Kobayashi length bounds on bordered surfaces and generalized integral points on abelian varieties

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Abstract

Let B be a compact Riemann surface and $B_0 \subset B$ a bordered hyperbolic subsurface obtained by removing finitely many disjoint closed disks. Fix a nontrivial loop α in B_0. For $s \geq 0$, let $L(\alpha, s)$ denote the supremum, over all finite subsets $S \subset B_0$ with $\#S \leq s$, of the minimal Kobayashi length of a loop in $B_0 \setminus S$ that is freely homotopic to α in B_0. Phung in [10] proved that $L(\alpha, s)$ grows at most linearly and at least as $\sqrt{s}/\log s$. We sharpen the upper bound to $O(\sqrt{s \log s})$, which determines $\lim_{s\to\infty} \frac{\log L(\alpha,s)}{\log s} = \frac{1}{2}$, answering a question raised in [10, Question 1.4]. As an application, we improve the counting bound for generalized integral points on abelian varieties over complex function fields: for an abelian variety of dimension n over $\mathbb{C}(B)$, Phung proved that the number of (s, B_0)-generalized integral points modulo the constant trace grows at most as s^{2nk}, where $k = \mathrm{rk}(\pi_1(B_0))$. We sharpen this to $s^{nk+\varepsilon}$ for every $\varepsilon > 0$, halving the exponent.

1 Introduction

1.1 Generalized integral points and the Lang–Vojta conjecture

Let k be an algebraically closed field of characteristic 0 and let B be a non-singular, irreducible, projective algebraic curve over k with function field $K = k(B)$. Let X be a smooth, integral, projective K-variety and fix a reduced effective divisor $D \subset X$. A *model* of (X, D) is a pair $(\mathcal{X}, \mathcal{D})$ where $\mathcal{X}$ is a normal k-variety equipped with a proper, flat morphism $f\colon \mathcal{X} \to B$ satisfying $\mathcal{X}_K \cong X$, and $\mathcal{D} \subset \mathcal{X}$ is a horizontal divisor with $\mathcal{D}_K \cong D$. Such a model always exists. We also define

$$\mathrm{Sing}(f) := \{b \in B\colon \mathcal{X}_b \text{ is singular}\}\,.$$

The set of rational points $X(K)$ corresponds bijectively to the sections of f: for any $P \in X(K)$, let $\sigma_P\colon B \to \mathcal{X}$ be the section obtained as the Zariski closure of P in $\mathcal{X}$. By abuse of notation we will simply write $\sigma_P \in X(K)$. Given a finite set $S \subset B$, the set of $(S, \mathcal{D})$-*integral points* of $\mathcal{X}$ is

$$\mathcal{X}(\mathcal{O}_{S,\mathcal{D}}) := \{\sigma_P \in X(K)\colon f(\sigma_P(B) \cap \mathcal{D}) \subseteq S\}\,,$$

that is, the set of sections whose image doesn't intersect the divisor $\mathcal{D}$ "above" $B \setminus S$.

Recall that the pair (X, D) is said to be of *log general type* when D is a normal crossings divisor and the log canonical bundle $\omega_X(D)$ is big. A special case is when $X = A$ is an abelian variety: the canonical bundle ω_A is trivial, so the log canonical bundle $\omega_A(D) \cong \mathcal{O}_A(D)$ is big whenever D is big.

The *geometric Lang–Vojta conjecture* predicts strong constraints for the integral points of pairs of log general type

Conjecture 1.1 (Geometric Lang–Vojta)**.** Let (X, D) be a pair of log general type over $K = \mathbb{C}(B)$ and let $(\mathcal{X}, \mathcal{D})$ be a model. Then there exist a proper closed subset $Z = Z(X_{\overline{K}}, D_{\overline{K}}) \subset X \setminus D$ and a constant $m = m(X_{\overline{K}}, D_{\overline{K}}) > 0$ with the following property: for every $\sigma_P \in \mathcal{X}(\mathcal{O}_{S,\mathcal{D}})$ with $\sigma_P(B) \not\subset \mathcal{Z}$, where $\mathcal{Z}$ denotes the Zariski closure of Z in $\mathcal{X}$, we have

$$\deg_B \sigma_P^* \mathcal{D} \;\leq\; m \max\{1,\, 2g(B) - 2 + \#S\}\,. \tag{1}$$

Roughly speaking it says that every integral point either lies in a fixed *exceptional* (proper) closed set Z or has height bounded *linearly* in the Euler characteristic $2g(B) - 2 + \#S$, with slope independent of the base.

The geometric Lang-Vojta conjecture has been settled in very few cases: when X is a curve; when $X = \mathbb{P}^2$ and D has a special shape; when $X = A$ is an abelian variety with trivial $K|k$ trace or when $k = \mathbb{C}$ and A is defined over $\mathbb{C}$ (constant case). For a quick review of the known cases we refer the reader to [10, Section 1] and we remark that for a general complex abelian variety — in particular, when $\mathrm{Tr}_{K/\mathbb{C}}(A) \neq 0$ and A is nonconstant — the conjecture remains open.

From now on we fix the base field $k = \mathbb{C}$, a model $(\mathcal{X}, \mathcal{D})$ of (X, D), and a Riemannian metric ρ on B inducing a path-length ℓ_ρ and a distance function $d_\rho(\cdot,\cdot)$. We shall use two distinct notions of "disk", and we keep them notationally separate throughout.

