

GENERALIZED INTEGRAL POINTS ON ABELIAN VARIETIES AND THE GEOMETRIC LANG–VOJTA CONJECTURE

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Abstract

Let $f: \mathcal{A} \rightarrow B$ be a family of abelian varieties over a compact Riemann surface B and fix an effective horizontal divisor $\mathcal{D} \subset \mathcal{A}$. We study (S, \mathcal{D}) -integral sections σ of the family \mathcal{A} where $S \subset B$ is arbitrary. These sections σ are algebraic and satisfy the geometric condition $f(\sigma(B) \cap \mathcal{D}) \subset S$. Developing the work of Parshin, we establish new quantitative results concerning the finiteness and the polynomial growth of large unions of (S, \mathcal{D}) -integral sections where S can vary and is required to be finite only in a thin analytic open subset of B . Such results are out of the range of purely algebraic methods and imply new evidence and interesting phenomena to the Geometric Lang–Vojta Conjecture.

1. Introduction

Let \bar{C} be a smooth projective curve defined over an algebraically closed field k of characteristic zero. Let S be a finite set of points of \bar{C} and denote $C = \bar{C} \setminus S$. Consider a smooth affine variety X of log general type over $k(\bar{C})$, that is, X can be embedded into a projective variety \bar{X} over $k(\bar{C})$ such that $\bar{X} \setminus X$ is a divisor of normal crossings and the divisor $\omega_{\bar{X}}(\bar{X} \setminus X)$ is big. Let $f: \mathcal{X} \rightarrow C$ be a model of X . Let \mathcal{D} be the hyperplane at infinity in a compactification $\bar{\mathcal{X}}$ of \mathcal{X} . A weak form of the *Geometric Lang–Vojta Conjecture* (cf. [20, F.3.5], [45, Conjecture 4.4], [26], cf. also [11], [12]) implies the following conjecture.

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Conjecture 1.1 (Lang–Vojta). *There exists a proper closed subset $Z \subset X$ and $m > 0$ with the following property. Let \mathcal{Z} be the Zariski closure of Z in \mathcal{X} . Then for every section $\sigma: C \rightarrow \mathcal{X}$ with $\sigma(C) \not\subset \mathcal{Z}$, let $\bar{\sigma}: \bar{C} \rightarrow \bar{\mathcal{X}}$ be the extension of σ , we have*

$$(1.1) \quad \deg_{\bar{C}} \bar{\sigma}^* \mathcal{D} \leq m(2g(\bar{C}) - 2 + \#\mathcal{S}).$$

In the strong form of the Geometric Lang–Vojta Conjecture, note that m should be independent of the curve \bar{C} . The bound (1.1) is known when \bar{X} is a curve and m is effective when \bar{X} is an elliptic curve (cf. [19, Corollary 8.5]). Several results are also known when $\bar{X} = \mathbb{P}^2$ in all arithmetic, analytic and algebraic settings (e.g. [11], [12], [43]). When X is the complement of an effective ample divisor in an abelian variety \bar{X} , the exceptional set Z can be taken to be empty (by an immediate induction using [46, Lemma 4.2.1]) and Conjecture 1.1 holds when the trace $\mathrm{Tr}_{k(\bar{C})/k}(\bar{X})$ (cf. [8], [9]) of \bar{X} is zero (cf. [6]) or when \bar{X} is defined over $k = \mathbb{C}$ (cf. [30]). Even in these cases, it is particularly difficult to obtain a purely algebraic proof: the proof in [6] uses the Kolchin differential theory while [30] is based on the tool of jet-differentials. No similar results were known for a general abelian variety.

The main goal of the paper is to provide new phenomena and evidence for Conjecture 1.1 when $\bar{\mathcal{X}} \rightarrow \bar{C}$ is a family of abelian varieties over a compact Riemann surface where $\mathcal{S} \subset \bar{C}$ can vary and is not necessarily finite (Theorem A, Theorem 11.1, Corollary 1.3, Corollary 11.2).

We develop the work of Parshin in [32] to formulate a hyperbolic-homotopic method as a substitute for the classical intersection theory. The technique allows us to address quantitative problems concerning (S, \mathcal{D}) -integral points, namely, rational points $P \in X(k(\bar{C}))$ whose induced section $\sigma_P \in \bar{\mathcal{X}}(\bar{C})$ satisfy $f(\sigma_P(\bar{C}) \cap \mathcal{D}) \subset S$. The hyperbolic part (Theorem B), which may be of independent interest, concerns the hyperbolic length of loops on compact Riemann surfaces. We obtain a reasonable estimation of the growth of integral sections of $\bar{\mathcal{X}} \rightarrow \bar{C}$ in the sense that it recovers corresponding results in all special known cases (cf. Section 2.3). Certain results on the topology of the intersection locus of sections with a horizontal divisor in an abelian fibration are also established (Theorem D).

Notation. We fix throughout the paper a compact Riemann surface B of genus g . Let $K = \mathbb{C}(B)$ be the function field of B . We equip B with an arbitrary Riemannian metric d . For every complex space X , the pseudo-Kobayashi hyperbolic metric on X is denoted by the symbol d_X (cf. Definition 7.3). Discs in Riemannian surfaces are assumed to have sufficiently piecewise smooth boundaries. The symbol $\#$ stands for cardinality.

1.1. Finiteness and growth of integral points. In [32, Theorem 3.2], Parshin proved the following general finiteness theorem with an elegant hyperbolic method:

Theorem 1.2 (Parshin). *Let $S \subset B$ be a finite subset. Let $f: \mathcal{A} \rightarrow B$ be a family of abelian varieties and let $\mathcal{D} \subset \mathcal{A}$ be an effective integral divisor which dominates B . Assume that D_K does not contain any translates of nonzero abelian subvarieties of \mathcal{A}_K . Then the set $\{P \in \mathcal{A}_K(K) : f(\sigma_P(B) \cap \mathcal{D}) \subset S\}$ is finite modulo the trace $\text{Tr}_{K/\mathbb{C}}(\mathcal{A}_K)(\mathbb{C})$.*

Note that by the Lang–Néron theorem [25], $\mathcal{A}_K(K)/\text{Tr}_{K/\mathbb{C}}(\mathcal{A}_K)(\mathbb{C})$ is an abelian group of finite rank. We establish the following generalization of Theorem 1.2 (cf. Section 8, see also Theorem 11.1 for an even stronger statement).

Setting (P). Let A be an abelian variety over K . Let $D \subset A$ be a reduced effective ample divisor of A which does not contain any translates of nonzero abelian subvarieties (e.g., when A is simple). Let \mathcal{D} be the Zariski closure of D in a proper flat model $f: \mathcal{A} \rightarrow B$ of A . Let $T \subset B$ be the finite subset consisting of $b \in B$ such that \mathcal{A}_b is not smooth.

Theorem A. *In Setting (P), let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. Then there exists $m > 0$ such that for all $s \in \mathbb{N}$, the set $I_s := \{P \in A(K) : \#f(\sigma_P(B_0) \cap \mathcal{D}) \leq s\}$ satisfies*

$$(1.2) \quad \#I_s \bmod \text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \leq m(s + 1)^{2 \dim A \cdot \text{rank } \pi_1(B_0)}.$$

Here, $\sigma_P: B \rightarrow \mathcal{A}$ is the induced section of $P \in A(K)$ and $\text{rank } \pi_1(B_0)$ denotes the minimal number of generators of the finitely generated group $\pi_1(B_0)$.

Theorem A implies a polynomial bound in $\#S$ on the number of (S, \mathcal{D}) -integral sections proved in [19] by Hindry-Silverman for elliptic curves or by Buium in [6] when $\text{Tr}_{K/\mathbb{C}}(A) = 0$ and in [30] by Noguchi-Winkelmann when A is defined over \mathbb{C} (Section 2.4). When A is an elliptic curve, results in [34] show that Theorem A can be strengthened where \mathcal{D} can also vary in a compact family of divisors on \mathcal{A} in the definition of union I_s .

To the limit of our knowledge, even finiteness results of (S, \mathcal{D}) -integral sections for certain $S \subset B$ countably infinite are not stated elsewhere before in the literature. To obtain such results, establishing a height bound as in traditional approaches, which depends on the cardinality of S , is clearly not sufficient. In the case of number fields, the closest related finiteness result seems to be a result of Silverman [39] (see also [33, Theorem 1.39]) for elliptic curves but it requires some strong restrictions on the set of sections.

1.2. Application to the Geometric Lang–Vojta Conjecture. Theorem A implies a certain extension of Conjecture 1.1 for polarized abelian varieties as follows.

Corollary 1.3. *In Setting (P), let $W \supset T$ be a finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. Then for every $s > 0$, there exists $M = M(\mathcal{A}, \mathcal{D}, B_0, s) > 0$ such that for every section $\sigma: B \rightarrow \mathcal{A}$ with $\#f(\sigma(B_0) \cap \mathcal{D}) \leq s$, one has $\deg_B \sigma^* \mathcal{D} < M$.*

When $\text{Tr}_{K/\mathbb{C}}(A)$ is nonzero and A is nonconstant, the above result was not known even under the hypothesis $W = T$. Note that we obtain in Corollary 11.2 a stronger statement.

Proof. We equip A/K with a symmetric ample line bundle L and consider the corresponding canonical Néron-Tate height function $\widehat{h}_L: A(K)/\text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \rightarrow \mathbb{R}_+$. Since L is ample, there exists $n \in \mathbb{N}^*$ such that the line bundle $L^{\otimes n} \otimes \mathcal{O}(-D)$ is very ample on A . By standard positivity properties of height theory (cf. [20], [10]), there exists a finite number $c > 0$ such that for every $P \in A(K)$ with $\sigma_P \in \mathcal{A}(B)$ the corresponding section, we have:

$$(1.3) \quad n\widehat{h}_L(P) + c > \deg_B \sigma_P^* \mathcal{O}(\mathcal{D}).$$

Let $s > 0$ and let $I_s := \{P \in A(K) : \#f(\sigma_P(B_0) \cap \mathcal{D}) \leq s\}$. As $I_s \bmod \text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is finite by Theorem A and as the canonical height \widehat{h}_L is invariant under translations by the trace, we can define a finite number $H := \max_{P \in I_s} \widehat{h}_L(P)$. For $\sigma \in I_s$, (1.3) implies that:

$$\deg_B \sigma^* \mathcal{D} \leq \sup_{\tau \in I_s} \deg_B \tau^* \mathcal{D} \leq n \max_{P \in I_s} \widehat{h}_L(P) + c \leq nH + c.$$

The conclusion follows by setting $M = nH + c > 0$. □

In particular, the Lefschetz Principle and Corollary 1.3 imply immediately that if we allow m to depend on $\#S$, Conjecture 1.1 is true for $X = \bar{X} \setminus D$ with moreover an empty exceptional set Z , where $\bar{X}/k(\bar{C})$ is an abelian variety and $D \subset \bar{X}$ is any effective ample divisor not containing any translates of nonzero abelian subvarieties of \bar{X} .

1.3. Hyperbolic length on Riemann surfaces. One of the new ingredients in the proof of Theorem A is the following linear bound on the hyperbolic length of loops in various complements of a Riemann surface. Let U be a finite union of disjoint closed discs in the Riemann surface B and denote $B_0 := B \setminus U$. We prove the following estimation (Section 5):

Theorem B. *Let $\alpha \in \pi_1(B_0) \setminus \{1\}$. Then there exists $L > 0$ with the following property. For any finite subset $S \subset B_0$, there exists a piecewise smooth loop $\gamma \subset B_0 \setminus S$ which represents the free homotopy class α in B_0 and satisfies $\text{length}_{d_{B_0 \setminus S}}(\gamma) \leq L(\#S + 1)$.*

Here, $d_{B_0 \setminus S}$ denotes the intrinsic Kobayashi hyperbolic metric on $B_0 \setminus S$ (Definition 7.3). More geometric information on the loop γ is given in Theorem 5.1 and Theorem 5.2. We sketch briefly below the role of Theorem B in the proof of Theorem A.

In Setting (P), let U be a certain finite union of disjoint closed discs in B and let $B_0 := B \setminus U$. For every finite subset $S \subset B_0$, the image of a holomorphic section $\sigma: B_0 \setminus S \rightarrow (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}$ is a totally geodesic subspace with respect to the Kobayashi hyperbolic metrics $d_{B_0 \setminus S}$ and $d_{(\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}}$. In particular, $\text{length}_{d_{(\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}}} \sigma(\gamma) = \text{length}_{d_{B_0 \setminus S}}(\gamma)$ for every loop $\gamma \subset B_0 \setminus S$. By Ehresmann’s theorem, $\mathcal{A}_{B_0} \rightarrow B_0$ is a fiber bundle in the differential category. Let $w_0 \in \mathcal{A}_{B_0}$ and $b_0 = f(w_0) \in B_0$. Every algebraic section $\sigma: B_0 \rightarrow \mathcal{A}_{B_0}$ induces a homotopy section i_σ of the short exact sequence:

$$0 \rightarrow \pi_1(\mathcal{A}_{b_0}, w_0) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{f_*} \pi_1(B_0, b_0) \rightarrow 0.$$

A reduction step of Parshin says that it is enough to bound the number of homotopy sections i_σ in order to bound the number of algebraic sections σ . Fix a system of generators $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$. A theorem of Green says that $(\mathcal{A} \setminus \mathcal{D})_{B_0}$ is hyperbolically embedded in \mathcal{A} . Therefore, the number of homotopy sections will be controlled if we can bound $\text{length}_{d_{(\mathcal{A} \setminus \mathcal{D})|_{B_0}}} \sigma(\gamma_i) = \text{length}_{d_{B_0}}(\gamma_i)$ for some representative loop γ_i of each α_i .

An $(S \cup U, \mathcal{D})$ -integral section $\sigma: B \rightarrow \mathcal{A}$ does not induce a section $B_0 \rightarrow \mathcal{A}_{B_0}$ in general but only induces a section $B_0 \setminus S \rightarrow (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}$. However, as $d_{B_0 \setminus S} \geq d_{B_0}|_{B_0 \setminus S}$, it suffices to bound $\text{length}_{d_{B_0 \setminus S}}(\gamma_i)$ for some loop $\gamma_i \subset B_0 \setminus S$ representing α_i for $i = 1, \dots, k$. Therefore, Theorem B plays a crucial role for the quantitative estimation of I_s in Theorem A.

We investigate some further related aspects of Theorem B. For each free homotopy class $\alpha \in \pi_1(B_0)$ and each $s \in \mathbb{N}$, we can associate a constant

$$(1.4) \quad L(\alpha, s) := \sup_{\#\mathcal{S} \leq s} \inf_{[\gamma] = \alpha} \text{length}_{d_{B_0 \setminus \mathcal{S}}}(\gamma) \in \mathbb{R}_+,$$

where S runs over all subsets of B_0 of cardinality at most s and γ runs over all loops in $B_0 \setminus S$ which represent α . Theorem B asserts that $L(\alpha, s)$ grows at most linearly in s . Moreover, we prove the following optimal lower bound (cf. Section 6, Remark 6.3 for the optimality):

Theorem C. *Given $\alpha \in \pi_1(B_0) \setminus \{1\}$, there exists $c > 0$ such that for every $s \in \mathbb{N}$:*

$$(1.5) \quad L(\alpha, s) \geq \frac{cs^{1/2}}{\ln(s+2)}.$$

It would be interesting to understand the asymptotic behavior of $L(\alpha, s)$ in terms of s :

Question 1.4. What are the limits:

$$(1.6) \quad \text{deg}^-(\alpha) := \liminf_{s \rightarrow \infty} \frac{\ln L(\alpha, s)}{\ln s}, \quad \text{deg}^+(\alpha) := \limsup_{s \rightarrow \infty} \frac{\ln L(\alpha, s)}{\ln s},$$

which correspond to the lower and upper polynomial growth degrees of $L(\alpha, s)$ in terms of s ?

