Sheet 4: Bump functions and analysis of extreme points

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The exercises denoted with an asterisk are difficult and can be ignored.

1. (Bump functions). Prove the following statements:

(a) The function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is of class $\mathcal{C}^\infty$ on its domain.

(b) Given any two real numbers $r_1 < r_2$ there exists a function $h \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that:

$$\begin{cases} h(t) = 1 & t \leq r_1 \\ 0 < h(t) < 1 & r_1 < t < r_2 \\ h(t) = 0 & t \geq r_2 \end{cases}$$

(c) Given any two positive real numbers $r_1 < r_2$ there exists a function $H \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ (for any n) such that:

$$\begin{cases} H(x) = 1 & \|x\| \leq r_1 \\ 0 < H(x) < 1 & r_1 < \|x\| < r_2 \\ H(x) = 0 & \|x\| \geq r_2 \end{cases}$$

(Such function H is called a *bump function* with respect to the radii r_1 and r_2)

***2.**(Smooth partition of unity). Let $S \subseteq \mathbb{R}^n$ be a subset and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of S, where I is a set of indices. *A smooth partition of unity for S relative to $\mathcal{U}$* is a collection of functions $\{\phi_i\}_{i \in I}$ with $\phi_i \in \mathcal{C}^\infty(S, \mathbb{R})$ satisfying the following conditions:

(i) $0 \leq \phi_i(x) \leq 1$, for all $i \in I$ and $x \in S$.

(ii) $\text{supp}(\phi_i) \subseteq U_i$ for all $i \in I$[1].

(iii) Any point $x \in S$ has a neighborhood that intersects only finitely many sets of the type $\text{supp}(\phi_i)$.

(iv) $\sum_{i \in I} \phi_i(x) = 1$ for all $x \in S$[2]

Show that a smooth partition of unity exists for any S and any open cover $\mathcal{U}$. *[Hint: Use exercise 1.(c)]*

3. Write explicitly the Euler-Lagrange equations for the stationary points functional defined by

$$f(t) \mapsto \int_a^b \dot{f}(t)(1 + t^2 \dot{f}(t))dt$$

4. Study the extreme points of the following functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_1(x, y) = (y - x^2)(y - 2x^2)$$

$$f_2(x, y) = \sin(x + y) - \cos(x - y)$$

$$f_3(x, y) = x^4 - 2x^2 + (e^x - y)^4$$

[1] Recall that $\text{supp}(\phi_i)$ is the topological closure of the set $\{x \in S \colon \phi_i(x) \neq 0\}$

[2] Note that the summation is actually finite thanks to (iii).