- A *(closed topological) disk* in B is a closed subset $\overline{\Delta} \subset B$ homeomorphic to the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leq 1\}$, with piecewise-$C^1$ boundary $\partial\overline{\Delta}$. These are the disks removed from B to form the bordered surface B_0; we denote them $\overline{\Delta}_1, \dots, \overline{\Delta}_t$. We do *not* require them to be metric balls—indeed the enlargements performed in §3 produce topological disks that are not metric balls.
- A *metric ball* (or *metric disk*) is a set of the form $\Delta_\rho(x_0, r) := \{b \in B \colon d_\rho(b, x_0) < r\}$, with closed version $\overline{\Delta}_\rho(x_0, r) := \{b \in B \colon d_\rho(b, x_0) \leq r\}$, for $x_0 \in B$ and $r > 0$. These appear only in the local integral estimates of §2.2 (e.g. Lemma 2.2) and carry genuine metric content. Any metric ball of sufficiently small radius is one instance of a topological disk.

The notion of $(S, \mathcal{D})$-integral point can be generalized by allowing S to vary while keeping only its cardinality $s = \#S$ bounded, and by requiring the intersection condition to hold only on the complement of finitely many disjoint closed disks. This notion of generalized integral points was introduced in [10] and we recall it below.

Definition 1.2. Let $t \in \mathbb{Z}_{\geq 0}$. Consider t disjoint disks $\overline{\Delta}_1, \dots, \overline{\Delta}_t$ in B such that $\mathrm{Sing}(f) \subset \bigsqcup_{i=1}^t \overline{\Delta}_i$ and distinct points of $\mathrm{Sing}(f)$ are contained in distinct disks. Set $B_0 := B \setminus \bigsqcup_{i=1}^t \overline{\Delta}_i$, for any $s \in \mathbb{Z}_{\geq 0}$

$$I(s, B_0) := \{\sigma_P \in X(K) \colon \#\left(f(\sigma_P(B_0) \cap \mathcal{D})\right) \leq s\}$$

is the set of *generalized* (s, B_0)*-integral points of* $(\mathcal{X}, \mathcal{D})$.

One immediately observes the inclusion

$$\bigcup_{\substack{S \subset B \\ \#S \leq s}} \mathcal{X}(\mathcal{O}_{S,\mathcal{D}}) \;\subseteq\; I(s, B_0)\,,$$

so $I(s, B_0)$ is potentially much larger than the integral points for any single fixed S. Understanding the growth of $\#I(s, B_0)$ as a function of s (with B_0 fixed) is the main object of this paper.

1.2 Main results

Building on Parshin's foundational work [9], Phung in [10] proved an important quantitative bound on $\#I(s, B_0)$ for abelian varieties, working in the following setting.

(P) Let A be an abelian variety of dimension n over $K = \mathbb{C}(B)$ and let $D \subset A$ be a reduced effective divisor. We fix a model $(\mathcal{A}, \mathcal{D})$, where the proper flat morphism is $f \colon \mathcal{A} \to B$ and $\sigma_O \colon B \to \mathcal{A}$ is a fixed section. We assume:

(i) There exist $t \in \mathbb{Z}_{\geq 0}$ disjoint disks $\overline{\Delta}_1, \ldots, \overline{\Delta}_t$ in B such that $\mathrm{Sing}(f) \subset \bigsqcup_{i=1}^t \overline{\Delta}_i$ and distinct points of $\mathrm{Sing}(f)$ are contained in distinct disks. We define

$$B_0 := B \smallsetminus \bigsqcup_{i=1}^{t} \overline{\Delta}_i$$

so that $f\colon \mathcal{A}_{B_0} \to B_0$ is a family of abelian varieties $(A_b, \sigma_O(b))_{b\in B_0}$.

(ii) B_0 is hyperbolic (this can always be achieved by taking $t \geq 3$).

(iii) $D \subset A$ does not contain any translates of nonzero abelian subvarieties.

Phung's main result is the following.

Theorem 1.3 (Phung [10, Theorem A]). *Assume that the setting (P) holds. Then there exists $m := m(\mathcal{A}, \mathcal{D}, B_0) \in \mathbb{R}_{>0}$ such that*

$$\# \left({I(s, B_0)} \big/ {\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})} \right) \leq m(s+1)^{2n\,\mathrm{rk}(\pi_1(B_0,b_0))}\,. \tag{2}$$

We point out that [10, Theorem A] is stated under the additional assumption that D is ample; ampleness, however, is used in [10] only for the height estimate of [10, Corollary 1.3] and plays no role in the proof of Theorem 1.3, which we recall in full in §3.
Note that the bound behaves coherently: if the number of removed disks t increases, then $\mathrm{rk}(\pi_1(B_0, b_0))$ is larger; on the other hand B_0 shrinks, so the set of generalized integral points grows as well.

The proof of Theorem 1.3 rests on a Parshin cocycle argument. Let $k := \mathrm{rk}(\pi_1(B_0, b_0))$ and fix generators $\alpha_1, \ldots, \alpha_k$. By Ehresmann's theorem, the fibration $\mathcal{A}_{B_0} \to B_0$ gives rise to a short exact sequence $1 \to \Gamma \to \pi_1(\mathcal{A}_{B_0}, w_0) \to G \to 1$, where $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \cong \mathbb{Z}^{2n}$ and $G = \pi_1(B_0, b_0)$. Each section σ_P induces a splitting of this sequence, and the difference between the splittings of σ_P and σ_O defines a 1-cocycle $c_P\colon G \to \Gamma$ whose cohomology class determines P modulo the trace and torsion. To count the possible cocycle classes, one bounds the lattice element $c_P(\alpha_j) \in \Gamma \cong \mathbb{Z}^{2n}$ for each generator by the displacement in the universal cover, which in turn is controlled by the h-length of the loop $\sigma_P(\gamma_j)$ in $\mathcal{A}_{B_0}$. A theorem of Green provides the comparison $\ell_h(\sigma_P(\gamma_j)) \leq c^{-1}\ell_S(\gamma_j)$, so the problem reduces to bounding the Kobayashi length of loops γ_j in $B_0 \smallsetminus S$ representing the generators α_j. Phung's linear bound $\ell_S(\gamma_j) = O(s)$ then yields the displacement bound $H(s) = O(s)$, and lattice counting in $\Gamma \cong \mathbb{Z}^{2n}$ gives at most $O(H(s)^{2n}) = O(s^{2n})$ possibilities per generator, hence the exponent $2nk$ in (2).