By Theorem B and Theorem C, we know that for every $\alpha \in \pi_1(B_0) \setminus \{1\}$, we have:

$$(1.7) \quad 1/2 \leq \text{deg}^-(\alpha) \leq \text{deg}^+(\alpha) \leq 1.$$

If we require α to belong to a certain base of $\pi_1(B_0)$ (cf. Section 3.1), the constant $L > 0$ in Theorem B depends only on U and the Riemann surface B (cf. Theorem 5.1).

2. Further applications and remarks

2.1. Intersection locus of sections with divisors. In Setting (P), assume in addition that $\text{Tr}_{K/\mathbb{C}}(A) = 0$. For every subset $R \subset A(K) \setminus D$, we define the intersection locus:

$$I(R, \mathcal{D}) := \cup_{P \in R} f(\sigma_P(B) \cap \mathcal{D}) \subset B.$$

We prove the following result which concerns the topology of the intersection of sections of an abelian scheme with a horizontal divisor (cf. Section 10).

Theorem D. *Assume that R is infinite. Then the following properties hold:*

- (i) $I(R, \mathcal{D})$ is countably infinite but it is not analytically closed in B ;
- (ii) the set $I(R, \mathcal{D})_\infty$ of limit points of $I(R, \mathcal{D})$ in B is uncountable;
- (iii) $I(R, \mathcal{D})_\infty$ is not contained in any union $W \supset T$ of disjoint closed discs in B such that distinct points of T are contained in distinct discs.

Here are some motivations for Theorem D. Assume that C is a smooth projective curve defined over a number field k . Let $F = k(C)$ and let $\Phi: \mathcal{E} \rightarrow C$ be a nonisotrivial Jacobian elliptic surface. Assume that \mathcal{D} is the zero section of \mathcal{E} and $R = \{nQ: n \in \mathbb{N}^*\} \subset \mathcal{E}_F(F)$ for some nontorsion point $Q \in \mathcal{E}_F(F)$. Let $I(R, \mathcal{D}) := \cup_{P \in R} \Phi(\sigma_P(C(\bar{k}) \cap \mathcal{D})) \subset C(\bar{k})$ where $\sigma_P \in \mathcal{E}(C)$ denotes the induced section of $P \in \mathcal{E}_F(F)$. Then by [47, Notes to chapter 3], $I(R, \mathcal{D})$ is analytically dense in $C(\mathbb{C})$.

Theorem D thus provides some evidence that analogous density results on the intersection locus could be true in higher dimensional abelian varieties over function fields. In fact, recent results in [13] imply that $I(R, \mathcal{D})$ is even equidistributed in $C(\mathbb{C})$ with respect to a certain Galois-invariant measure. Another supporting result is recently given in [35, Corollary C] in the same context of Theorem D but without the hypothesis $\text{Tr}_{K/\mathbb{C}}(A) = 0$.

2.2. Application to the generic emptiness of integral points. As the space $B^{(s)}$, $s \geq 1$, of subsets $S \subset B$ of cardinality at most s has positive dimension, the finiteness of the union I_s of (S, \mathcal{D}) -integral sections (Theorem A) implies that for a general choice of such S , there are very few (S, \mathcal{D}) -integral points. We can show an even stronger property (cf. Section 9):

Corollary A. *Let the notation be as in Theorem A. Assume $\text{Tr}_{K/\mathbb{C}}(A) = 0$ and \mathcal{D} horizontally strictly nef, i.e., $\mathcal{D} \cdot C > 0$ for all curves $C \subset \mathcal{A}$ not contained in a fiber. Then for $s \in \mathbb{N}$:*

- (i) *there exists a finite subset $E \subset B$ such that for any $S \subset B \setminus E$ with $\#(S \cap B_0) \leq s$, the set of (S, \mathcal{D}) -integral sections is empty. We can choose E such that*

$$(2.1) \quad \#E \cap B_0 \leq ms(s + 1)^{2 \dim A \cdot \text{rank } \pi_1(B_0)},$$

- (ii) *there exists a Zariski proper closed subset $\Delta \subset B^{(s)}$ such that for any $S \subset B$ of cardinality s whose image $[S] \in B^{(s)} \setminus \Delta$, there are no (S, \mathcal{D}) -integral sections.*

The subsets $S \subset B$ satisfying Corollary A(i) can be taken as $S = (B \setminus (B_0 \cup E)) \cup N = (U \setminus E) \cup N$ where $N \subset B_0 \setminus E$ is any finite subset of cardinality at most s . Since B_0 can be taken arbitrarily thin (cf. Theorem A), such subsets S are very large. In this sense, we find that Corollary A(i) is quite surprising.

2.3. Growth degree of the set of integral points in Theorem A. We explain below that the exponent $2 \dim A \text{rank } \pi_1(B_0)$ in the bound (1.2) is as reasonable, up to a factor of $1/2$, as we can possibly expect. In Setting (P), let $t = \#T$ and let r be the Mordell-Weil rank of A . Suppose that W is a disjoint union of t closed discs centered at the points of T so that $\text{rank } \pi_1(B_0) = 2g - 1 + t$. For $s \in \mathbb{N}$, let $J_s := \{P \in A(K): \#f(\sigma_P(B) \cap \mathcal{D}_z) \leq s\} \subset I_s$.

Assume first that A is a nonisotrivial elliptic curve so that its trace is zero. Note that $\frac{1}{2} \deg f_{A/K} \leq t \leq \deg f_{A/K}$ where $f_{A/K}$ is the conductor divisor of A over K (cf. [40, Ex. III.3.36]). Shioda's result in [38, Theorem 2.5] provides a very general bound $r \leq 2(2g - 2 + t)$. Height theory [19, Corollary 8.5] tells us that $\#J_s$ is bounded by a polynomial in s of degree

$$\frac{r}{2} \leq (2g - 1 + t) = \frac{1}{2}(2 \operatorname{rank} \pi_1(B_0)).$$

Therefore, up to a factor of $1/2$, Theorem A implies the known polynomial growth for the set $J_s \subset I_s$. More generally, when A is a traceless abelian variety, the Ogg-Shafarevich formula (cf. [37], [31], see also [18]) implies that $r \leq 2 \dim A \cdot (2g - 2 + t)$. In this case, best results using height theory (due to Buium [6], see Section 2.4) tell us that $\#J_s$ is bounded by a polynomial in s of degree $\frac{r}{2} \leq \dim A \cdot (2g - 2 + t)$. As $\operatorname{vol}_d(W)$ can be arbitrarily closed to $\operatorname{vol}_d(B)$, our union I_s is *a priori* much larger than J_s . However, Theorem A assures that the polynomial growth degree of $\#I_s$ is still at most $O(\dim A \cdot \operatorname{rank} \pi_1(B_0))$ just as J_s .

Suppose that \mathcal{A} is a constant family of simple abelian varieties. Then $T = \emptyset$ and Noguchi-Winkelmann's results (cf. [30], see Section 2.4) for a constant ample divisor $\mathcal{D} = D \times B$ imply that modulo $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, the cardinality of J_s is bounded by a polynomial in s of degree $\frac{r}{2} \leq 2g \dim A$ (cf. [34, Theorem J] for more general \mathcal{D}). Theorem A thus also improves known upper bounds on the growth degree to the much larger set $I_s \supset J_s$.

2.4. On the Geometric Lang–Vojta Conjecture. Keep the notation as in Theorem A. By a counting argument on the lattice $\Gamma = A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, a bound on the canonical height of (S, \mathcal{D}) -integral sections, if it is a polynomial p in $\#S$, would imply a polynomial bound of the form $(cp(\#S)^{1/2} + 1)^{\operatorname{rank} \Gamma}$ on the number of such integral sections (modulo the trace).

Conversely, to obtain with height theory a polynomial bound in function of $\#S$ on the set of (S, \mathcal{D}) -integral sections with $S \subset B$ finite, it is necessary to establish a uniform bound on the intersection multiplicities (as predicted by Conjecture 1.1). Such uniform bounds are only available when $\operatorname{Tr}_{K/\mathbb{C}}(A) = 0$ ([6], or [19] for elliptic curves) or when the family $\mathcal{A} \rightarrow B$ is trivial ([30] for \mathcal{D} constant, or [35] for general \mathcal{D}).

2.5. Other remarks. Theorem B and Theorem C are in a sense *orthogonal* to various remarkable results on the growth of the counting functions $c_X(L)$, $s_X(L)$ for the number of certain type of closed geodesics of length $\leq L$ on a complete hyperbolic bordered Riemann surface X of finite area (cf., for example, [27], [2], [21], [29]).

Let X be a complete hyperbolic Riemann surface of genus g with n cusps. A *partition* on X is a set of $3g - 3 + n$ pairwise disjoint simple closed geodesics. These curves *do not* generate the free homotopy group $\pi_1(X)$. Then Bers' theorem (cf. [7, Theorem 5.2.6]) asserts that X admits a partition with geodesics of hyperbolic length bounded by $13(3g - 3 + n)$. Hence, Bers' theorem applies to surfaces of *finite area*. On the other hand, punctured Riemann surfaces $B_0 \setminus S$ (as in Theorem B, Theorem C) have *infinite area* (whenever U is not finite) and look like the punctured Poincaré disc locally around the punctured points. Therefore, Bers' theorems do not apply to the punctured Riemann surfaces $B_0 \setminus S$ which are equipped with the intrinsic hyperbolic metric $d_{B_0 \setminus S}$ (cf. Definition 7.3). Moreover, our proofs of Theorem B and of Theorem C work with the very definition of the pseudo-Kobayashi hyperbolic metric and do not require any tools from hyperbolic trigonometry.

3. Preliminaries for Theorem A and Theorem B

Here are some main technical difficulties that we must tackle in the proof of Theorem B:

- (1) when S *varies*, the hyperbolic metric on $B_0 \setminus S$ is not the same nor induced by a *single* given metric on B_0 ; note that the analysis of the hyperbolic metric on the punctured complex plane $\mathbb{C} \setminus \{a_1, \dots, a_s\}$ ($s \geq 2$) is a nontrivial research area (cf., e.g. [28], [5], [42]);
- (2) the base point b_0 should not be fixed since otherwise, an accumulation of many points of the set S near b_0 would increase to infinity the hyperbolic length of loops based at b_0 ;
- (3) a certain construction on loops needs to be carried out to obtain a bound which is *linear* in $\#S$ but independent of the choice of S in B_0 . One should consider only certain classes of “nice” loops and avoid pathological loops such as Peano curves.

To deal with the last point, we shall describe a detailed algorithm on simple loops called Procedure (*) given in Lemma 3.7 (see also Lemma 4.1 for the global case).

3.1. Simple base of loops. Let X be an orientable connected compact surface possibly with boundary. A loop $\gamma: [0, 1] \rightarrow X$ is called *simple* if it is non-nullhomotopic and injective on $[0, 1[$. Then $\pi_1(X, x_0)$, for every $x_0 \in X$, admits a canonical system of generators $\alpha_1, \dots, \alpha_k$ such that each α_i is represented by a simple piecewise smooth closed loop $\gamma_i: [0, 1] \rightarrow X \setminus \partial X$ based at x_0 . Every such system of generators $\alpha_1, \dots, \alpha_k$ is called a *simple base* of $\pi_1(X, x_0)$.

Remark 3.1. Given a simple piecewise smooth loop γ in a compact Riemannian surface (X, d) with $\gamma \cap \partial X = \emptyset$. Since γ only has a finite number of singular points, for every $x \in \gamma$ and $\varepsilon > 0$ smaller than the injectivity radius of X at x , we can find a contractible closed region $\Delta \subset V(x, \varepsilon)$ where $V(x, \varepsilon)$ is the closed disc of d -radius ε , such that $x \in \Delta$ and

- (a) $\partial\Delta$ is a non-self-intersecting smooth loop homeomorphic to a circle in $V(x, \varepsilon)$;
- (b) $\gamma \cap \Delta$ contains only one connected branch of γ .

Definition 3.2. Let B be a compact Riemannian surface equipped with a Riemannian metric d . Let U be finite union of disjoint closed discs in B and $b_0 \in B_0 := B \setminus U$. Fix a simple base $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$ and a collection $\{c_{b_0b}\}$ consisting of bounded d -length and smooth directed paths contained in B_0 such that c_{b_0b} goes from b_0 to $b \in B_0$ for each $b \in B_0$. A loop γ_i represents a class $\alpha_i \in \pi_1(B_0, b_0)$ up to a single conjugation (with respect to the fixed collection of paths $\{c_{b_0b}\}$) if γ_i represents the conjugation of the class $\alpha_i \in \pi_1(B_0, b_0)$ by the change of base points from b to b_0 using the specific chosen path c_{b_0b} , i.e., $[c_{b_0b}^{-1} \circ \gamma_i \circ c_{b_0b}] = \alpha_i \in \pi_1(B_0, b_0)$.

3.2. Preliminary lemmata. We denote by $\Delta(x, r) \subset \mathbb{C}$ the open complex disc centered at a point $x \in \mathbb{C}$ and of radius $r > 0$. For a complex space X , the infinitesimal Kobayashi–Royden pseudo-metric λ_X on X corresponding to the pseudo-Kobayashi hyperbolic metric d_X can be defined as follows. For $x \in X$ and every vector v in the tangent cone of X at x , we define

$$(3.1) \quad \lambda_X(x, v) := \inf \frac{2}{R},$$

where the minimum is taken over all $R > 0$ for which there exists a holomorphic map $f: \Delta(0, R) \rightarrow X$ such that $f'(0) = v$. Note that when $x \in X$ is regular, the tangent cone of X at x is the same as the tangent space $T_x X$. We begin with the following simple estimation: fix a Riemannian metric d on a compact Riemann surface B . For every $z \in B$ and $r > 0$, let $D(z, r) := \{b \in B: d(b, z) < r\}$. Define the d -unit tangent space of B by $T_1 B := \{(z, v) \in TB: |v|_d = 1\}$. By a direct argument using the compactness of B , we have:

Lemma 3.3. *There exist $c(B, d), r(B, d) > 0$ such that for every $(z, v) \in T_1 B$ and every $0 < r < r(B, d)$, we have $\lambda_{D(z,r)}(z, v) \leq \frac{c(B,d)}{r}$.*

Let the constants $c(B, d), r(B, d) > 0$ be as in Lemma 3.3. For every subset $\Omega \subset B$ and every $r > 0$, we define $D(\Omega, r) := \{b \in B : d(b, \Omega) < r\} \subset B$. Using the distance-decreasing property of the pseudo-Kobayashi hyperbolic metric (cf. Lemma 7.4), we obtain:

Corollary 3.4. *Let $\gamma \subset B$ be a piecewise smooth closed curve. Then for $0 < r < r(B, d)$:*

$$\text{length}_{d_{D(\gamma, r)}}(\gamma) \leq \frac{c(B, d)}{r} \text{length}_d(\gamma).$$

Lemma 3.5. *Let (M, d) be a compact Riemannian surface possibly with boundary. Then there exist constants $r_0 = r_0(M, d) > 0, c_0 = c_0(M, d) > 0$ such that for every disc $D(x, r)$ of d -radius $r \leq r_0$ with $x \in M$, one has $\text{length}_d \partial D(x, r) \leq c_0 r$ and $\text{vol}_d(D(x, r)) \leq c_0 r^2$.*

Proof. It follows from the compactness of M and Bertrand-Diguët-Puiseux’s theorem [41]. □

3.3. Procedure (*) for simple loops. Given a subset Ω of a metric space (X, d) and let $r \geq 0$, we recall the notation $V(\Omega, r) = \{x \in X : d(x, \Omega) \leq r\}$ and $D(\Omega, r) = \{x \in X : d(x, \Omega) < r\}$. Let (B, d) be a compact Riemannian surface with boundary. Let $\gamma \subset B \setminus \partial B$ be *simple* piecewise smooth loop based at $b_0 \in B \setminus \partial B$. Denote by $\text{rad}(\gamma, d)$ the infimum of the injectivity radii in $(B \setminus \partial B, d)$ at all points $x \in \gamma$. Since γ is compact, $\text{rad}(\gamma, d) > 0$ and

$$L = L(B, d, \gamma) := \min(d(\gamma, \partial B), \text{rad}(\gamma, d)) > 0.$$

We shall consider $a > 0$ small enough (depending only on γ, B, d) such that:

- (P1) $4a < L(B, d, \gamma)$.
- (P2) every point $x \in \gamma$ admits a simply connected open neighborhood U_x in $B \setminus \partial B$ such that $V(x, 2a) \subset U_x$ and that $\gamma \cap U_x$ contains exactly one connected branch of γ .

