbb{P}^k = Sym^k(\mathbb{P}^1)$. The fiber over a point $t_1 + \cdots + t_k$ (where the t_i are distinct points on the projective line) is just the product $C_{t_1} \times C_{t_2} \times \cdots \times C_{t_k}$. So this is a fibration where the fibers over an open subset of the base are k-dimensional tori. Any section s of g naturally induces a section $s^{[k]}$ of $g^{[k]}$, and non-torsion induces non-torsion.

We are now in a position to give explicit examples of the situation considered in the paper.

Theorem A.2. *For each $k \geq 1$ there exist algebraic varieties X of dimension $2k$ with two fibrations ϕ and ϕ' from X to $\mathbb{P}^k$, such that ϕ resp. ϕ' induces an abelian scheme structure without a fixed part on an open subset U resp. U' of X. Each of these fibrations has an extra non-torsion section. Moreover the multiples of these sections are Zariski-dense in U, U'.*

Proof. Take S a quartic in $\mathbb{P}^3$ containing two skew lines l and l', inducing fibrations ϕ and ϕ', and consider $\phi^{[k]}$ and $\phi'^{[k]}$ on $X = S^{[k]}$. ☐

Another, maybe slightly less well-known construction is as follows, see [22]. Take S a complete intersection of three quadrics in $\mathbb{P}^5$. This is again a K3 surface. We can arrange for S to contain a rational normal cubic C and to contain no lines. Let H be a hyperplane section divisor, then $(H - C)^2 = 0$, so curves residual to C in a hyperplane section are of square zero and genus one, this gives a fibration of S, and C induces a multisection of degree 5. Lift this fibration to $S^{[2]}$ as before, call it ϕ. Remark that a point of $S^{[2]}$ is either a pair of distinct points of S or a point together with a tangent direction. Through each pair of points of S, or a point with a tangent direction, there is a unique line l, and it does not intersect S at any extra points (indeed, since S is an intersection of quadrics, the line would be contained in S otherwise). The quadrics containing S are parameterized by a projective plane $\mathbb{P}(V)$, and those among them which contain l, by a line in this plane, so we have a natural map from $S^{[2]}$ to the dual projective plane $\mathbb{P}(V^*)$, and a fiber is naturally identified to the set of lines contained in the intersection of two quadrics, known to be an abelian surface generically (when this intersection is smooth), see e.g. [42]. So we have another fibration called ϕ'.

Proposition A.3. *The curve $C^{[2]}$ viewed as a subvariety of $S^{[2]}$ induces a (possibly rational[7]) section of ϕ'.*

Proof. Indeed the intersection of two sufficiently general quadrics from $\mathbb{P}(V)$ and the projective space $\mathbb{P}^3$ generated by C is a union of C and one of its secant lines l, so that $C \cap l$ gives a distinguished point in each fiber of ϕ'. ☐

[7]By a rational section we mean a section defined over a dense open subset of the base.

Note, though, that the first fibration does not have a natural section arizing from this geometric construction. However one can impose a section, e.g. by requiring S to contain another rational normal cubic C' intersecting C at two points: then C' induces a section of ϕ and $C^{[2]}$ induces a section of $\phi^{[2]}$. One may remark that there is also an abstract existence result, which follows from the Torelli theorem for K3 surfaces and Nikulin's results on lattice embedding: for any nondegenerate even lattice Λ of signature $(1, \rho - 1)$, $\rho \leq 10$, there exists a K3 surface with Neron-Severi group Λ (see [34]).

Once two fibrations are constructed, the existence of parabolic automorphisms preserving each one can be deduced in the same way as in Proposition 1: indeed the description of the Neron-Severi group and of the MBM classes on $S^{[2]}$ is well-known (the latter are the classes of square -2 and those classes of square -10 which have even pairing with all other classes in $H^2(S^{[2]}, \mathbb{Z})$, see [23] for statements, [5] for an easy proof). We check the existence of a parabolic automorphism preserving ϕ on S, and of a parabolic automorphism preserving ϕ' on $S^{[2]}$. The details are left to the reader.

As a final remark, let us mention that many more examples can be constructed in an "abstract" way, by choosing a convenient lattice Λ of low rank (but at least three), so that there is an IHS manifold of one of the four known deformation types (e.g. deformation equivalent to the Hilbert scheme of a K3 surface) X with Neron-Severi lattice Λ. As the description of the MBM classes is available, by choosing the lattice carefully it is possible to arrange for two Beauville-Bogomolov isotropic nef classes with no, or few, orthogonal MBM classes. Since the Lagrangian conjecture is verified, this gives two lagrangian fibrations ϕ, ϕ', and by Hodge-theoretic Torelli, two groups of parabolic automorphisms P resp. P' preserving each. One then may study the locus of points with finite orbit with respect to the group generated by some $f \in P$ and $f' \in P'$.

Note also that IHS manifolds with two transversal lagrangian fibrations have been constructed in [24]; as the ambient space there has Picard rank two, there are no automorphisms which are interesting for us, but a suitable modification of the construction could certainly yield some. The construction of [24] is entirely based on the Torelli theorem, so it is not explicit.

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E. Amerik, Université Paris-Saclay, Laboratoire de Mathématiques d'Orsay, France and National Research University Higher School of Economics, Laboratory of Algebraic Geometry and its Applications, Russia
E-mail address: ekaterina.amerik@gmail.com

P. Dolce, Westlake University, Institute for theoretical sciences, China
E-mail address: dolce@westlake.edu.cn

F. Tropeano, Università degli Studi Roma Tre, Italy
E-mail address: francesco.tropeano@uniroma3.it