When D is moreover ample, Theorem 1.3 implies that for every $\sigma_P \in I(s, B_0)$ there exists a constant $M = M(\mathcal{A}, \mathcal{D}, B_0, s) > 0$ such that $\deg_B \sigma_P^*\mathcal{D} < M$ (see [10, Corollary 1.3]). This is a weak form of the geometric Lang–Vojta conjecture for generalized integral points, weak in two compatible senses. On the one hand it holds for the *larger* set $I(s, B_0)$ of generalized integral points, rather than for the integral points attached to a single fixed S. On the other hand the conclusion is correspondingly weaker: Conjecture 1.1 demands a bound that is *linear* in $\#S$ with slope m *independent of* s, whereas $M = M(s)$ is only known to be finite for each fixed s, with a priori no control on its growth as $s \to \infty$ — the issue is thus the absence of a uniform linear rate, not the mere presence of s. We also stress that [10, Corollary 1.3] assumes D *ample*, which is strictly stronger than the log general type (D big) hypothesis of Conjecture 1.1; as recalled above, ampleness enters only in this corollary and plays no role in Theorem 1.3.

The exponent $2n\,\mathrm{rk}(\pi_1(B_0))$ in (2) can be compared with known results for the smaller set

$$J(s) := \{\sigma_P \in A(K)\colon \#\left(f(\sigma_P(B) \cap \mathcal{D})\right) \leq s\} \;\subseteq\; I(s, B_0)\,.$$

When $n = 1$ (elliptic curves), classical height-theoretic bounds of Hindry–Silverman [5, Corollary 8.5] together with Shioda's bound [12, Theorem 2.5] for the Mordell–Weil rank show that $\#J(s)$ is bounded by a polynomial in s of degree at most $\mathrm{rk}(\pi_1(B_0))$. This is half the exponent appearing in Phung's bound for the much larger set $I(s, B_0)$.

The main result of this paper eliminates this discrepancy:

Theorem 1.4. *Assume that the setting (P) holds. Then for every $\varepsilon > 0$ there exists $m := m(\mathcal{A}, \mathcal{D}, B_0, \varepsilon) \in \mathbb{R}_{>0}$ such that*

$$\# \left({I(s, B_0)} \big/ {\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})} \right) \leq m(s+1)^{n\;\mathrm{rk}(\pi_1(B_0,b_0))+\varepsilon}\,. \tag{3}$$

The key ingredient behind Theorem 1.4 is a new, essentially optimal, upper bound for the Kobayashi length of loops on punctured bordered surfaces (Theorem 1.5). Let us briefly describe the context. Let B_0 be the bordered hyperbolic surface introduced above. For a finite set $S \subset B_0$, the punctured surface $B_0 \smallsetminus S$ carries its own Kobayashi metric, and the induced length ℓ_S of paths in $B_0 \smallsetminus S$ is larger than the length of the same paths measured in the Kobayashi metric of B_0: the new cusps created by the punctures stretch the metric, so loops become longer as the cardinality of S grows. Given a loop α representing a nontrivial class in $\pi_1(B_0, b_0)$, it is natural to ask how the minimal Kobayashi length of a representative of α in $B_0 \smallsetminus S$ grows with the number of punctures. This is captured by the quantity

$$L(\alpha, s) := \sup_{\substack{S \subset B_0 \\ \#S \leq s}} \inf \{\ell_S(\gamma) : \gamma \subset B_0 \smallsetminus S \text{ is a loop freely homotopic to } \alpha \text{ in } B_0\} ,$$

which measures the worst-case minimal Kobayashi length over all configurations of at most s punctures. Phung [10, Theorems B, C] proved that $L(\alpha, s)$ grows at least as $c\sqrt{s}/\log(s+2)$ and at most linearly in s. As explained above, the linear upper bound $L(\alpha, s) = O(s)$ is what produces the factor of 2 in the exponent of (2): replacing $H(s) = O(s)$ by $H(s) = O(\sqrt{s \log s})$ in the lattice counting reduces the exponent from $2nk$ to $nk + \varepsilon$, for every $\varepsilon > 0$ (the logarithmic factor being absorbed into the ε). We prove:

Theorem 1.5. *Fix a loop $\alpha \subset B_0$ representing a nontrivial class in $\pi_1(B_0, b_0)$. There exists $C > 0$, depending only on α, B_0, and ρ, such that*

$$L(\alpha, s) \leq C\sqrt{(s+1)\log(s+2)} \quad \textit{for all } s \geq 0 . \tag{4}$$

Combined with Phung's lower bound, this determines the exact growth rate (Corollary 2.6):

$$\lim_{s \to +\infty} \frac{\log L(\alpha, s)}{\log s} = \frac{1}{2} . \tag{5}$$

This answers [10, Question 1.4] which asked about the asymptotic behavior of $\frac{\log L(\alpha,s)}{\log s}$.