By a direct compactness argument, it is not hard to see that:

Lemma 3.6. *There exists a $a > 0$ depending only on γ, B, d which satisfies (P1) and (P2).*

Lemma 3.7 (Procedure (*)). *Suppose that $a > 0$ satisfies (P1) and (P2). Then for every finite subset $S \subset B$ of cardinality $s > 0$ such that $d(b_0, S) = \min_{x \in S} d(b, x) > a/s$, there exists a piecewise smooth loop $\gamma' \subset B \setminus D(S \cup \partial B, a/s)$ based at b_0 of the same homotopy class in $\pi_1(B, b_0)$ as γ and such that $\text{length}_d(\gamma') \leq \text{length}_d(\gamma) + c_0(B, d)a$.*

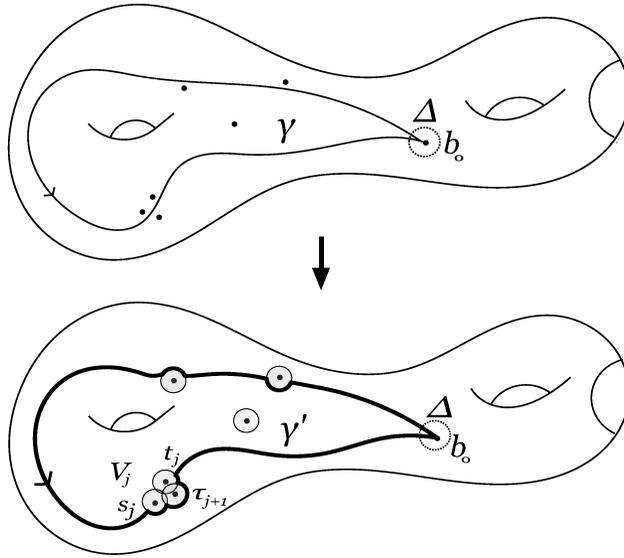


FIGURE 1. Procedure (*) applied to γ (with $\Delta = V(b_0, a/s)$)

Proof. We decompose $V(S, a/s) = \cup_{x \in S} V(x, a/s)$ as a disjoint union of connected components V_1, \dots, V_m . Then each V_j is path connected. Let $n_j = \#S \cap V_j$, then:

$$(3.2) \quad n_1 + \dots + n_m \leq s = \#S.$$

By the triangle inequality, $\text{diam}_d(V_j) \leq n_j 2a/s \leq s \times 2a/s = 2a$. Consider the following m -step algorithm. Define $\gamma_0 := \gamma$. Given a curve $\gamma_j \subset B$ where $j \in \{0, \dots, m - 1\}$. Denote by $s_j, t_j \in \gamma_j$ the first and the last points, if they exist, on the intersection $\gamma_j \cap V_{j+1}$. If there are no such points, we set $\gamma_{j+1} = \gamma_j$ and continue the algorithm. Otherwise, we replace the directed part of γ_j between s_j and t_j by any directed simple curve τ_{j+1} lying on the boundary of V_j which connects s_j and t_j (cf. Figure 1). This is possible since V_j is path connected. Define γ_{j+1} the resulting curve and continue until we reach γ_m .

As $\min_{x \in S} d(b, x) > a/s$ by hypothesis, $b \notin V_j$ for every j and thus the base point $b_0 \in \gamma$ is not modified at any step. The loops $\gamma_1, \dots, \gamma_m$ based at b_0 are piecewise smooth. Moreover,

$$(3.3) \quad \text{length}_d(\gamma_{j+1}) \leq \text{length}_d(\gamma_j) + \text{length}_d(\tau_{j+1}), \quad j = 0, \dots, m - 1.$$

As V_1, \dots, V_m are disjoint, an induction shows that $s_j, t_j \in \gamma \cap \gamma_j$ and each of the curves τ_1, \dots, τ_j (when defined) is either nonmodified or does not appear in γ_{j+1} at the j -th step for every $0 \leq j \leq m - 1$. Hence, γ_m contains some pairwise disjoint curves $\tau_{i_1}, \dots, \tau_{i_k}$ and γ_m coincides with γ outside the directed paths $\sigma_{i_p} \subset \gamma$ between s_{i_p-1} and t_{i_p-1} where $p = 1, \dots, k$. It follows that there exists a homotopy in B with base point b_0 between γ_m and γ .

Since $\text{diam}V(S, a/s) \leq s \times 2a/s = 2a$ and $4a < d(\gamma, \partial B)$ by (P1), one has $d(\gamma_m, \partial B) > 2a$. Thus $d(\gamma_m, S \cup \partial B) \geq a/s$ as $s \geq 1$. Hence, $\gamma' = \gamma_m$ is a piecewise smooth loop based at b_0 homotopic to γ and $\gamma' \subset B \setminus D(S \cup \partial B, a/s)$. Let $c_0 = c_0(B, d) > 0$ be as in Lemma 3.5 then:

$$(3.4) \quad \text{length}_d(\tau_j) \leq \sum_{x \in S \cap V_j} \text{length}_d \partial V(x, a/s) \leq n_j \times c_0 a/s.$$

It follows from (3.2), (3.3), and (3.4) that

$$\begin{aligned} \text{length}_d(\gamma') &\leq \text{length}_d(\gamma) + \sum_j \text{length}_d(\tau_j) \\ &\leq \text{length}_d(\gamma) + \sum_j n_j \times c_0 a/s \leq \text{length}_d(\gamma) + c_0 a. \end{aligned}$$

The proof of the lemma is complete. □

Let $U \subset B \setminus \partial B$ be a simply connected open neighborhood of b_0 such that $\gamma \cap U$ contains exactly one connected branch of γ . Let $\Delta \subset B \setminus \partial B$ be a closed subset such that:

- (a) $b_0 \subset \Delta \subset U$;
- (b) for every $x \in \Delta$, there exists a simple loop $\gamma_x \subset B$ based at x such that γ_x and γ coincide as paths over $B \setminus \Delta$.

Lemma 3.8. *There exists $A > 0$ such that for every $x \in \Delta$, the constant A satisfies (P1) and (P2) for the loop γ_x in the Riemannian surface (B, d) .*

Proof. As $\Gamma = \gamma \cup \Delta \subset B \setminus \partial B$ is a compact subset, $L := \min(d(\Gamma, \partial B), \text{rad}(\Gamma, d)) > 0$ where $\text{rad}(\Gamma, d)$ is the infimum of the injectivity radii in $(B \setminus \partial B, d)$ at all points $x \in \Gamma$. For every $x \in \Delta$, we have $\gamma_x \subset \Gamma$ by Condition (b) above and thus $L(B, d, \gamma_x) \geq L$. Suppose on the contrary that for every $n \geq 5$, there exist $x_n \in \Delta$ and $z_n \in \gamma_x$ such that for all simply connected open neighborhood U_n of x_n in B with $V(z_n, 2L/n) \subset U_n$, the restriction $\gamma_x \cap U_n$ has more than two connected components. By the compactness of Γ , we can suppose, up to passing to a subsequence, that $z_n \rightarrow z$ as $n \rightarrow \infty$ for some $z \in \Gamma$. We distinguish two cases. First, assume that $z \in U$. Then for $n \gg 1$, we have $z_n \in U$ and $V(z_n, 2L/n) \subset U$. This is a contradiction since $U \subset B \setminus \partial B$ is simply connected by hypothesis and $\gamma_x \cap U$ has only one connected branch of γ_x by Condition (b). If $z \in B \setminus U$ then we also find a

contradiction since γ_x and γ coincide as paths over $B \setminus \Delta$ and Δ is closed and contained in U . □

4. Riemannian lengths of special loops with varying base points

We describe a global version of Procedure (*) given in Lemma 3.7. Let (B, d) be a compact Riemannian surface. Let V be a finite union of disjoint closed discs in B . Let $U = V \setminus \partial V$. Let $b_0 \in B_0 := B \setminus U$ and fix a base of simple generators $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$ (Section 3.1).

Lemma 4.1. *Let $\varepsilon > 0$. Then there exist constants $a, H > 0$ with the following property. For every $x \in B_0$ such that $d(x, \partial B_0) \geq \varepsilon$, there exist simple piecewise smooth loops $\gamma_1, \dots, \gamma_k \subset B_0$ based at x representing $\alpha_1, \dots, \alpha_k$ up to a single conjugation (Definition 3.2) such that a verifies (P1) and (P2) for the data (γ_i, B_0, d) and $\text{length}_d(\gamma_i) \leq H$ for every $i \in \{1, \dots, k\}$.*

Proof. For each $b \in B_0$ with $d(b, \partial B_0) \geq \varepsilon$, we can choose simple piecewise smooth loops $\gamma_{b,1}, \dots, \gamma_{b,k} \subset B_0 \setminus \partial B_0$ based at b representing the classes $\alpha_1, \dots, \alpha_k$ respectively up to a single conjugation. Let $H_b := \max_{1 \leq i \leq k} (\text{length}_d \gamma_{b,i}) > 0$.

By Lemma 3.6, there exists $a_b > 0$ satisfying (P1) and (P2) for $(\gamma_{b,i}, B_0, d)$ for all i . Thus, for each $i \in \{1, \dots, k\}$, b admits a simply connected neighborhood $U_{b,i} \subset B_0 \setminus \partial B_0$ such that $V(b, 2a_b) \subset U_{b,i}$ and $\gamma \cap U_{b,i}$ contains only one connected branch of $\gamma_{b,i}$ (cf. Remark 3.1).

Consider a small enough closed region $\Delta_b \subset V(b, a_b) \subset B_0 \setminus \partial B_0$ containing b such that $\partial \Delta_b$ is a non-self-intersecting smooth loop homeomorphic to a circle in $V(b, a_b)$ and $\gamma_{b,i} \cap \Delta_b$ contains only one connected branch of γ for every $i = 1, \dots, k$. Let $l_b := \text{length}_d(\partial \Delta_b) > 0$.

Since $B' = \{b \in B_0 : d(x, \partial B_0) \geq \varepsilon\}$ is compact, there exists a finite subset $I \subset B'$ such that $B' \subset \cup_{b \in I} \Delta_b$.

Consider the following construction for every $x \in B'$. We can choose $b \in I$ such that $x \in \Delta_b$. For each $i \in \{1, \dots, k\}$, let $s_{b,i}, t_{b,i} \in \gamma_{b,i} \cap \Delta_b$ be respectively the first and the last intersections of $\gamma_{b,i}$ with the boundary $\partial \Delta_b$. Note that $s_{b,i} \neq t_{b,i}$. Let $\sigma \subset \Delta_b$ be any maximal geodesic segment passing through x . Let $s, t \in \sigma \cap \partial \Delta_b$ be the two extremal points of σ such that s and $s_{b,i}$ do not lie on distinct arcs delimited by $t_{b,i}$ and t on $\partial \Delta_b$ (cf. Figure 2). The two directed geodesic segments δ_s from x to s and δ_t from t to x do not intersect except at x . Replace $\gamma_{b,i} \cap \Delta_b$ by the union of the directed arc $T_i \subset \partial \Delta_b$ from $t_{b,i}$ to t not containing $s_{b,i}$, with the paths δ_t, δ_s and the directed arc $S_i \subset \partial \Delta_b$ from s to $s_{b,i}$ not containing $t_{b,i}$. Denote the resulting loop with base point x by $\gamma_{x,i}$.

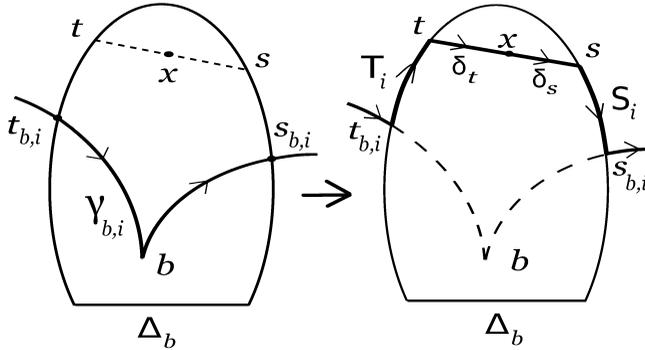


FIGURE 2. Local modification of $\gamma_{b,i} \cap \Delta_b$

By construction, the loop $\gamma_{x,i}$ is simple and piecewise smooth. Moreover, the restrictions of $\gamma_{b,i}$ and $\gamma_{x,i}$ to $B_0 \setminus \Delta_b$ coincide. As $\Delta_b \subset U_{b,i}$ and $U_{b,i}$ is simply connected and contains only one branch of $\gamma_{b,i}$, the loops $\gamma_{b,i}$ and $\gamma_{x,i}$ are homotopic up to a conjugation induced by the change of base points. By setting $H := \max_{b \in I} (H_b + l_b + 2a_b) > 0$, we find that:

$$\begin{aligned} \text{length}_d(\gamma_{x,i}) &= \text{length}_d(\gamma_{x,i}|_{B \setminus \Delta_b}) \\ &\quad + \text{length}_d(T_i) + \text{length}_d(S_i) + \text{length}_d(\delta_t) + \text{length}_d(\delta_s) \\ &\leq \text{length}_d(\gamma_{b,i}) + \text{length}_d(\partial\Delta_b) + \text{length}_d(\sigma) \\ &\leq H_b + l_b + 2a_b \leq H \qquad \qquad \qquad (\text{as } \sigma \subset \Delta_b \subset V(b, a_b)). \end{aligned}$$

By construction, $\Delta_b \subset V(b, a_b) \subset U_{b,i}$ for every $b \in I$ and $i \in \{1, \dots, k\}$. Therefore, Lemma 3.8 applied to $\gamma_{b,i}$, b , U_b , and Δ_b implies that there exists $A_{b,i} > 0$ such that for every $x \in \Delta_b$, the constant $A_{b,i}$ satisfies Properties (P1) and (P2) for the loop $\gamma_{x,i}$ in the Riemannian surface (B_0, d) . As I is finite, we can define $a := \min_{b \in I} \min_{1 \leq i \leq k} A_{b,i} > 0$.

