Sheet 1: Review of normed vector spaces and linear transformations

March 17, 2024

1) Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a real vector space X are said equivalent if there exists $m, M \in \mathbb{R}_{>0}$ such that

$$m\|x\| \leq \|x\|' \leq M\|x\|\,, \quad \forall x \in X$$

Show that if X is finite dimensional then any two norms are equivalent.
[Hint: show that the equivalence of norms is actually an equivalence relation. So it is enough to show that any norm $\|\cdot\|$ is equivalent to the euclidean norm. The inequality $\leq$ follows by the axioms. Finally the real number m can be obtained as the minimum value attained by $\|\cdot\|$ on the euclidean unit sphere]

2) Let $X_1, \dots, X_n, Y$ be finite dimensional normed real vector spaces. Show that any multilinear transformation $L\colon X_1 \times \ldots \times X_n \to Y$ is bounded.
[Hint: first prove the claim using on each vector space the norm defined by summing the absolute values of the components with respct to a choice of a basis. Then use the previous exercise to conclude.]

3) Let $\mathbb{P}[x]$ be the vector space of polynomials in the variable x with real coefficients. Endow $\mathbb{P}[x]$ with the following inner product

$$\langle p(x), q(x)\rangle := \int_0^1 p(x)q(x)dx\,.$$

Calculate the norm of the linear transformation of $\mathbb{P}[x]$ given by $m_x : p(x) \mapsto xp(x)$.

4) Let $T : X \to Y$ be a bounded linear transformation between normed real vector spaces. Assume that there is a constant $C > 0$ such that

$$\|Tx\| \geq C\|x\| \quad \forall x \in X\,.$$

Show that the set theoretical inverse transformation $T^{-1}\colon Y \to X$ exists and moreover show that it is linear and bounded.

5) Consider the space $C^\infty[0,1]$ of smooth functions (with domain $[0,1]$ and codomain $\mathbb{R}$) endowed with the following norm:

$$\|f\| := \sup_{x\in[0,1]} f(x)\,.$$

Is the linear transformation of $C^\infty[0,1]$ given by $f \mapsto \frac{df}{dx}$ a bounded linear transformation?