The proof of Theorem 1.5 proceeds in two steps. First, we establish the bound under the assumption that α admits a smooth *simple* representative $\gamma \Subset B_0$. We consider the Fermi strip $T_\delta(\gamma)$, a thin tubular neighborhood of γ foliated by parallel curves γ_u (the simplicity of γ ensures that the exponential map from the normal bundle is a diffeomorphism for δ small enough). Phung's linear upper bound $L(\alpha, s) = O(s)$ in [10] is obtained by estimating the Kobayashi length of a single loop directly; our improvement replaces this pointwise approach with an integral averaging argument. The key idea is to bound the L^p-norm of the distortion function λ_S (the supremal directional ratio of the Kobayashi–Royden metric of $B_0 \setminus S$ to the background metric ρ) over the entire strip, for a variable exponent $1 < p < 2$. This is achieved by a Voronoi decomposition of the strip into cells centered at the punctures: near each puncture, λ_S blows up like the inverse of the distance, and the L^p-integrability for $p < 2$ (but not for $p = 2$) controls the singularity. The crucial gain over the linear bound is that each of the s punctures contributes only $O(1/(2-p))$ to the L^p-mass of λ_S on the strip, so the total mass is $O(s/(2-p))$; Hölder's inequality then converts this L^p-bound into an upper bound of order $(s/(2-p))^{1/p}$—sublinear in s—on the Kobayashi length of a typical parallel curve γ_{u_0}, which is freely homotopic to α in B_0 and avoids S. Optimising the exponent as $p = 2 - 1/\log(s+2)$ balances the divergence of the L^p-norm as $p \to 2^-$ against the sharpness of the Hölder estimate, yielding the $O(\sqrt{(s+1)\log(s+2)})$ bound.

In the second step, the general case is reduced to the simple case via a word decomposition argument: B_0 has the homotopy type of a compact orientable surface of genus g with t boundary circles, so $\pi_1(B_0, b_0)$ is a free group admitting a basis $\alpha_1, \ldots, \alpha_k$ of *smooth simple loops* (the generators around the disks, together with the standard genus generators when $g \geq 1$); any nontrivial $\alpha \in \pi_1(B_0, b_0)$ can be written as a word $\alpha_{i_1}^{\epsilon_1} \cdots \alpha_{i_m}^{\epsilon_m}$ in this basis. Applying the simple case to each basis loop and concatenating the resulting parallel curves at a common base point (which can be arranged, see Proposition 2.5 below) yields a representative of α avoiding S whose Kobayashi length is at most m times the bound for a single basis loop.

Theorem 1.4 follows from Theorem 1.5 by substituting the improved length estimate into the Parshin cocycle argument outlined above. The cocycle construction requires loops representing the generators $\alpha_1, \ldots, \alpha_k$ that share a *common base point* and are conjugated to the generators by a *single* path. Theorem 1.5, applied independently to each generator, produces loops with the optimal Kobayashi length bound but with potentially different base points. In Proposition 2.5 we show that, by choosing smooth simple representatives of $\alpha_1, \ldots, \alpha_k$ that share a common tangent direction at b_0 (so that their Fermi strips can be controlled by a single averaging parameter), a common base point can be arranged without degrading the asymptotic bound. The improved displacement bound $H(s) = O(\sqrt{s \log s})$ then propagates through the lattice counting argument unchanged, halving the exponent.

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2 Kobayashi lengths estimates

2.1 Kobayashi pseudo-metric

Let X be a connected complex manifold and let TX be its tangent bundle. Throughout, TX denotes the *real* tangent bundle $T^{\mathbb{R}}X$, regarded as a complex vector bundle (of complex rank $\dim_{\mathbb{C}} X$) via the complex structure J, so that $\lambda v := (\operatorname{Re}\lambda)\, v + (\operatorname{Im}\lambda)\, Jv$ for $\lambda \in \mathbb{C}$ and $v \in T^{\mathbb{R}}_{X,x}$; this is canonically $\mathbb{C}$-isomorphic to the holomorphic tangent bundle $T^{1,0}X$ and we use the two interchangeably. In particular it is *not* the complexification $T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$, whose complex rank is twice as large.

Definition 2.1. A *Finsler pseudo-metric* on X is a function

$$\begin{aligned} F\colon TX &\to \mathbb{R}_{\geq 0} \\ (x,v) &\mapsto F(x,v) \end{aligned}$$

satisfying the homogeneity condition $F(x, \lambda v) = |\lambda|\, F(x,v)$ for all $\lambda \in \mathbb{C}$. If $F(x,v) > 0$ for all $x \in X$ and $v \in T_{X,x} \setminus \{0\}$, then we say that F is a *Finsler metric*.

Throughout, by a *path* we mean a piecewise C^1 map $\gamma\colon [0,1] \to X$; this entails no loss of generality for the distances defined below. Given a Borel measurable Finsler pseudo-metric F and a path γ, the function $t \mapsto F(\gamma(t), \gamma'(t))$ is measurable, and we define the length of γ as

$$\ell_F(\gamma) := \int_0^1 F(\gamma(t), \gamma'(t))\, dt \ \in [0, +\infty].$$

When F is *upper semicontinuous* we have $\ell_F(\gamma) < +\infty$, since F is then locally bounded above and $\gamma([0,1])$ is compact. Given a set of paths Γ we set

$$\ell_F(\Gamma) := \inf_{\gamma \in \Gamma} \ell_F(\gamma)\,.$$

We endow X with the structure of extended pseudo-metric space by setting

$$d_F(x,y) := \inf_{\gamma} \ell_F(\gamma)\,, \quad \text{where the infimum runs over all paths } \gamma \text{ joining } x \text{ and } y\,.$$

Moreover, if Y, Z are two subsets of X, we put

$$\operatorname{diam}_F(Y) := \sup_{y,y' \in Y} d_F(y,y')\,, \quad d_F(Y,Z) := \inf_{y \in Y,\, z \in Z} d_F(y,z)\,.$$

One can show that a *continuous* Finsler metric induces a metric d_F. Any Hermitian metric h on X induces a Finsler metric $F_h(x,v) = \sqrt{h_x(v,v)}$, hence a distance which we denote by d_h.