It is clear that $H, a > 0$ are independent of $x \in B'$ and verify the desired properties. □

5. Proof of Theorem B

Let U be a finite union of disjoint closed discs in a compact Riemann surface B . Let $b_1 \in B_1 := B \setminus U$. Fix a simple base $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_1, b_1)$ (see Section 3.1). Let d be a Riemannian metric on B . We prove the following slightly stronger version of Theorem B.

Theorem 5.1. *Let the hypotheses be as above. Then there exist $A > 0$ and $L > 0$ satisfying the following properties. For any finite subset $S \subset B_1$, there exist $b \in B_1 \setminus S$ and piecewise smooth loops $\gamma_i \subset B_1 \setminus S$ based at b representing α_i up to a single conjugation and such that $V(\gamma_i, A(\#S)^{-1}) \subset B_1 \setminus S$ and $\text{length}_{d_{B_1 \setminus S}}(\gamma_i) \leq L(\#S + 1)$ for $i = 1, \dots, k$.*

Proof. Fix $\varepsilon > 0$ small enough so that $\text{vol}_d(B_\varepsilon) > 0$ where $B_\varepsilon = \{b \in B_0 : d(x, \partial B_0) \geq \varepsilon\}$. Let $a, H > 0$ be the constants given by Lemma 4.1 applied to (B_0, d) , to the constant $\varepsilon > 0$ and to the base $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_1) = \pi_1(B_1, b_1)$. Let $c_0 = c_0(B_0, d)$, $r_0 = r_0(B_0, d)$, $c = c(B, d) > 0$, and $r = r(B, d) > 0$ be the constants given by Lemma 3.5 and Lemma 3.3 applied to (B_0, d) and (B, d) respectively. Let us define:

$$(5.1) \quad A := \frac{1}{2} \min \left(a, r, r_0, \sqrt{\frac{\text{vol}_d(B_\varepsilon)}{4c_0}} \right) > 0, \quad L := c \left(\frac{H}{A} + c_0 \right) > 0.$$

Let $S \subset B_0$ be a finite subset of cardinality $s \geq 1$. We find that:

$$\begin{aligned} \text{vol}_d V(S, 2A/s) &\leq \sum_{x \in S} \text{vol}_d(V(x, 2A/s)) && \text{(as } V(S, 2A/s) \cup_{x \in S} V(x, 2A/s)) \\ &\leq s \times c_0 \times (2A/s)^2 = 4c_0 A^2/s && \text{(by Lemma 3.5 and } 2A/s \\ &&& \leq r_0 \text{ by (5.1))} \\ &\leq 4c_0 A^2 \leq \frac{\text{vol}_d(B_\varepsilon)}{4} < \text{vol}_d(B_\varepsilon) && \text{(since } s \geq 1 \text{ and by (5.1)).} \end{aligned}$$

It follows that there exists $b \in B_\varepsilon \subset B_1$ such that $b \cap V(S, 2a/s) = \emptyset$, i.e., $d(b, S) \geq 2a/s$. Therefore, Lemma 4.1 implies that there exist simple piecewise smooth loops $\sigma_1, \dots, \sigma_k \subset B_1$ based at b representing $\alpha_1, \dots, \alpha_k$ respectively up to a single conjugation with:

$$(5.2) \quad \text{length}_d(\sigma_i) \leq H, \quad \text{for every } i \in \{1, \dots, k\}.$$

Moreover, as $0 < A < a$ by (5.1), we infer from Lemma 4.1 that A verifies the properties (P1) and (P2) for the data (σ_i, B_0, d) for every $i \in \{1, \dots, k\}$.

Since $d(b, S) = \min_{x \in S} d(b, x) \geq 2a/s > A/s$ by (5.1), we can apply Lemma 3.7 to the loops $\sigma_1, \dots, \sigma_k$ to obtain piecewise smooth loops $\gamma_1, \dots, \gamma_k \subset B_0 \setminus V(S \cup \partial B_0, A/s)$ based at b which are of the same homotopy classes as $\sigma_1, \dots, \sigma_k$ in $\pi_1(B_0, b_1)$ respectively and satisfy:

$$(5.3) \quad \text{length}_d(\gamma_i) \leq \text{length}_d(\sigma_i) + c_0A, \quad i = 1, \dots, k.$$

In particular, $V(\gamma_i, A/s) \subset B_1 \setminus S$ for every i . Consequently, we find that:

$$\begin{aligned} \text{length}_{d_{B_1 \setminus S}}(\gamma_i) &\leq \text{length}_{d_{V(\gamma_i, A/s)}}(\gamma_i) && \text{(by Lemma 7.4 as } D(\gamma_i, A/s) \\ &&& \subset B_1 \setminus S) \\ &\leq \frac{c(B, d)}{A/s} \text{length}_d(\gamma_i) && \text{(by Corollary 3.4 and } A/s \leq A \\ &&& < r \text{ by (5.1))} \\ &\leq \frac{cs}{A}(\text{length}_d(\sigma_i) + c_0A) && \text{(by (5.3))} \\ &\leq \frac{cs}{A}(H + c_0A) = Ls && \text{(by (5.2) and (5.1)).} \end{aligned}$$

The conclusion follows since by construction, the loops $\gamma_1, \dots, \gamma_k \subset B_1 \setminus S$ represent the homotopy classes $\alpha_1, \dots, \alpha_k$ respectively up to a single conjugation. □

A slight modification of the above proof allows us to show the following generalization of Theorem B and Theorem 5.1 by allowing certain bounded moving discs besides S :

Theorem 5.2. *Let the hypotheses be as in Theorem 5.1 and let $p \geq 1$. Then there exist $L, R > 0$ with the following property. For every finite subset $S \subset B_0$ and every union Z of p discs in B each of d -radius R , there exist $b \in B_0 \setminus (S \cup Z)$ and piecewise smooth loops $\gamma_1, \dots, \gamma_k \subset B_0 \setminus S$ based at b which represent respectively $\alpha_1, \dots, \alpha_k$ in $\pi_1(B_0, b_0)$ up to a single conjugation and such that $\text{length}_{d_{B_0 \setminus (S \cup Z)}}(\gamma_i) \leq L(\#S + 1)$.*

Proof. We adopt the proof of Theorem 5.1. Let $R := A/(4p)$. The constant $A' := A/4 < A < a$ still satisfies the conclusion of Lemma 4.1. Then we can simply take R and $4L > 0$. Indeed, for every p discs Z in B each of d -radius R , the only two modifications needed are the following. First, by the same area argument, we can find $b \in B_\varepsilon \subset B_1$ such that $b \cap (V(S, 2a/s) \cup Z) = \emptyset$. The second change lies in the use of Lemma 3.7 in the last step: we apply the procedure (*) described in the proof of Lemma 3.7 for the decomposition into connected components of the closed set $V(S, A'/s) \cup Z$ instead of the set $V(S, A'/s)$. □

6. Proof of Theorem C

Let $\Omega = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. Denote by $T_1\mathbb{CP}^1$ the unit tangent space with respect to the Fubini-Study metric d_{FS} on \mathbb{CP}^1 given by $d_{FS}z = |dz|/(1 + |z|^2)$ where z is the affine coordinate on \mathbb{CP}^1 . The next result of Ahlfors ([1, Theorem 1-12], notably (1-24)) says that the hyperbolic metric on Ω near the cusp 0 behaves as the hyperbolic metric of the punctured unit disc:

Theorem 6.1 (Ahlfors). *There exist $\delta > 0$ and $C > 0$ such that for every $(z, v) \in T_1\mathbb{CP}^1$ with $z \in \Omega$ and $d_{FS}(z, 0) < \delta$, we have $|\ln \lambda_\Omega(z, v) + \ln d_{FS}(z, 0) + \ln \ln(d_{FS}(z, 0))^{-1}| < C$.*

We shall need the following auxiliary lemma for the proof of Theorem C.

Lemma 6.2. *There exists $r_0 > 0$ such that for every $s \geq 2$, we can cover the Riemann sphere \mathbb{CP}^1 by s closed discs of d_{FS} -radius $r_0s^{-1/2}$.*

Proof. Let $\Delta_2 = \{z \in \mathbb{C} : |z| \leq 2\} \subset \mathbb{C}$. Denote d the Euclidean metric on \mathbb{C} . For each $\varepsilon > 0$, let $N(\varepsilon)$ denote the minimum number of closed discs of d -radius ε in \mathbb{C} which can cover Δ_2 . Then Kershner’s theorem (cf. [22]) tells us that $\lim_{s \rightarrow \infty} s^{-1}N(s^{-1/2}) = 8\pi 3^{1/2}/9$.

Note that $N(s^{-1/2})$ is an increasing function in s . It follows that there exists a real number $c > 1$ such that $N(s^{-1/2}) < cs$ for all $s \geq 1$. Replacing s by s/c , we deduce that for every $s \geq c$, there exists a covering of Δ_2 by at most s discs of d -radius $(c/s)^{1/2}$. In particular, since $(c/s)^{1/2} \leq 1$ for $s \geq c$, we can find a subset of $k \leq s$ discs D_1, \dots, D_k which cover Δ_1 and whose centers z_1, \dots, z_k belong to Δ_2 . Let p_N be the stereographic projection from the north pole of \mathbb{CP}^1 onto \mathbb{C} . We obtain a cover of the Southern hemisphere of \mathbb{CP}^1 by $p_N^{-1}(D_1), \dots, p_N^{-1}(D_k)$. As $d_{FS}z = |dz|/(1 + |z|^2) \geq |dz|/5$ for every $z \in \Delta_2$, every set $p_N^{-1}D_i$, where $i = 1, \dots, k$, is contained in the disc centered at $p_N^{-1}(z_i)$ of d_{FS} -radius $5(c/s)^{1/2}$. By symmetry, we obtain a cover of \mathbb{CP}^1 by $2s$ discs of d_{FS} -radius $\leq 5(c/s)^{1/2}$ for every $s \geq c$. □

Proof of Theorem C. Consider an arbitrary ramified cover $\pi : B \rightarrow \mathbb{CP}^1$ of B to the Riemann sphere. Let d_{FS} be the Fubini-Study metric on \mathbb{CP}^1 . We denote by \tilde{d} the induced metric on $B \setminus R_\pi$ where $R_\pi \subset B$ is the branch locus of π which is a finite subset.

Since $\alpha \in \pi_1(B_0) \setminus \{1\}$, it is well-known that there exists $c_0 > 0$ such that every loop $\gamma \subset B_0$ representing α has bounded \tilde{d} -length from below by c_0 , i.e., $\text{length}_{\tilde{d}}(\gamma) > c_0$ (cf., for example, [7, Theorem 1.6.11]). In particular, it follows that

$$(6.1) \quad \text{length}_{d_{FS}}(\pi(\gamma)) = \deg(\pi)^{-1} \text{length}_{\tilde{d}}(\gamma) > c_0 \deg(\pi)^{-1}.$$

By Lemma 6.2, there exists $r_0 > 0$ such that for every $s \geq 2$, we can cover the Riemann sphere $\mathbb{C}\mathbb{P}^1$ in a certain regular manner by s discs of d_{FS} -radius $r_0 s^{-1/2}$. Let $Z \subset \mathbb{C}\mathbb{P}^1$ be the set containing the centers of these discs. For each point $z \in Z$, let $z' \in \mathbb{C}\mathbb{P}^1$ be any point on the equator relative to z as a pole. Consider the stereographic projection P_w from the opposite pole $w \in \mathbb{C}\mathbb{P}^1$ of z to the complex plane such that $P_w(z) = 0$ and $P_w(z') = 1$. Then P_w is a biholomorphic isometry with respect to the induced Fubini-Study metric d_{FS} :

$$(6.2) \quad \tilde{P}_w : \mathbb{C}\mathbb{P}^1 \setminus \{w, z, z'\} \rightarrow \Omega = \mathbb{C}\mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Define $T = \{w, z, z' : z \in Z\}$ and $S = \pi^{-1}Z \subset B$. Then $\pi(B_0 \setminus S) \subset \mathbb{C}\mathbb{P}^1 \setminus T$ and:

$$(6.3) \quad \#S \sim 3 \deg(\pi)s.$$

For every $x \in \mathbb{C}\mathbb{P}^1 \setminus T$, we can find by construction some $z \in Z$ such that $d_{FS}(x, z) < r_0 s^{-1/2}$. By Theorem 6.1 applied to $x \in \mathbb{C}\mathbb{P}^1 \setminus \{w, z, z'\} \simeq \Omega$, we deduce that for all s large enough so that $r_0 s^{-1/2} < \delta$ and for every unit vector $v \in (T_1 \mathbb{C}\mathbb{P}^1)_x$, we have:

$$(6.4) \quad \begin{aligned} \lambda_{\mathbb{C}\mathbb{P}^1 \setminus T}(x, v) &\geq \lambda_{\mathbb{C}\mathbb{P}^1 \setminus \{w, z, z'\}}(x, v) && \text{(by Lemma 7.4)} \\ &\gtrsim d_{FS}(x, z)^{-1} (\ln d_{FS}(x, z)^{-1})^{-1} && \text{(by (6.2) and Theorem 6.1)} \\ &\gtrsim s^{1/2} (\ln(s))^{-1} && \text{(as } d_{FS}(x, z) < r_0 s^{-1/2} \text{)}. \end{aligned}$$

Now let $\gamma \subset B_0 \setminus S$ be any piecewise smooth loop representing α and is parametrized by a map $f : [0, 1] \rightarrow B_0 \setminus S$. We find that:

$$\begin{aligned} \text{length}_{B_0 \setminus S}(\gamma) &= \int_0^1 \lambda_{B \setminus S}(f(t), f'(t)) dt \\ &\geq \int_0^1 \lambda_{\mathbb{C}\mathbb{P}^1 \setminus T}(\pi \circ f(t), (\pi \circ f)'(t)) dt && \text{(by Lemma 7.4)} \\ &\gtrsim \int_0^1 s^{1/2} (\ln(s))^{-1} |(\pi \circ f)'(t)|_{d_{FS}} dt && \text{(by (6.4))} \\ &= s^{1/2} (\ln(s))^{-1} \text{length}_{d_{FS}}(\pi(\gamma)) \\ &\gtrsim \#S^{1/2} (\ln(\#S + 1))^{-1} && \text{(by (6.1) and (6.3)).} \end{aligned}$$

□

Remark 6.3. The lower asymptotic polynomial growth $s^{1/2}$ in Theorem C is optimal. Indeed, assume that $B = \mathbb{C}\mathbb{P}^1$. Then in the above proof, we can use directly the bi-Lipschitz metric equivalence in Theorem 6.1 instead of the inequality (6.4). Therefore, for the choice given in the above proof of certain uniform distribution of the s points on $\mathbb{C}\mathbb{P}^1$, the asymptotic polynomial growth $s^{1/2}$ of the function $L(\alpha, s)$ is attained.

7. Parshin’s homotopy reduction and metric properties of hyperbolic spaces

7.1. The homotopy reduction step of Parshin. In Setting (P), let $t_A \in \mathbb{N}$ be the cardinality of $(A(K)/\text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}))_{tors}$ (cf. [25]). Without loss of generality, we suppose in the rest of the paper that $A[m] \subset A(K)$ for some integer $m \geq 2$. The proof of Theorem A combines the *homotopy* reduction step due to Parshin (cf. Proposition 7.2) with the estimation given in Theorem B on the *hyperbolic* lengths of loops in complements of the Riemann surface B .

The bridges connecting the above two blocks in Theorem A are the following: the first one is the *Fundamental Lemma* of the geometry of groups, which we formulated in Proposition A.8 and in Lemma A.9 (cf. Appendix A). In particular, Lemma A.9 can be regarded as an analogous counting tool of a Counting Lemma of Minkovski frequently used in height theory. The second one is a theorem of Green (cf. Theorem 7.5) which allows us to transfer hyperbolic metric information from the Riemann surface B to the family \mathcal{A} .

The approach of Parshin in his Theorem 1.2 is based on Proposition 7.1 which is stated without proof in [32].

Proposition 7.1 (Parshin). *Let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 := B \setminus W$. Let $b_0 \in B_0$ and denote $\Gamma = H_1(\mathcal{A}_{b_0}, \mathbb{Z})$, $G = \pi_1(B_0, b_0)$. Assume $A[m] \subset A(K)$ for some integer $m \geq 2$. Then we have a natural commutative diagram of homomorphisms:*

$$(7.1) \quad \begin{array}{ccccc} A(K)/mA(K) & \xleftarrow{\delta} & H^1(\widehat{G}, A[m]) & & \\ & \uparrow & \searrow \simeq & & \\ A(K) & \xrightarrow{\alpha} & H^1(G, \Gamma) & \xrightarrow{\beta} & H^1(G, A[m]). \end{array}$$

Here, \widehat{G} denotes as usual the profinite completion of the group G . Let $\omega \in W$ be the finite subset containing the centers of the discs in W and such that $T \subset \omega$. By the theory of étale fundamental groups, $\widehat{G} = \text{Gal}(K_\omega/K)$ where K_ω/K is the maximal Galois extension of K which is unramified outside of ω . We first indicate below how Proposition 7.1 allows us to reduce the problem to the finiteness of certain morphisms between certain fundamental groups. In what follows, we keep the notation as in Proposition 7.1.

Since $\mathcal{A}_{B_0} \rightarrow B_0$ is a proper submersion, it is a fiber bundle by Ehresmann’s fibration theorem (cf. [15]). It follows that we have an exact sequence of fundamental groups induced by the fiber bundle $A_{b_0} \rightarrow \mathcal{A}_{B_0} \rightarrow B_0$ of $K(\pi, 1)$ -spaces where we denote $A_{b_0} := \mathcal{A}_{b_0}$:

$$(7.2) \quad 0 \rightarrow \pi_1(A_{b_0}, w_0) = H_1(A_{b_0}, \mathbb{Z}) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{\rho=f_*} \pi_1(B_0, b_0) \rightarrow 0.$$

To fix the ideas, w_0 is chosen here and in the rest of the paper to be the neutral element of the group A_{b_0} , which also lies on the zero section of \mathcal{A}_{B_0} .

Every rational point $P \in A(K)$ induces (cf. (7.8)) a homotopy conjugacy class of sections $i_P: \pi_1(B_0, b_0) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0)$ of the exact sequence (7.2). The quantitative version of the homotopy reduction step of Parshin can be stated as follows:

Proposition 7.2. *Let the notation be as in Proposition 7.1. Modulo the translations by the trace $\text{Tr}_{K/\mathbb{C}} A(\mathbb{C})$, every conjugacy class of a section i of the exact sequence (7.2) is induced by at most $t_A = \#(A(K)/\text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}))_{tors}$ rational points $P \in A(K)$.*

Proof. (Cf. [32, Proposition 2.1]) Let $P, Q \in A(K)$ and assume that the conjugacy classes of i_P and i_Q of the exact sequence (7.2) are equal as homomorphisms $\pi_1(B_0, b_0) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0)$. Up to making a finite base change $B' \rightarrow B$ étale outside of T , we can suppose without loss of generality that $A[m] \subset A(K)$ for some integer $m \geq 2$ (this assumption is only necessary in Proposition 7.1). It follows that $\alpha(P) = \alpha(Q)$. Since α is a homomorphism, $\alpha(P - Q) = 0$. Proposition 7.1 then implies that $\delta(P - Q) = 0$ and $P - Q \in mA(K)$. Thus, we have $P - Q = mR$ for some $R \in A(K)$. Observe that $m\alpha(R) = \alpha(mR) = \alpha(P - Q) = 0$ in the torsion free abelian group $H^1(G, H_1(A_{b_0}, \mathbb{Z}))$. We deduce that $\alpha(R) = 0$ since $m \neq 0$. Therefore, by an immediate induction, the same argument shows that $P - Q \in m^k A(K)$ for every $k \in \mathbb{N}$. But since $\Omega := A(K)/\text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is a finitely generated abelian group, we must have $P - Q \text{ mod } \text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \in \Omega_{tors}$ because $m \geq 2$. As the latter set is finite, the conclusion follows as $t_A = \#\Omega_{tors}$ by definition. \square

Proof of Proposition 7.1. Let \mathcal{A}_0 be the restriction of \mathcal{A} over B_0 . Let $\mathcal{L} = \mathcal{N}_{\sigma_O(B_0)/\mathcal{A}_0}$ be the complex Lie algebra of \mathcal{A}_0 , viewed as a vector bundle over B_0 . Identifying the kernel Γ of the relative exponential map $\mathcal{L} \rightarrow \mathcal{A}_0$ with $(R^1 f_* \mathbb{Z})^\vee$, we obtain a canonical short exact sequence of locally constant analytic sheaves over B_0 :

$$(7.3) \quad 0 \rightarrow (R^1 f_* \mathbb{Z})^\vee \rightarrow \mathcal{L} \rightarrow \mathcal{A}_0^{an} \rightarrow 0.$$

When the trace of \mathcal{A} is zero, we obtain a map $\mathcal{A}_0^{an}(B_0) \rightarrow H^1(B_0, (R^1 f_* \mathbb{Z})^\vee)$ whose restriction to the set of rational sections $A(K) \subset \mathcal{A}_0(B_0)$ is injective and this is already good enough for the proof of Proposition 7.1. In general, consider the multiplication-by- m B_0 -morphism $[m]: \mathcal{A}_0 \rightarrow \mathcal{A}_0$. The induced map $[m]: \mathcal{L} \rightarrow \mathcal{L}$ on \mathcal{L} is an isomorphism with inverse $[m^{-1}]: \mathcal{L} \rightarrow \mathcal{L}$ given

by the multiplication by m^{-1} . Notice that we assume $\mathcal{A}_0[m] \subset A(K) \subset \mathcal{A}(B_0)$. The map $[m]$ and the sequence (7.3) induce the following commutative diagram in the analytic category.

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathcal{A}_0[m] & & \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{A}_0[m] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (R^1 f_* \mathbb{Z})^\vee & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{A}_0 \longrightarrow 0 \\
 (7.4) & & \downarrow m & & \downarrow \simeq & & \downarrow m \\
 0 & \longrightarrow & (R^1 f_* \mathbb{Z})^\vee & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{A}_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Q & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

By the snake lemma, $\mathcal{A}_0[m] \simeq Q$. Let $S \supset T$ be the centers of the discs in W and let K_S be the maximal Galois extension of K unramified outside of S . Then $\text{Gal}(K_S/K) \simeq \widehat{G}$ where $G = \pi_1(B_0; b_0)$. The natural isomorphism $H^1(\widehat{G}, A[m]) \simeq H^1(G, A[m])$ induced by the injection $G \rightarrow \widehat{G}$ (cf. [36, I.2.6.b]). The Kummer exact sequence $0 \rightarrow A(m) \rightarrow A(\bar{K}) \xrightarrow{m} A(\bar{K}) \rightarrow 0$ gives rise to an exact sequence of Galois cohomology (thus of algebraic nature)

$$(7.5) \quad A(K) \xrightarrow{m} A(K) \rightarrow H^1(\widehat{G}, A[m]).$$

It is well-known that the category of local systems on an Eilenberg-MacLane $K(\pi, 1)$ -space X (i.e., $\pi_i(X) = 0$ for all $i > 0$) is equivalent to the category of π -modules. As \mathcal{A}_0 and B_0 are $K(\pi, 1)$ -spaces, there are canonical isomorphisms

$$(7.6) \quad H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \simeq H^1(G, \Gamma), \quad H^1(B_0, \mathcal{A}_0[m]) \simeq H^1(G, A[m]),$$

where $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \simeq (R^1 f_* \mathbb{Z})_{b_0}^\vee$. Here, the group G acts naturally on Γ by monodromy (cf. [33]) and acts trivially on $A[m]$.

Combining (7.5) with the cohomology long exact sequences induced by Diagram (7.4), we obtain a natural commutative diagram:

$$\begin{array}{ccccc}
 A(K) & \longrightarrow & \mathcal{A}_0(B_0) & \longrightarrow & H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \\
 \downarrow m & & \downarrow m & & \downarrow \simeq \\
 (7.7) \quad A(K) & \longrightarrow & \mathcal{A}_0(B_0) & \longrightarrow & H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \\
 \downarrow & & \downarrow & & \downarrow \\
 H^1(\widehat{G}, A[m]) & \xrightarrow{\simeq} & H^1(B_0, \mathcal{A}_0[m]) & \longrightarrow & H^1(B_0, Q).
 \end{array}$$

By decomposing the first column into $A(K)/mA(K) \hookrightarrow H^1(\widehat{G}, A[m])$ and using the isomorphisms in (7.6), we obtain as claimed the commutative diagram (7.1).

The detailed descriptions of the involved homomorphisms α, β, γ and the monodromy action of G are given in [33, Appendix 8]. □

7.2. The homomorphism α and the monodromy action of G . Keep the notation as in Proposition 7.1. As A_{b_0} is a torus, we can fix a collection of smooth geodesics $l_{w_0,w}: [0, 1] \rightarrow A_{b_0}$ such that $l_{w_0,w}(0) = w_0$ and $l_{w_0,w}(1) = w \in A_{b_0}$. Now, each section $\sigma_P: B_0 \rightarrow \mathcal{A}_{B_0}$ induces naturally a section $i_P: \pi_1(B_0, b_0) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0)$ of (7.2) as follows. Take any loop γ of B_0 based at b_0 , we define the section i_P by the formula:

$$(7.8) \quad i_P([\gamma]) = [l_{w_0, \sigma_P(b_0)}^{-1} \circ \sigma_P(\gamma) \circ l_{w_0, \sigma_P(b_0)}] \in \pi_1(\mathcal{A}_{B_0}, w_0).$$

As a convention, we concatenate oriented paths as above, as opposed to the usual composition of homotopy classes, so the multiplication order reverses. As i_P, i_O are sections of ρ , the difference $i_P - i_O$ satisfies $\rho(i_P - i_O) = \rho(i_P) - \rho(i_O) = 0$. Therefore, $\text{Im}(i_P - i_O) \subset \text{Ker } \rho = H_1(A_{b_0}, \mathbb{Z})$. We have just defined a map $i_P - i_O: \pi_1(B_0, b_0) \rightarrow H_1(A_{b_0}, \mathbb{Z})$. This map is well-defined modulo a principal crossed homomorphism induced by different choices of the paths $l_{w_0,w}$ (these choices of paths also give rise to the conjugation class of i_P).

By the exact sequence (7.2), it is not hard to check that $i_P - i_O$ is a 1-cocycle of the group $G = \pi_1(B_0, b_0)$ with coefficients in $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \simeq \mathbb{Z}^{2 \dim A}$. By this way, we obtain a natural induced natural homomorphism of groups:

$$(7.9) \quad \alpha: A(K) \rightarrow H^1(G, \Gamma), \quad P \mapsto i_P - i_O,$$

where the monodromy G -action on Γ is given by conjugation as follows. Let $\lambda: [0, 1] \rightarrow A_{b_0}$ be a loop with $\lambda(0) = \lambda(1) = w_0$. Let $\gamma: [0, 1] \rightarrow B_0$ be a loop in B_0 with $\gamma(0) = \gamma(1) = b_0$. Let $\gamma' = \sigma_O \circ \gamma$. By (7.2), $\gamma' \circ \lambda \circ \gamma'^{-1}$ defines an element in $\pi_1(\mathcal{A}_{B_0}, w_0)$ denoted $[\gamma] \cdot [\lambda]$. It is clear that $[\gamma] \cdot [\lambda]$ depends only on the homotopy classes $[\lambda]$ and $[\gamma]$ (with base points).

7.3. Some metric properties of hyperbolic manifolds. We follow closely [32]. Let X be a complex manifold. The *pseudo-Kobayashi hyperbolic metric* $d_X: X \times X \rightarrow \mathbb{R}$ is defined as follows. Let ρ be the Poincaré metric on the unit disc $\Delta = \{z \in \mathbb{C}: |z| = 1\}$.

Let $x, y \in X$. Consider the data L consisting of a sequence of points $x_0 = x, x_1, \dots, x_n = y$ in X , a sequence of holomorphic maps $f_i: \Delta \rightarrow X$ and the pairs $(a_i, b_i) \in \Delta^2$ for $i = 0, \dots, n$ such that $f_i(a_i) = x_i$ and $f_i(b_i) = x_{i+1}$. Let $H(x, y; L) = \sum_{i=0}^n \rho(a_i, b_i)$.

Definition 7.3 (Cf. [23]). For $x, y \in X$, we define $d_X(x, y) := \inf_L H(x, y; L)$.

If $d_X(x, y) > 0$ for all distinct $x, y \in X$, i.e., when d_X is a metric, X is called a *hyperbolic* manifold. Recall the fundamental distance-decreasing property (cf. [24, Proposition 3.1.6]):

Lemma 7.4. *Let $f: X \rightarrow Y$ be a holomorphic map of complex manifolds. Then for all $x, y \in X$, $d_Y(f(x), f(y)) \leq d_X(x, y)$. In particular, if $X \subset Y$, we have $d_Y|_X \leq d_X$.*

Proof. For every data $L = \{x_i, f_i, a_i, b_i\}$ associated with the points x, y , we have a data $f(L) = \{f(x_i), f \circ f_i, a_i, b_i\}$ associated with the points $f(x), f(y)$ and $H(x, y; L) = H(f(x), f(y); f(L))$. The lemma now follows from the definition. □

A complex space X is *Brody hyperbolic* if there is no nonconstant holomorphic map $\mathbb{C} \rightarrow X$.

Theorem 7.5 (Green). *Let X be a relatively compact open subset of a complex manifold M . Let $D \subset X$ be a closed complex subspace. Denote by \bar{X}, \bar{D} the closures of X and D in M . Assume that \bar{D} and $\bar{X} \setminus \bar{D}$ are Brody hyperbolic. Then $X \setminus D$ is hyperbolic and we have $d_{X \setminus D} \geq \rho|_{X \setminus D}$ for some Hermitian metric ρ on M . In particular, if M is compact and λ is any Riemannian metric on $|M|$ then there exists $c > 0$ such that $d_{X \setminus D} \geq c\lambda|_{X \setminus D}$.*

Proof. See [16, Theorem 3]. □

Theorem 7.6 (Green). *Let $X \subset A$ be a complex subspace and let D be a hypersurface of a complex torus A . Then the following hold:*

- (i) *X is hyperbolic if and only if X does not contain any translates of a nonzero complex subtorus of A ;*
- (ii) *if D does not contain any translates of nonzero subtori of T then $A \setminus D$ is complete hyperbolic.*

Proof. See [16, Theorems 1-2]. □

Thanks to the distance-decreasing property of the pseudo-Kobayashi hyperbolic metric, we have the following important property of sections (cf. [32]).