An important example of a Finsler pseudo-metric is the *Kobayashi–Royden pseudo-metric*

$$\kappa_X(x,v) := \inf \left\{ \frac{2}{R} \in \mathbb{R}_{>0} \colon \exists\, f \in \operatorname{Hol}(\Delta(0,R), X) \text{ such that } f(0) = x,\ f'(0) = v \right\},$$

where $\operatorname{Hol}(\Delta(0,R), X)$ is the set of holomorphic maps from the open disk of radius R centered at 0 to X. Royden [11, Proposition 3] shows that κ_X is upper semicontinuous, hence Borel measurable. Since κ_X is intrinsic to X, the induced topological pseudo-metric is denoted by d_X and called the *Kobayashi pseudo-metric*. We say that X is *Kobayashi hyperbolic* if d_X is a metric.

If X carries a Hermitian metric h whose holomorphic sectional curvature is bounded above by a negative constant, then by the Ahlfors–Schwarz lemma one has $\kappa_X \geq c\, \|\cdot\|_h$ for some constant $c > 0$; in particular $d_X \geq c\, d_h$ and X is Kobayashi hyperbolic. In dimension ≥ 2 the converse is a delicate matter and is not known to hold in general. On a Riemann surface, however, it holds: if X is Kobayashi hyperbolic its universal cover is the disk, so X is biholomorphic to $\Gamma\backslash\mathbb{H}$ for a torsion-free Fuchsian group $\Gamma \cong \pi_1(X)$ (no elliptic elements, parabolics allowed). The curvature -1 Poincaré metric $ds^2 = (dx^2+dy^2)/y^2$ on $\mathbb{H}$ descends to X, and since holomorphic covering maps are infinitesimal isometries for κ_X, the descended metric *is* the Kobayashi metric. Thus on Riemann surfaces "admits a Hermitian metric of holomorphic sectional curvature bounded above by a negative constant" and "Kobayashi hyperbolic" coincide.

From the definition one deduces the distance-decreasing property of the Kobayashi–Royden metric: if $\phi\colon X \to Y$ is holomorphic, then

$$\kappa_Y(\phi(x), d_x\phi(v)) \leq \kappa_X(x,v)\,, \quad \forall (x,v) \in TX\,.$$

This in turn implies the distance-decreasing property of the Kobayashi distance

$$d_Y(\phi(x), \phi(x')) \leq d_X(x,x')\,, \quad \forall x,x' \in X\,.$$

2.2 Growth of loop lengths in terms of the cusps

We fix a compact Riemann surface B of genus g endowed with a Riemannian metric ρ. Let $\overline{\Delta}_1,\ldots,\overline{\Delta}_t$ be disjoint disks on B and define $B_0 := B \smallsetminus \sqcup_{i=1}^t \overline{\Delta}_i$. We assume that $t \geq 3$ so that B_0 is hyperbolic. Let $S \subseteq B_0$ be a finite set (possibly empty), put $s := \#S$, and let κ_S be the Kobayashi–Royden metric of the hyperbolic surface $B_0 \smallsetminus S$, with induced length and distance ℓ_S, d_S. We define the *distortion function*

$$\lambda_S(b) := \sup_{v\in T_bB\smallsetminus\{0\}} \frac{\kappa_S(b,v)}{\sqrt{\rho_b(v,v)}}\,, \qquad b \in B_0 \smallsetminus S\,, \tag{6}$$

where $T_bB = T_b^{\mathbb{R}}B$ is the *real* tangent plane (real dimension 2), on which both $\kappa_S(b,\cdot)$ and $\sqrt{\rho_b(\cdot,\cdot)}$ are defined. The ratio is homogeneous of degree 0, so the supremum may be taken over the compact ρ-unit circle, where it is the upper semicontinuous function $\kappa_S(b,\cdot)$; thus $\lambda_S(b)$ is finite and attained, and λ_S is Borel on $B_0 \smallsetminus S$. In a holomorphic coordinate $z = x+iy$, identifying $a\partial_x + b'\partial_y \leftrightarrow (a+ib')\partial_z$, $\mathbb{C}$-homogeneity gives $\kappa_S(b, a\partial_x + b'\partial_y) = \kappa_S(b,\partial_z)\sqrt{a^2+b'^2}$, while ρ_b has eigenvalues $\mu_1(b) \leq \mu_2(b)$ in the frame (∂_x,∂_y). Maximizing the ratio over the Euclidean unit circle thus minimizes $\rho_b(v,v)$, giving

$$\lambda_S(b) = \frac{\kappa_S(b,\partial_z)}{\sqrt{\mu_1(b)}}\,.$$

If ρ is conformal ($\mu_1 = \mu_2$, or equivalently ρ is hermitian) the ratio of Equation (6) is independent of v (so that the supremum on v is superfluous) and λ_S is smooth; we do not assume this.

Standing assumption for §2.2 (lifted in §2.3). Throughout this subsection, we fix a loop α in B_0 that doesn't represent a trivial class in $\pi_1(B_0,b_0)$ *and that admits a smooth simple representative* $\gamma \Subset B_0$ freely homotopic to α (such loops exist abundantly: for instance, the basis loops of $\pi_1(B_0,b_0)$ described in §2.3 are smooth simple). Existence of a smooth representative for any free homotopy class is given by [6, Theorem 6.26]; the simplicity is part of our standing assumption here, and the general case (no simplicity hypothesis) is reduced to this one in §2.3.