Lemma 7.7. *Let $f: X \rightarrow Y$ be a holomorphic map between complex spaces. Suppose that $\sigma: Y \rightarrow X$ is a holomorphic section. Then $\sigma(Y)$ is a totally geodesic subspace of X , i.e., for all $x, y \in Y$, one has $d_Y(x, y) = d_X(\sigma(x), \sigma(y))$.*

Proof. It is a consequence of distance-decreasing property of the pseudo-Kobayashi hyperbolic metric Lemma 7.4: $d_Y(x, y) = d_Y(f(\sigma(x)), f(\sigma(y))) \leq d_X(\sigma(x), \sigma(y)) \leq d_Y(x, y)$. □

8. Growth of generalized integral sections and proof of Theorem A

8.1. Some geometry of Riemann surfaces. In the proof of Theorem A, we shall need the following general auxiliary lemma to control the geometry of a countable closed subset (note that our lemma is more general than the last lemma in [32]).

Lemma 8.1. *Let R be a closed countable subset of a compact surface B equipped with a metric ρ . Let $T \subset B$ be a finite subset. Then for every $\varepsilon > 0$, $R \cup T$ is contained in a finite union Z of disjoint closed discs each of radius at most ε such that $\text{vol}_\rho Z \leq \varepsilon$ and such that any two distinct points in T are contained in distinct discs.*

Proof. Write $T = \{t_1, \dots, t_p\}$ and enumerate $R \cup T = (x_n)_{n \geq 1}$ such that $x_1 = t_1, \dots, x_p = t_p$. Let $\delta = (\varepsilon/c)^{1/2} > 0$ where $c \gg 1$ is some large constant to be chosen later. We define by recurrence $(y_n)_{n \geq 1} \subset R$ such that each y_n is contained in a small closed disc V_n of B of radius $r_n < \delta/2^n$. For $n = 1$, let $y_1 := x_1 = t_1$ and let $D \subset B$ be the closed disc of radius $\delta/2$ centered at y_1 . Since R is countable and $]0, \delta/2[$ is uncountable, there exists a closed disc $V_1 \subset D$ of radius $r_1 \in]0, \delta/2[$ also centered at x_1 such that $\partial V_1 \cap R = \emptyset$ and $V_1 \cap T = \emptyset$. Similarly, we can find successively for $i = 2, \dots, p$ a closed disc V_i of radius $r_i \in]0, \delta/2^i[$ such that $t_i \in V_i$, $\partial V_i \cap R = V_i \cap T = \emptyset$ and $V_i \cap (V_1 \cup \dots \cup V_{i-1}) = \emptyset$.

For $n = k + 1 > p$, let $m \geq 1$ be the smallest integer such that $x_m \notin V_1 \cup \dots \cup V_k$. Define $y_{k+1} := x_m$ and let $V_{k+1} \subset B$ be a closed disc of radius $r_{k+1} \in]0, \delta/2^{k+1}[$ centered at y_{k+1} such that $\partial V_{k+1} \cap R = \emptyset$ and $V_{k+1} \cap (V_1 \cup \dots \cup V_k) = \emptyset$. Such V_{k+1} exists since R is countable. By construction, $R \cup T \subset \cup_{n \geq 1} (V_n \setminus \partial V_n)$. As $R \cup T \subset B$ is closed and B is compact, $R \cup T$ is compact. Hence, $R \cup T$ is contained in a finite union Z of disjoint closed discs $Z = V_{n_1} \cup \dots \cup V_{n_q}$. It follows that $\{1, \dots, m\} \subset \{n_1, \dots, n_q\}$ since $t_i \in V_i$ but $t_i \notin V_j$ for all $i \in \{1, \dots, p\}$ and $j \neq i$. Thus, distinct points in T belong to distinct discs of Z .

By compactness of B , there exists $c_0 > 0$ such that for every small enough $r > 0$, we have for every point $b \in B$ (cf. Lemma 3.5) that $\text{vol}_\rho D(b, r) \leq c_0 r^2$. Therefore, for all $c \gg c_0$:

$$\text{vol } Z \leq \sum_{n \geq 1} \text{vol } V_n < c_0 \sum_{n \geq 1} (\delta/2^n)^2 < c_0 \frac{\varepsilon}{c} \sum_{n \geq 1} 4^{-n} < \varepsilon. \quad \square$$

8.2. Proof of Theorem A. We can now give the proof of the main result of the paper. We follow closely the strategy of Parshin [32] and carefully make all the arguments effective.

Proof of Theorem A. Let $n := \dim A$. Up to enlarging W , we can clearly suppose that W contains exactly $t = \#T$ disjoint closed discs $(W_t)_{t \in T}$ centered at the points of T . Define

$$(8.1) \quad \varepsilon := \frac{1}{3} \min(\text{rad}(B, d), \min\{d(W_u, W_t), u, t \in T, u \neq t\}),$$

where $d(W_u, W_t)$ denotes the d -distance between W_u and W_t and $\text{rad}(B, d) > 0$ denotes the injectivity radius of B . Since the discs W_t 's are disjoint and closed, we have $\varepsilon > 0$.

Consider the nonhyperbolic locus

$$(8.2) \quad V := \{b \in B : \mathcal{D}_b \text{ is not hyperbolic}\}.$$

Then V is an analytic closed subset of B since hyperbolicity is an analytic open property on the base in a proper holomorphic family (cf. [4]). Observe that $V \subset Z(\mathcal{A}, \mathcal{D})$ by Theorem 7.6 where $Z(\mathcal{A}, \mathcal{D})$ is defined in Lemma B.2. Since $\mathcal{D}_K = D$ does not contain any translates of nonzero abelian subvarieties of A , it follows that $Z(\mathcal{A}, \mathcal{D})$ and thus V are at most countable by Lemma B.2. We can thus apply Lemma 8.1 to obtain a finite union Z_ε of disjoint closed discs each of d -radius at most ε and such that $V \subset Z_\varepsilon$. Then it follows from the definition of ε in (8.1) that we can enlarge the discs W_t 's into larger disjoint closed discs W'_t 's such that $B'_0 := B \setminus (\cup_{t \in T} W'_t) \subset B_0$ is a deformation retract of B_0 and $Z_\varepsilon \subset \cup_{t \in T} W'_t$. In particular, $\pi_1(B'_0) = \pi_1(B_0)$. Clearly, it suffices to prove the theorem for $B'_0 \subset B_0$. Hence, up to replacing B_0 by B'_0 , we can suppose that $V \subset W$.

Note that B_0 is an unbordered hyperbolic Riemann surface. Let $\alpha_1, \dots, \alpha_k$ be a fixed system of simple generators of the fundamental group $\pi_1(B_0, b_0)$ for some $b_0 \in B_0$ (cf. Section 3.1). Notice that $k = \text{rank}(\pi_1(B_0))$.

Fix a Hermitian metric ρ on \mathcal{A} . Let $P \in I_s$, i.e., $P \in A(K)$ such that P is an (S, \mathcal{D}) -integral section for some $S \subset B$ with $\#(S \cap B_0) \leq s$.

By Theorem 5.1, there exist $b \in B_0$ and simple loops $\gamma_1, \dots, \gamma_k$ based at b representing respectively the homotopy classes $\alpha_1, \dots, \alpha_k$ up to a single conjugation by using a fixed collection of chosen paths $(c_{b_0, b})_{b \in B_0}$ in B_0 (cf. Definition 3.2) such that $\gamma_j \subset B_0 \setminus S$ and that

$$(8.3) \quad \text{length}_{d_{B_0 \setminus S}}(\gamma_j) \leq L(s + 1), \quad \text{for every } j = 1, \dots, k,$$

for some constant $L > 0$ independent of s, S, b and P .

By Theorem 7.6 and by our reduction to the case $V \subset W$, the varieties \mathcal{D}_b and $\mathcal{A}_b \setminus \mathcal{D}_b$ are hyperbolic for every $b \in B_0$. As we can always suppose that there is at least one closed disc (of strictly positive radius) in the union W , we can assume that B_0 is hyperbolic.

Let $h: \mathbb{C} \rightarrow (\mathcal{A} \setminus \mathcal{D})|_{B_0}$ be a holomorphic map. The holomorphic map $f \circ h: \mathbb{C} \rightarrow B_0$ must be constant since B_0 is hyperbolic. Thus, h factors through $\mathcal{A}_b \setminus \mathcal{D}_b$ for some $b \in B_0$. Since $b \in B_0$, we have $b \notin V$ by the definition of B_0 . Then by the definition of V (cf. (8.2)), Theorem 7.6 implies that $\mathcal{A}_b \setminus \mathcal{D}_b$ is hyperbolic. It follows that h is constant. Therefore, up to enlarging slightly furthermore W if necessary, we see that the analytic closure of $(\mathcal{A} \setminus \mathcal{D})|_{B_0}$ in \mathcal{A} is Brody hyperbolic. Similarly, the analytic closure $\mathcal{D}|_{B_0}$ is also Brody hyperbolic. Hence, Theorem 7.5 implies that there exists $c > 0$ such that $d_{(\mathcal{A} \setminus \mathcal{D})|_{B_0}} \geq c\rho|_{(\mathcal{A} \setminus \mathcal{D})|_{B_0}}$.

Now, let $\sigma_P: B \rightarrow \mathcal{A}$ be the corresponding section of the rational point P . Notice that for every $j \in \{1, \dots, k\}$, we have by the definition of (S, \mathcal{D}) -integral sections that:

$$\sigma_P(\gamma_j) \subset \sigma_P(B_0 \setminus S) \subset (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}.$$

It follows that for every $j \in \{1, \dots, k\}$, we have:

$$\begin{aligned} \text{length}_\rho(\sigma_P(\gamma_j)) &\leq c^{-1} \text{length}_{d_{(\mathcal{A} \setminus \mathcal{D})|_{B_0}}}(\sigma_P(\gamma_j)) \\ &\leq c^{-1} \text{length}_{d_{(\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}}}(\sigma_P(\gamma_j)) \quad (\text{as } (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S} \\ &\hspace{15em} \subset (\mathcal{A} \setminus \mathcal{D})|_{B_0}) \\ &= c^{-1} \text{length}_{d_{B_0 \setminus S}}(\gamma_j) \quad (\text{by Lemma 7.7}) \\ (8.4) \quad &\leq c^{-1} L(s + 1) \quad (\text{by (8.3)}). \end{aligned}$$

Let σ_O be the zero section of $\mathcal{A} \rightarrow B$. Denote $w_0 = \sigma_O(b_0) \in A_{b_0}$ where $A_{b_0} := \mathcal{A}_{b_0} \subset \mathcal{A}$. Recall the following short exact sequence associated with the locally trivial fibration $\mathcal{A}_{B_0} \rightarrow B_0$ (cf. (7.2) which follows Proposition 7.1):

$$(8.5) \quad 0 \rightarrow \pi_1(A_{b_0}, w_0) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0) \rightarrow \pi_1(B_0, b_0) \rightarrow 0.$$

The zero section of \mathcal{A} induces a section $i_O: \pi_1(B_0, b_0) \rightarrow \pi_1(\mathcal{A}_{B_0}, w_0)$ of (8.5) which in turn induces a semi-direct product $\pi_1(\mathcal{A}_{B_0}, w_0) = \pi_1(A_{b_0}, w_0) \rtimes_{\varphi} \pi_1(B_0, b_0)$. Here, $\pi_1(B_0, b_0)$ acts on $\pi_1(A_{b_0}, w_0)$ by conjugation (see Section 7.2), which is the monodromy action denoted by:

$$(8.6) \quad \varphi: \pi_1(B_0, b_0) \rightarrow \text{Aut}(\pi_1(A_{b_0}, w_0)), \quad \alpha \mapsto \varphi_{\alpha}.$$

Let δ_0 be the diameter of the analytic closure of \mathcal{A}_{B_0} in \mathcal{A} with respect to the metric ρ . Let $(c_{b_0, b})_{b \in B_0}$ be the collection of smooth directed paths of bounded d -length going from b_0 to every point $b \in B_0$. Then by the compactness of the closure of \mathcal{A}_{B_0} , the ρ -lengths of the induced collection of directed smooth paths $\{\sigma_O(c_{b_0, b})\}$ are also uniformly bounded, say, by a constant $\delta'_0 > 0$.

We deduce from (8.4) that the conjugacy class of the section i_P of (8.5) associated with σ_P admits a representative (also denoted i_P) which sends the basis $(\alpha_j)_{1 \leq j \leq k}$ to some homotopy classes in $\pi_1(\mathcal{A}_{B_0}, w_0)$ which admit representative loops of ρ -lengths bounded by

$$(8.7) \quad H(s) := c^{-1}L(s+1) + 2(\delta_0 + \delta'_0).$$

The term $2(\delta_0 + \delta'_0)$ corresponds to the upper bound for the conjugation induced by the change of base points of the loops $\sigma_O(\gamma_i)$'s from $\sigma_P(b)$ to $w_0 = \sigma_O(b_0)$ using a short path living in the fiber A_b of ρ -length bounded by δ_0 which goes from $\sigma_P(b)$ to $\sigma_O(b)$ and the other path $\sigma_O(c_{b_0, b})$ whose ρ -length is bounded by δ'_0 by construction. Notice that one cannot simply use the path $\sigma_P(c_{b_0, b})$ (and another path from $\sigma_P(b_0)$ to w_0) since we have no control on its ρ -length.

Via the semi-direct product $\pi_1(\mathcal{A}_{B_0}, w_0) = \pi_1(A_{b_0}, w_0) \rtimes_{\varphi} \pi_1(B_0, b_0)$, we can write $i_P(\alpha_j) = (\beta_j, \alpha_j)$ where $\beta_j \in \pi_1(A_{b_0}, w_0)$ for every $j \in \{1, \dots, k\}$.

As already remarked above, we can replace B_0 by $B_0 \cup \partial B_0$ without loss of generality. Thus, $(\mathcal{A}_{B_0}, \rho)$ can be regarded as a compact Riemannian manifold with boundary. Let $\pi: \tilde{\mathcal{A}}_0 \rightarrow \mathcal{A}_{B_0}$, $u: \mathbb{R}^{2n} \rightarrow A_{b_0}$, and $v: \Delta \rightarrow B_0$ be the universal covering maps. We have $\tilde{\mathcal{A}}_0 \simeq \mathbb{R}^{2n} \times \Delta$ and a commutative diagram where the composition of the top row is π :

$$\begin{array}{ccc} \tilde{\mathcal{A}}_0 \simeq \mathbb{R}^{2n} \times \Delta & \xrightarrow{u \times \text{Id}} & A_{b_0} \times \Delta \longrightarrow \mathcal{A}_{B_0} \\ & \text{pr}_2 \downarrow & \downarrow f_{B_0} \\ & \Delta & \xrightarrow{v} B_0. \end{array}$$

The right-most square is the pullback of the fiber bundle $\mathcal{A}_{B_0} \rightarrow B_0$ over the contractible open unit disc Δ (in the category of differential manifolds).

Fix a point $\tilde{w} = (\tilde{x}_0, \tilde{y}_0) \in \mathbb{R}^{2n} \times \Delta$ in the fiber $\pi^{-1}(w_0)$ above $w_0 \in \mathcal{A}_{B_0}$. Denote $x_0 = u(\tilde{x}_0) = w_0 \in A_{b_0}$ and $y_0 = v(\tilde{y}_0) = b_0$.

Let $j \in \{1, \dots, k\}$. We claim that the deck transformation $i_P(\alpha_j)w_0 = (\beta_j, \alpha_j)w_0 \in \mathbb{R}^{2n} \times \Delta$ is then simply given by the couple of deck transformations

$$(\varphi_{\alpha_j}(\beta_j)x_0, \alpha_j y_0) \in \mathbb{R}^{2n} \times \Delta$$

(cf. the monodromy action φ in (8.6)) with respect to the chosen points $\tilde{w} \in \pi^{-1}(w_0)$, $\tilde{x}_0 \in u^{-1}(x_0)$ and $\tilde{y}_0 \in v^{-1}(y_0)$ in the universal cover. Indeed, let $\gamma: [0, 1] \rightarrow \mathcal{A}$ be the path representing (β_j, α_j) . Let $\tilde{\gamma}: [0, 1] \rightarrow \mathbb{R}^{2n} \rightarrow \Delta$ be the lifting of γ such that $\tilde{\gamma}(0) = (\tilde{x}_0, \tilde{y}_0) = \tilde{w}$. Define $\gamma' = (u \times \text{Id}) \circ \tilde{\gamma}: [0, 1] \rightarrow A_{b_0} \times \Delta$. Then γ' is the lifting of γ such that $\gamma'(0) = (w_0, \tilde{y}_0)$. Note that α_j is represented by $f_{B_0} \circ \gamma': [0, 1] \rightarrow B_0$. By the definition of the monodromy action φ (via the Homotopy Lifting Property [44, page 45]), the homotopy class of γ' in $\pi_1(A_{b_0}, w_0)$ under the projection $A_{b_0} \times \Delta \rightarrow A_{b_0}$ is $\varphi_{\alpha_j}(\beta_j)$. The claim thus follows.

The metric ρ on \mathcal{A}_{B_0} pull backs to a geodesic Riemannian metric $\tilde{\rho}$ on $\mathbb{R}^{2n} \times \Delta \simeq \tilde{\mathcal{A}}_0$. The metric $\tilde{\rho}$ induces a geodesic Riemannian metric $d_j = \tilde{\rho}|_{\mathbb{R}^{2n} \times \{\alpha_j y_0\}}$ on $\mathbb{R}^{2n} \times \{\alpha_j y_0\}$ for every $j = 1, \dots, k$. Then $u: \mathbb{R}^{2n} \rightarrow A_{b_0}$ makes A_{b_0} into a compact geodesic Riemannian manifold with the induced metric d_j for every $j = 1, \dots, k$. By construction and by (8.7), we find that:

$$d_j(\varphi_{\alpha_j}(\beta_j)x_0, \tilde{x}_0) \leq \tilde{\rho}((\beta_j, \alpha_j)w_0, \tilde{w}) \leq H(s).$$

Therefore, as $\pi_1(A_{b_0}, w_0) \simeq \mathbb{Z}^{2n}$ is an abelian group of finite rank, Lemma A.9(i) and Proposition A.8 imply that for every $j = 1, \dots, k$, there exists a constant $m_j > 1$ independent of s such that there are at most $m_j(H(s) + 1)^{2n}$ possibilities for $\varphi_{\alpha_j}(\beta_j)$ and thus for $i_P(\alpha_j) = (\beta_j, \alpha_j)$ as well.

Let $m_0 = (\max_{1 \leq j \leq k} m_j)^k > 1$. We deduce that the number of (conjugacy classes of) sections i_P of (8.5), where σ_P is an (S, \mathcal{D}) -integral section, is at most $m_0(H(s) + 1)^{2nk}$. We can therefore conclude from Proposition 7.2 that:

$$\#(I_s \text{ mod } \text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})) \leq N(s + 1)^{2nk}, \quad \text{for all } s \geq 0,$$

where $N = t_A m_0 (c^{-1}L + 2(\delta_0 + \delta'_0))^{2nk}$ and $t_A = \#(A(K) / \text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}))_{tors}$. □

9. Proof of Corollary A

Proof of Corollary A. For (i), the hypothesis $\text{Tr}_{K/\mathbb{C}}(A) = 0$ and Theorem A imply that the union of all (S, \mathcal{D}) -integral sections of A , where $S \subset B$ with $\#S \cap B_0 \leq s$, is finite. Let $P_1, \dots, P_q \in A(K)$ be all such integral points where $q \leq m(s + 1)^r$ for $m = m(\mathcal{A}, B_0)$ given in Theorem A and

$r = 2 \dim A \cdot \text{rank } \pi_1(B_0)$. For each $i = 1, \dots, q$, $f(\sigma_{P_i}(B) \cap \mathcal{D}) \cap B_0 = S_i$ for some finite subset $S_i \subset B_0$ of cardinality at most s . In particular, for every $i = 1, \dots, q$, we have $\sigma_{P_i}(B) \not\subset \mathcal{D}$ so that $\sigma_{P_i}(B) \cap \mathcal{D}$ is finite as σ_{P_i} is algebraic. Hence, we can define the finite intersection loci in B by $E := \cup_{i=1}^q f(\sigma_{P_i}(B) \cap \mathcal{D}) \subsetneq B$. Moreover, we have:

$$\#E \cap B_0 = \# \cup_{i=1}^q S_i \leq qs \leq m(s+1)^r s.$$

We claim that E verifies the point (i). Indeed, let $S \subset B \setminus E$ be any subset such that $\#S \cap B_0 \leq s$ and suppose on the contrary that $P \in A(K)$ is an (S, \mathcal{D}) -integral point. Then $P = P_j$ for some $1 \leq j \leq q$ by the definition of the P_i 's. Hence, $f(\sigma_P(B) \cap \mathcal{D}) \subset E$. But P is (S, \mathcal{D}) -integral so that $f(\sigma_P(B) \cap \mathcal{D}) \subset S$. As $S \cap E = \emptyset$, we deduce that $f(\sigma_P(B) \cap \mathcal{D}) = \emptyset$ and thus $\text{deg}_B \sigma_P^* \mathcal{D} = 0$. This is a contradiction since \mathcal{D} is strictly nef by hypothesis. We conclude the proof of Corollary A(i).

For (ii), let E and r be given in Corollary A(i). Let $\Delta \subset B^{(s)}$, where $B^{(s)} = B^s / \mathfrak{S}_s$ is the s -th symmetric product, be the image of the $(s-1)$ -dimensional closed subset $E \times B^{s-1} \subset B^s$ under the quotient map $p: B^s \rightarrow B^{(s)}$. Since p is a finite morphism of algebraic schemes, Δ is an $(s-1)$ -dimensional algebraic closed subset of $B_0^{(s)}$. Now let $[S] \in B^{(s)} \setminus \Delta$ and let $S = \text{supp}[S] \subset B$. Hence $\#S \leq s$ and it follows from the construction of Δ that $S \cap E = \emptyset$.

We claim that none of the P_i 's is an (S, \mathcal{D}) -integral point. Indeed, if P_i is (S, \mathcal{D}) -integral then $\sigma_{P_i}(B) \cap \mathcal{D}$ is nonempty since \mathcal{D} is strictly nef by hypothesis.

On the other hand, $f(\sigma_{P_i}(B) \cap \mathcal{D}) \subset S \cap E = \emptyset$ by the definition of E . We arrive at a contradiction and the claim is proved. The proof of Corollary A is thus completed. □

Remark 9.1. Corollary A(ii) can be made quantitative. Let $B_0^{(s)} = B_0^s / \mathfrak{S}_s \subset B^{(s)}$ be the s -th symmetric product of B_0 . Let $E_0 = E \cap B_0 \subset B$. Let $\Delta_0 = \Delta \cap B_0^{(s)}$. Then Δ_0 is the union of $\#E_0 \leq ms(s+1)^r$ closed subspaces $E_0 \times B_0^{s-1} \subset B_0^s$. Then Δ_0 is an $(s-1)$ -dimensional algebraic closed subspace of $B_0^{(s)}$. Let $[S_0] \in B_0^{(s)} \setminus \Delta_0$ and let $S_0 = \text{supp}[S_0] \subset B_0$. Then $\#S_0 \leq s$ and it follows that $S_0 \cap E_0 = \emptyset$. For every $P \in A(K)$, $\sigma_P(B) \cap \mathcal{D} \neq \emptyset$ as \mathcal{D} horizontally strictly nef. Hence, none of the P_i 's is an (S_0, \mathcal{D}) -integral point by the definition of E . It follows that there is no (S_0, \mathcal{D}) -integral point of \mathcal{A} whenever $[S_0] \in B_0^{(s)} \setminus \Delta_0$.

10. Proof of Theorem D

Proof of Theorem D. By the Lang–Néron theorem, $A(K)$ is a finitely generated abelian group since $\mathrm{Tr}_{K/\mathbb{C}}(\mathcal{A}_K) = 0$. In particular, the subset $R \subset A(K)$ is at most countable. On the other hand, for every $P \in R$, the set $\sigma_P(B) \cap \mathcal{D}$ is finite since $P \notin D$. It follows that the set $I(R, \mathcal{D})$ is at most countable. Let $V := \{b \in B : \mathcal{D}_b \text{ is not hyperbolic}\} \subset B$. Then V is an analytically closed subset of B (cf. [4]) which is at most countable by Lemma B.2 since $V \subset Z(\mathcal{A}, \mathcal{D})$ by Theorem 7.6 where $Z(\mathcal{A}, \mathcal{D})$ is defined in Lemma B.2.

For (i), we suppose on the contrary that $I(R, \mathcal{D})$ is analytically closed in B . We fix an arbitrary finite disjoint union W of closed discs centered at the points of T (points of bad reduction of the family $\mathcal{A} \rightarrow B$). Then the same argument at the beginning of the proof of Theorem A implies that we can enlarge the discs in W to contain $V \cup I(R, \mathcal{D})$ so that they are still closed, disjoint and contain points of T separately.

Then Theorem A applied for $B_0 = B \setminus W$ tells us that $A(K)$ and thus $R \subset A(K)$ contains only finitely many (W, \mathcal{D}) -integral points, since $\mathrm{Tr}_{K/\mathbb{C}}(\mathcal{A}) = 0$. On the other hand, as $I(R, \mathcal{D}) \subset W$, every $P \in R$ is a (W, \mathcal{D}) -integral point. It follows that R must be a finite subset, which is a contradiction to the assumption that R is infinite. Hence, $I(R, \mathcal{D})$ is not analytically closed in B and in particular, it must be infinite. The proof of (i) is thus completed. Observe that the same argument proves also (iii).

The set of limit points $I(R, \mathcal{D})_\infty$ is closed in the analytic topology. For (ii), suppose also on the contrary that $I(R, \mathcal{D})_\infty$ is countable. Then (i) implies that $\overline{I(R, \mathcal{D})} = I(R, \mathcal{D}) \cup I(R, \mathcal{D})_\infty$ is a countable and analytically closed subset of B . Therefore, the same argument as above shows that R is a finite subset of $A(K)$ which is again a contradiction. This proves (ii). \square

11. Some generalizations

By using Theorem 5.2, the proof of Theorem A can be easily modified, *mutatis mutandis*, to obtain the following stronger result in which we allow furthermore the intersection of integral sections with D to happen over some bounded moving discs in B_0 :

Theorem 11.1. *In Setting (P), let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$ and $p \in \mathbb{N}$. Then there exist $m, r > 0$ such that for all $s > 0$*

and $I_s^p := \cup_Z \{P \in A(K) : \#f(\sigma_P(B_0 \setminus Z) \cap \mathcal{D}_z) \leq s\}$ with the union over all unions Z of p discs of d -radius r in B , one has

$$\#I_s^p \bmod \text{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \leq m(s + 1)^{2 \dim A \cdot \text{rank } \pi_1(B_0)}.$$

As an application, we obtain the following generalization of Corollary 1.3. The proof is similar to that of Corollary 1.3 but we use Theorem 11.1 instead of Theorem A.

Corollary 11.2. *In Setting (P), let $W \supset T$ be a finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$ and $p, s \in \mathbb{N}$. Then there exist $r, M = M(\mathcal{A}, \mathcal{D}, B_0, p, s) > 0$ such that for every union Z of p discs of d -radius r in B and every section $\sigma : B \rightarrow \mathcal{A}$ with $\#f(\sigma(B_0 \setminus Z) \cap \mathcal{D}) \leq s$, one has $\deg_B \sigma^* \mathcal{D} < M$.*

Appendix A. Geometry of the fundamental groups

We concisely collect standard results on the geometry of the fundamental groups which are necessary for the proof of Theorem A. For the convenience of future works, we give the main statements Proposition A.8 and Lemma A.9 which are slightly more general than what we actually need. We first recall the following notion of quasi-isometry introduced by Gromov:

Definition A.1 (Gromov). Let X, Y be metric spaces. Let $L, A > 0$. We say that a map $f : X \rightarrow Y$ (not necessarily continuous) is (L, A) -quasi-isometry if:

- (1) (equivalence) $\frac{1}{L}d(x, y) - A \leq d(f(x), f(y)) \leq Ld(x, y) + A$, for all $x, y \in X$;
- (2) (quasi-surjective) there exists $R > 0$ such that $f(X) \cap B(y, R) \neq \emptyset$ for every $y \in Y$ where $B(y, R) = \{z \in Y : d(z, y) < R\}$ is the ball of radius R centered at y .

Note that being *quasi-isometric to* is an equivalence relation.

Definition A.2. Let G be a finitely generated group and let S be a finite generating subset. The function $d_S : G \times G \rightarrow \mathbb{N}$ given by

$$(g, h) \mapsto \begin{cases} 0 & \text{if } g = h, \\ \min\{n : g^{-1}h = s_1 \cdots s_n \text{ for some } s_1, \dots, s_n \in S \cup S^{-1}\} & \text{otherwise} \end{cases}$$

defines a metric on G called the *word metric* associated with the generating set S . We denote also $B((G, d_S), R) = \{g \in G : d_S(g, 1_G) < R\}$ or simply $B(G, R)$ if d_S is fixed.

The following sufficient condition of quasi-isometry is due to Milnor and Svarc. It is sometimes called the Fundamental Lemma of Geometric Group Theory [3, Proposition I.8.19].

Theorem A.3 (Milnor-Svarc). *Let (X, d) be a geodesic metric space and G a finitely generated group acting on X by isometries, i.e., $d(gx, gy) = d(x, y)$. Let $x_0 \in X$ and assume:*

- (i) *(cobounded action) there exists $R > 0$ such that translates of $B(x_0, R)$ cover X , i.e.,*

$$X = \cup_{g \in G} (gB(x_0, R)) = \cup_{g \in G} B(gx_0, R);$$

- (ii) *(metrically proper action) for any $r > 0$, $\{g \in G : B(x_0, r) \cap gB(x_0, r) \neq \emptyset\}$, is finite.*

Then G is finitely generated and the map $p: G \rightarrow X$, $p(g) = gx_0$, is a quasi-isometry where G is equipped with an arbitrary word metric associated with a finite system of generators.