Parametrize γ by its ρ-arc-length and let $\eta(\tau)$ be the unit normal vector of γ at τ; here and throughout we fix the convention that $\eta(\tau)$ is the rotation of the unit tangent vector $\gamma'(\tau)$ by $+\pi/2$, with respect to ρ and the orientation of the Riemann surface B. Since γ is smooth and simple and compactly contained in B_0, for $\delta \in \mathbb{R}_{>0}$ sufficiently small the exponential map from the normal bundle is a diffeomorphism, yielding:

$$\begin{aligned} \mathbb{R}/\ell_\rho(\gamma)\mathbb{Z} \times [-\delta,\delta] &\to \{b \in B_0 \colon d_\rho(b,\gamma) \leq \delta\} =: T_\delta(\gamma) \Subset B_0 \\ (\tau,u) &\mapsto \exp_{\gamma(\tau)}(u\eta(\tau)) \end{aligned}$$

The coordinates (τ,u) are called *Fermi coordinates*, and $T_\delta(\gamma)$ is the *Fermi strip* with respect to γ. For any $u \in [-\delta,\delta]$ we define the *parallel curve* $\gamma_u\colon [0,\ell_\rho(\gamma)] \to B_0$, $\gamma_u(\tau) := \exp_{\gamma(\tau)}(u\eta(\tau))$.

In Fermi coordinates the induced metric on each parallel curve γ_u is $\rho|_{\gamma_u} = \rho_{\tau\tau}\, d\tau^2$, so its ρ-length is

$$\ell_\rho(\gamma_u) = \int_0^{\ell_\rho(\gamma)} \sqrt{\rho_{\tau\tau}(\tau,u)}\, d\tau\,.$$

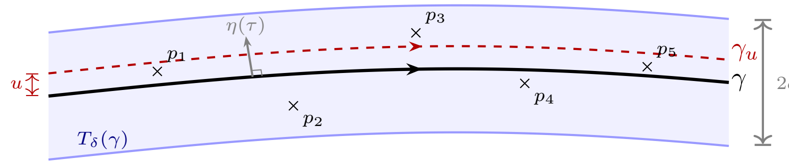

Figure 1: The Fermi strip $T_\delta(\gamma)$ around γ, with punctures $p_1, \ldots, p_5 \in S$ and the parallel curve γ_u.

Write the Riemannian area form in Fermi coordinates as $dA_\rho = J(\tau, u)\, d\tau\, du$, where

$$J(\tau, u) := \sqrt{\rho_{\tau\tau}\rho_{uu} - \rho_{\tau u}^2}$$

is the square root of the determinant of the metric ρ in the coordinates (τ, u). We claim that, for δ small enough, J and $\sqrt{\rho_{\tau\tau}}$ are uniformly comparable on the strip. Indeed, by the properties of Fermi coordinates ([7, Proposition 5.26]), the metric at $u = 0$ satisfies $\rho_{\tau u}(\tau, 0) = 0$ and $\rho_{uu}(\tau, 0) = 1$, so $J(\tau, 0) = \sqrt{\rho_{\tau\tau}(\tau, 0)}$. Now $\rho_{\tau\tau}(\cdot, 0)$ is continuous and attains a positive minimum on the compact circle $[0, \ell_\rho(\gamma)] \times \{0\}$; by uniform continuity of $\rho_{\tau\tau}$ there is a $\delta > 0$—which we also shrink, if necessary, so that the exponential map of the normal bundle is a diffeomorphism onto $T_\delta(\gamma)$ —such that $\rho_{\tau\tau} > 0$ on the whole strip $[0, \ell_\rho(\gamma)] \times [-\delta, \delta]$. Then J and $\sqrt{\rho_{\tau\tau}}$ are continuous and strictly positive there, so the ratio $J/\sqrt{\rho_{\tau\tau}}$ is a continuous positive function on a compact set. It follows that there exists $\kappa_0 = \kappa_0(\rho, \gamma, \delta) \geq 1$ such that

$$\kappa_0^{-1}\sqrt{\rho_{\tau\tau}(\tau, u)} \leq J(\tau, u) \leq \kappa_0\sqrt{\rho_{\tau\tau}(\tau, u)} \qquad \forall\, (\tau, u) \in [0, \ell_\rho(\gamma)] \times [-\delta, \delta]\,. \tag{7}$$

In particular, for any measurable $f \geq 0$:

$$\kappa_0^{-1}\int_{-\delta}^{\delta}\int_0^{\ell_\rho(\gamma)} f(\tau, u)\sqrt{\rho_{\tau\tau}(\tau, u)}\, d\tau\, du \;\leq\; \int_{T_\delta(\gamma)} f\, dA_\rho \;\leq\; \kappa_0\int_{-\delta}^{\delta}\int_0^{\ell_\rho(\gamma)} f(\tau, u)\sqrt{\rho_{\tau\tau}(\tau, u)}\, d\tau\, du\,. \tag{8}$$

Lemma 2.2. *There exists a constant $A := A(\rho, B_0, \alpha) \in \mathbb{R}_{>0}$ such that for every $1 \leq p < 2$ and every finite set $S \subset B_0$:*

$$\int_{T_\delta(\gamma)} \lambda_S^p\, dA_\rho \leq \frac{A(s+1)}{2-p}\,.$$

Proof. Throughout this proof, $\Delta(\cdot, \cdot) = \Delta_\rho(\cdot, \cdot)$ denotes a *metric* ball for the Riemannian distance d_ρ (not a removed topological disk of B_0). Assume $S = \{p_1, \ldots, p_s\} \neq \emptyset$. Consider the Voronoi decomposition of $T_\delta(\gamma)$ induced by S: for each $p_j \in S$, define

$$V_j := \{b \in T_\delta(\gamma) \colon d_\rho(b, p_j) \leq d_\rho(b, p_i)\ \forall i \neq j\}\,.$$

Note that the punctures p_j need not lie inside $T_\delta(\gamma)$. Nevertheless, the cells cover the strip: $T_\delta(\gamma) = \bigcup_{j=1}^s V_j$ (some cells V_j may be empty, and distinct cells may overlap along tie sets).

Step 1: Pointwise and area bounds. Put $T := T_\delta(\gamma)$ and fix $b \in T \setminus S$. Set $D_b := d_\rho(b, S) > 0$. By [10, Lemma 3.3], there exist constants $c_1, r_1 > 0$ (depending only on B and ρ) such that for every $b \in B$, every $v \in T_bB \setminus \{0\}$, and every $0 < r < r_1$:

$$\kappa_{\Delta(b,r)}(b, v) \;\leq\; \frac{c_1}{r}\sqrt{\rho_b(v, v)}\,.$$

By [10, Lemma 3.5], there exist constants $c_2, r_2 > 0$ such that $\mathrm{Area}_\rho(\Delta(q, r)) \leq c_2\, r^2$ for all $q \in B$ and $0 < r \leq r_2$. Set $D := \min\{d_\rho(T, \partial B_0),\, r_1,\, r_2\} > 0$ and $R_b := \min\{D_b, D\}$. Then $\Delta(b, R_b) \subset B_0 \setminus S$ (it lies in B_0 because $R_b \leq D \leq d_\rho(T, \partial B_0)$, and avoids S because $R_b \leq D_b$), so the decreasing property of the Kobayashi–Royden metric (applied to the inclusion $\Delta(b, R_b) \hookrightarrow B_0 \smallsetminus S$) gives, for every $v \in T_bB \smallsetminus \{0\}$, the bound $\kappa_{B_0\setminus S}(b, v) \leq \kappa_{\Delta(b,R_b)}(b, v)$. Taking the supremum over v and using that [10, Lemma 3.3] holds uniformly in v:

$$\lambda_S(b) \;=\; \sup_{v\neq 0}\frac{\kappa_{B_0\setminus S}(b, v)}{\sqrt{\rho_b(v, v)}} \;\leq\; \sup_{v\neq 0}\frac{\kappa_{\Delta(b,R_b)}(b, v)}{\sqrt{\rho_b(v, v)}} \;\leq\; \frac{c_1}{R_b} \;=\; \frac{c_1}{\min\{D_b,\, D\}}\,. \tag{9}$$

Moreover, since $D \leq r_2$:

$$\mathrm{Area}_\rho(\Delta(q, r)) \;\leq\; c_2\, r^2 \qquad \text{for all } q \in B \text{ and } r \leq D\,. \tag{10}$$

Step 2: Integral over Voronoi cells. For $b \in V_j$, the point p_j is the nearest element of S to b, so $D_b = d_\rho(b, p_j)$. We split V_j into two regions:

$$V_j^{\mathrm{near}} := \{b \in V_j \colon d_\rho(b, p_j) \le D\}\,, \qquad V_j^{\mathrm{far}} := \{b \in V_j \colon d_\rho(b, p_j) > D\}\,.$$

On V_j^{near} we have $R_b = D_b = d_\rho(b, p_j)$, so (9) gives $\lambda_S(b) \le c_1/d_\rho(b, p_j)$. By the "layer cake representation" in measure theory ($\int f^p\, d\mu = p\int_0^\infty t^{p-1}\mu(\{f > t\})\, dt$, see [8, Theorem 1.13]) and the substitution $r = 1/t$:

$$\begin{aligned}\int_{V_j^{\mathrm{near}}} \lambda_S^p \, dA_\rho &\le c_1^p \int_{V_j^{\mathrm{near}}} \frac{dA_\rho}{d_\rho(b, p_j)^p} \\ &= p\, c_1^p \int_0^{+\infty} t^{p-1}\, \mathrm{Area}_\rho\big\{b \in V_j^{\mathrm{near}} \colon d_\rho(b, p_j)^{-1} > t\big\}\, dt \\ &= p\, c_1^p \int_0^{+\infty} \frac{\mathrm{Area}_\rho(V_j^{\mathrm{near}} \cap \Delta(p_j, r))}{r^{p+1}}\, dr \\ &\le p\, c_1^p \int_0^{D} \frac{\mathrm{Area}_\rho(\Delta(p_j, r))}{r^{p+1}}\, dr + p\, c_1^p \int_D^{+\infty} \frac{\mathrm{Area}_\rho(T)}{r^{p+1}}\, dr \\ &\le \frac{p\, c_1^p\, c_2\, D^{2-p}}{2-p} + \frac{c_1^p\, \mathrm{Area}_\rho(T)}{D^p}\,, \end{aligned} \tag{11}$$

where the fourth line uses $V_j^{\mathrm{near}} \cap \Delta(p_j, r) \subset \Delta(p_j, r)$ for $r \le D$, and $\mathrm{Area}_\rho(V_j^{\mathrm{near}} \cap \Delta(p_j, r)) \le \mathrm{Area}_\rho(T)$ for $r \ge D$. Moreover the fifth line uses the area bound (10) and $\int_D^{+\infty} r^{-(p+1)}\, dr = D^{-p}/p$. Note that the integral $\int_0^D r^{1-p}\, dr$ converges since $p < 2$.