Lemma A.4. *Every finitely generated group G has at most exponential growth, i.e., for every finite generating subset $S \subset G$, there exists $N > 1$ such that for every $R > 0$, we have*

$$(A.1) \quad \#B((G, d_S), R) \leq N^R.$$

Proof. We will show that every $N \geq 2s + 1$ satisfies the inequality (A.1), where $s = \#S$. Moreover, the equality holds only if G is a free group of finite rank. Indeed, by the definition of the word metric d_S , we have for every $R > 0$ that $B((G, d_S), R) \subset \left\{ \prod_{j=1}^{\lfloor R \rfloor} s_j : s_j \in S \cup S^{-1} \cup \{1_G\} \right\}$ where $\lfloor R \rfloor$ denotes the largest integer smaller than or equal to R . It follows that

$$\begin{aligned} \#B((G, d_S), R) &\leq \# \left\{ \prod_{j=1}^{\lfloor R \rfloor} s_j : s_j \in S \cup S^{-1} \cup \{1_G\} \right\} \\ &\leq (\#S + \#S^{-1} + 1)^{\lfloor R \rfloor} \leq (2s + 1)^R \end{aligned}$$

and the lemma is proved by taking $N = 2s + 1$. □

We mention here the famous theorem of Gromov classifying groups of polynomial growth.

Theorem A.5 (Gromov). *Let G be a finitely generated group. Fix a finite generating system $S \subset G$ and consider the corresponding word metric d_S . Let $n(L)$ be the number of elements $g \in G$ such that $d_S(g, 1_G) \leq L$. Then there exist $a, r > 0$ such that $n(L) \leq a(L + 1)^r$ for all $L > 0$ if and only if G is virtually nilpotent, i.e., G admits a finite index subgroup H which is nilpotent, i.e., $H_m = 0$ for some $m \geq 0$ where $H_{k+1} := [H_k, H]$, $H_0 = H$.*

Proof. See [17, Main Theorem]. □

We recall without proof the following standard theorem.

Theorem A.6. *Let M be a compact Hausdorff space which admits a finite cover by open simply connected sets, and which is locally path connected (i.e., there is a base for the topology consisting of path connected sets). Then $\pi_1(M)$ is finitely generated. In particular, this holds for all compact semi-locally simply connected spaces and thus for all compact Riemannian manifolds with or without boundary.*

Definition A.7 (Equivalence of growth functions). Two increasing functions $f, g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are said to have the *same order of growth* if there exist constants $a, c > 0$ such that for all $r > 0$, we have $f(r) \leq cg(ar)$ and $g(r) \leq cf(ar)$.

Hence, if f, g have the same order of growth, then f is bounded from above (resp. from below) by a polynomial of degree m if and only if so is g . The main statement is the following.

Proposition A.8. *Let (M, d) be a compact connected Riemannian manifold with boundary. Let $\pi: \tilde{M} \rightarrow M$ be the universal cover of M . Then for any point $x_0 \in M$ not lying on the boundary, the map $p: \pi_1(M, x_0) \rightarrow \pi^{-1}(x_0)$ given by $g \mapsto gx_0$ is a bijective quasi-isometry. In particular, the growths of $\pi_1(M, x_0)$ are the same whether calculated with respect to the induced geometric norm by d on $\pi_1^{-1}(x_0)$ or with respect to the algebraic word norm associated with an arbitrary finite system of generators of $\pi_1(M, x_0)$.*

Proof. By Theorem A.6, $\pi_1(M)$ is a finitely generated group. Note that M is also a geodesic metric space. Fix $\tilde{x}_0 \in \pi^{-1}(x_0)$. Theorem A.3 implies that the deck transformation action of $\pi_1(M, x_0)$ on the universal cover \tilde{M} induces a quasi-isometry between $(\pi_1(M, x_0), d_{word})$ and (\tilde{M}, \tilde{d}) , say, an (L, A) -quasi-isometry. Here d_{word} denotes the word metric of the group $\pi_1(M, x_0)$ associated with some finite generating set and \tilde{d} denotes the induced metric of d on \tilde{M} . Since the action of $\pi_1(M, x_0)$ commutes with $\pi: \tilde{M} \rightarrow M$, the same quasi-isometry $g \mapsto gx_0$ gives us a bijective (L, A) -quasi-isometry between $(\pi_1(M, x_0), d_{word})$ and the fiber $\pi^{-1}(x_0)$ equipped with the induced metric $\tilde{d}|_{\pi^{-1}(x_0)}$:

$$L^{-1}d_{word}(g, h) - A \leq \tilde{d}|_{\pi^{-1}(x_0)}(gx_0, hx_0) \leq Ld_{word}(g, h) + A,$$

for every $g, h \in \pi_1(M, x_0)$. Hence, for every $D > 0$, we have:

$$\begin{aligned} \#B(\pi_1(M, x_0), L^{-1}D + A) &\leq \#B((\pi^{-1}(x_0), \tilde{x}_0), D) \\ &\leq \#B(\pi_1(M, x_0), LD + LA). \end{aligned}$$

Here, $B(\pi_1(M, x_0), r)$ denotes the ball of d_{word} -radius r in $\pi_1(M, x_0)$ centered at 0. Similarly, $B((\pi^{-1}(x_0), \tilde{x}_0), r)$ denotes the ball of $\tilde{d}|_{\pi^{-1}(x_0)}$ -radius r in $\pi^{-1}(x_0)$ centered at \tilde{x}_0 . Since $LD+A$ and $LD + LA$ are fixed linear

functions in D , the growths of $\pi^{-1}(x_0)$ and of $\pi_1(M, x_0)$ are clearly of the same order. □

An important application of Proposition A.8 is Lemma A.9 which can be seen as the analogue of the Minkovski Counting Lemma:

Lemma A.9. *Let (M, d) be a connected compact Riemannian manifold with boundary. Let $x_0 \in M$. For every $L > 0$, define $n(L)$ to be the number of homotopy classes in $\pi_1(M, x_0)$ which admit some representative loops of length at most L with respect to the metric d . Then:*

- (i) *if $\pi_1(M)$ is virtually nilpotent, there exist $a, r > 0$ such that $n(L) \leq a(L + 1)^r$ for all $L > 0$. If $\pi_1(M)$ is abelian, r can be chosen to be the rank of $\pi_1(M)$;*
- (ii) *In general, there exists $p > 0$ such that $n(L) \leq \exp(p(L + 1))$ for all $L > 0$.*

Proof. Let $\pi: \tilde{M} \rightarrow M$ be the universal cover of M and fix $\tilde{x}_0 \in \pi^{-1}(x_0)$. Then \tilde{M} is a connected geodesic Riemannian manifold with metric \tilde{d} which makes π into a local isometry. The length of the loops in (M, x_0) is the same as the length of their lifts in \tilde{M} . The geometric metric on $\pi^{-1}(x_0)$ is induced by \tilde{d} . Hence, a loop γ in M based at x_0 of length at most L is uniquely determined by a point $[\gamma]x_0 \in \pi^{-1}(x_0)$ such that $\tilde{d}([\gamma]x_0, \tilde{x}_0) \leq L$. By Theorem A.3, $\pi_1(M, x_0)$ is quasi-isometric to the fiber $\pi^{-1}(x_0)$. Hence, they have the same order of growth (Proposition A.8) where the metric on $\pi^{-1}(x_0)$ is induced by \tilde{d} while the one on $\pi_1(M, x_0)$ is the word metric with respect to a finite system of generators (which exists since $\pi_1(M, x_0)$ is finitely generated by Theorem A.6). Since the growth of $\pi_1(M, x_0)$ is at most exponential (cf. Lemma A.4), the point (ii) is proved. The above discussion and Theorem A.5 of Gromov imply the point (i) except for the second statement. If $\Gamma = \pi_1(M, x_0)$ is abelian then $r = \text{rank } \Gamma \geq 0$ is finite (by Theorem A.6). Hence, $\Gamma = \mathbb{Z}g_1 \oplus \dots \oplus \mathbb{Z}g_r \oplus \Gamma_{tors}$ for some $g_1, \dots, g_r \in \Gamma$. Recall the word metric d_S on Γ where $S = \{g_1, \dots, g_r\}$. It is easy to see that:

$$\begin{aligned} \#\{g \in \Gamma: d_S(g, 1_\Gamma) \leq L\} &\leq \#\Gamma_{tors} \cdot \#\left\{g = \sum_{j=1}^r n_j g_j: -L \leq n_j \leq L\right\} \\ &\leq \#\Gamma_{tors} (2L + 1)^r. \end{aligned}$$

The conclusion follows by Proposition A.8. □

Appendix B. Hilbert schemes of algebraic groups

By the theory of Hilbert schemes of subvarieties developed by Grothendieck and Altman-Kleiman, we have the following properties needed in the proof of Theorem 1.2 and Theorem A.

Lemma B.1. *Let $\pi: G \rightarrow S$ be a (quasi-)projective group scheme over a scheme S . Let $\mathcal{D} \subset G$ be a closed subscheme and let $\mathcal{V} = G \setminus \mathcal{D}$. Consider the contravariant functors $\mathcal{F}_{G/S}: \text{Sch}_S \rightarrow \text{Ens}$ defined for $T \rightarrow S$ an S -scheme by*

$$\mathcal{F}_{G/S}(T) := \left\{ \begin{array}{l} \text{(connected) group subschemes of } G_T, \text{ that are} \\ \text{flat, proper, and of finite presentation over } T \end{array} \right\}.$$

Then the following holds:

- (a) $\mathcal{F}_{G/S}$ is representable by a locally of finite type S -scheme denoted also by $\mathcal{F}_{G/S}$;
- (b) There exist natural immersions of S -schemes

$$\mathcal{F}_{G/S}, \text{Mor}_S(S, G), \text{Hilb}_{\mathcal{D}/S}^{\dim>0}, \text{Hilb}_{\mathcal{V}/S}^{\dim>0} \subset \text{Hilb}_{G/S},$$

where $\text{Hilb}_{X/S}^{\dim>0}$ denotes the complement in $\text{Hilb}_{X/S}$ of the S -relative Hilbert schemes of points, i.e., of zero dimensional closed subschemes, of an S -scheme X .

- (c) The S -scheme $\mathcal{F}_{G/S, \mathcal{D}} := \text{Hilb}_{\mathcal{D}/S}^{\dim>0} \times_{\text{Hilb}_{G/S}} (\mathcal{F}_{G/S} \times_S \text{Mor}_S(S, G))$ represents the contravariant functor $\text{Sch}_S \rightarrow \text{Ens}$ given by

$$T \mapsto \left\{ \begin{array}{l} \text{translates of } (\varphi: H \rightarrow G_T) \in \mathcal{F}_{G/S}(T) \text{ by a } T\text{-section } \sigma: T \rightarrow G_T \\ \text{such that } \dim H > 0 \text{ and } \text{Im}(\sigma, \varphi) \subseteq \mathcal{D}_T \end{array} \right\},$$

where $\sigma.\varphi(h) := \sigma(\pi_T \circ \varphi(h))\varphi(h)$ for every $h \in H$. That is, $\mathcal{F}_{G/S, \mathcal{D}}$ is the moduli space of translates of positive dimensional group subschemes of G_s contained in \mathcal{D}_s for $s \in S$;

- (d) Similarly, the S -scheme $\mathcal{F}_{G/S, \mathcal{V}} := \text{Hilb}_{\mathcal{V}/S}^{\dim>0} \times_{\text{Hilb}_{G/S}} (\mathcal{F}_{G/S} \times_S \text{Mor}_S(S, G))$ is the moduli space of translates of positive dimensional group subschemes of G_s that have empty intersection with \mathcal{D}_s for $s \in S$;
- (e) The schemes $\mathcal{F}_{G/S, \mathcal{D}}$ and $\mathcal{F}_{G/S, \mathcal{V}}$ have only countably many irreducible components.

Proof. For (a), see [14, Exposé XI, Remarque 3.13]. The existence of other schemes in (b) is standard since G/S is quasi-projective and so are $\mathcal{D}/S, \mathcal{V}/S$ as $\mathcal{D} \subset G$ is assumed to be closed. Observe that $\text{Hilb}_{G/S}$ is the disjoint union of quasi-projective S -schemes $\text{Hilb}_{G/S}^{q(x)}$ with $q(x) \in \mathbb{Q}[x]$ runs over all numerical polynomials of degree $\leq \dim G_s$ where G_s is a general fiber. Moreover, the S -schemes $\mathcal{F}_{G/S}, \text{Mor}_S(S, G), \text{Hilb}_{\mathcal{D}/S}^{\dim>0}$ and $\text{Hilb}_{\mathcal{V}/S}^{\dim>0}$ are also stratified by the same Hilbert polynomials and that $\text{Hilb}_{\mathcal{D}/S}^{\dim>0}$ and $\text{Hilb}_{\mathcal{V}/S}^{\dim>0}$ do not take

into account zero degree polynomials. Since each S -scheme of finite type has finitely many irreducible components, it follows from the stratification by Hilbert polynomials that all Hilbert schemes in the lemma have only countably many irreducible components. In particular, this proves (e). The assertions (c) and (d) are also clear. \square

Lemma B.2. *Let B be a Dedekind scheme with fraction field K . Suppose that $\pi: G \rightarrow B$ is a quasi-projective B -group scheme. Let $\mathcal{D} \subset G$ be a closed subset with generic fiber $D = \mathcal{D}_K \subset G_K$. Then:*

- (i) *if D does not contain any translates of positive dimensional \overline{K} -algebraic subgroups of G_K , then the following exceptional set is countable*

$$Z(G, \mathcal{D}) := \{t \in B \text{ closed point: } \exists x \in G_t, \exists H \subset G_t \text{ a subgroup,} \\ \dim H > 0, xH \subset \mathcal{D}_t\};$$

- (ii) *if $G_K \setminus D$ does not contain any translates of positive dimensional \overline{K} -algebraic subgroups of G_K , then the following exceptional set is countable*

$$U(G, \mathcal{D}) := \{t \in B \text{ closed point: } \exists x \in G_t, \exists H \subset G_t \text{ a subgroup,} \\ \dim H > 0, xH \subset G_t \setminus \mathcal{D}_t\}.$$

Proof. For (i), let X be an irreducible component of the Hilbert B -scheme $\mathcal{F}_{(G/B), \mathcal{D}}$ defined in Lemma B.1. Then X is a scheme of finite type over B . We claim that the induced morphism $f_X: X \rightarrow B$ is not dominant. The point (i) will then be proved since $\mathcal{F}_{(G/B), \mathcal{D}}$ does not dominate B . Indeed, as $\mathcal{F}_{(G/B), \mathcal{D}}$ contains only countably many irreducible components each of which being of finite type over B (cf. Lemma B.1(c)), the image of $\mathcal{F}_{(G/B), \mathcal{D}}$ in B is thus a countable subset of closed points of B by Chevalley’s theorem and the conclusion follows.

Suppose on the contrary that X dominates B . Then we can find a 1-dimensional irreducible closed subscheme $C \subset X$ such that C dominates B . Up to replacing X by C , we can thus assume that $\dim X = 1$. Let η be the generic point of X and let $V \hookrightarrow \mathcal{F}_{(G/B), \mathcal{D}} \times_B G$ be the universal group scheme. Then $L = \kappa(\eta) \subset \overline{K}$ is a finite extension of K . On the other hand, it follows from the functorial property of Hilbert schemes that V_L is the universal group scheme of G_L/L avoiding \mathcal{D}_L but $\mathcal{F}_{(G_L/L), \mathcal{D}_L}$ is empty by the hypothesis of (i). This contradiction shows that X cannot dominate B and (i) is proved. The proof of (ii) is similar. \square

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