On V_j^{far} we have $R_b = D$, so $\lambda_S(b) \le c_1/D$ and

$$\int_{V_j^{\mathrm{far}}} \lambda_S^p\, dA_\rho \le \frac{c_1^p}{D^p}\, \mathrm{Area}_\rho(T)\,. \tag{12}$$

Set $c_3 := \max\big\{2\max\{1, c_1^2\}\, c_2 \max\{1, D\},\ 2\max\{c_1/D,\, c_1^2/D^2\}\, \mathrm{Area}_\rho(T)\big\}$. Since $1 \le p < 2$ we have $c_1^p \le \max\{1, c_1^2\}$, $D^{2-p} \le \max\{1, D\}$, and $c_1^p/D^p \le \max\{c_1/D,\, c_1^2/D^2\}$, so combining (11) and (12):

$$\int_{V_j} \lambda_S^p\, dA_\rho \le \frac{c_3}{2-p} + c_3 \le \frac{2c_3}{2-p}\,, \tag{13}$$

where the last inequality uses $1 \le 1/(2-p)$ for $p \ge 1$.

Step 3: Summing over cells. Since the cells V_j cover T and $\lambda_S^p \ge 0$, subadditivity of the integral gives (cells with $V_j = \emptyset$ contribute zero):

$$\int_T \lambda_S^p\, dA_\rho \le \sum_{j=1}^{s} \int_{V_j} \lambda_S^p\, dA_\rho \le \frac{2c_3 s}{2-p}\,.$$

If $s = 0$ (i.e. $S = \emptyset$) on T we have

$$\lambda_S(b) = \sup_{v \ne 0} \frac{\kappa_{B_0}(b, v)}{\sqrt{\rho_b(v, v)}} \le M := \sup_{b \in T}\, \sup_{v \ne 0} \frac{\kappa_{B_0}(b, v)}{\sqrt{\rho_b(v, v)}} < +\infty\,;$$

where the finiteness follows from $T \Subset B_0$. In this case we have

$$\int_T \lambda_S^p\, dA_\rho \le M^p\, \mathrm{Area}_\rho(T) \le \max\{1, M^2\}\, \mathrm{Area}_\rho(T)\,.$$

Setting $A := \max\{2c_3,\, \max\{1, M^2\}\, \mathrm{Area}_\rho(T)\}$ the claim is proved. □

Let us now introduce the main object of this section and then prove the main result.

Definition 2.3. Fix a loop α in B_0 that doesn't represent a trivial class in $\pi_1(B_0, b_0)$. Let Γ_S^α denote the set of loops in $B_0 \setminus S$ that are freely homotopic to α in B_0. We define

$$L(\alpha, s) := \sup_{\substack{S \subset B_0 \\ \#S \le s}} \ell_S(\Gamma_S^\alpha)\,.$$

Note that since α is chosen (non-trivially) in B_0 and the free homotopies are checked in the non-cuspidal hyperbolic surface B_0, we have $\ell_S(\Gamma_S^\alpha) > 0$ by [1, Proposition 2.24].

Proof of Theorem 1.5 under the standing simplicity assumption. Consider the smooth simple loop $\gamma \Subset B_0$ freely homotopic to α, fixed by the standing assumption at the beginning of the section. Put $T := T_\delta(\gamma)$ parametrized with Fermi coordinates (τ, u) as above. Set

$$L_0 := \sup_{|u|\le\delta} \ell_\rho(\gamma_u) < +\infty\,.$$

Fix an arbitrary $S \subset B_0$ with $\#S = s$ and a parameter $1 < p < 2$. Define $q := \frac{p}{p-1}$ (the conjugate exponent, so $1/p + 1/q = 1$). Recall that in Fermi coordinates $\gamma_u'(\tau) = \partial_\tau|_{(\tau,u)}$ and $\rho_{\tau\tau}(\tau, u) = \rho_{\gamma_u(\tau)}(\gamma_u'(\tau), \gamma_u'(\tau))$, so by the definition of λ_S as a directional supremum,

$$\kappa_S(\gamma_u(\tau), \gamma_u'(\tau)) \le \lambda_S(\gamma_u(\tau)) \sqrt{\rho_{\gamma_u(\tau)}(\gamma_u'(\tau), \gamma_u'(\tau))} = \lambda_S(\tau, u) \sqrt{\rho_{\tau\tau}(\tau, u)}\,.$$

For each $u \in]-\delta, \delta[$ with $\gamma_u \cap S = \emptyset$, Hölder's inequality with respect to the measure $d\mu = \sqrt{\rho_{\tau\tau}}\, d\tau$ then gives:

$$\begin{aligned}\ell_S(\gamma_u) &= \int_0^{\ell_\rho(\gamma)} \kappa_S(\gamma_u(\tau), \gamma_u'(\tau))\, d\tau \\ &\le \int_0^{\ell_\rho(\gamma)} \lambda_S(\tau, u) \sqrt{\rho_{\tau\tau}(\tau, u)}\, d\tau \\ &\le \left(\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau, u)^p \sqrt{\rho_{\tau\tau}}\, d\tau\right)^{1/p} \left(\int_0^{\ell_\rho(\gamma)} \sqrt{\rho_{\tau\tau}}\, d\tau\right)^{1/q} \\ &= \left(\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau, u)^p \sqrt{\rho_{\tau\tau}}\, d\tau\right)^{1/p} \ell_\rho(\gamma_u)^{1/q}\,.\end{aligned} \tag{14}$$

By Equation (8) and Lemma 2.2 we get:

$$\int_{-\delta}^{\delta} \int_0^{\ell_\rho(\gamma)} \lambda_S(\tau, u)^p \sqrt{\rho_{\tau\tau}}\, d\tau\, du \;\le\; \kappa_0 \int_T \lambda_S^p\, dA_\rho \;\le\; \frac{\kappa_0 A(s+1)}{2-p}\,. \tag{15}$$

The set $E := \{u \in]-\delta, \delta[: \gamma_u \cap S \ne \emptyset\}$ has at most s points, hence has measure zero. Define $g(u) := \int_0^{\ell_\rho(\gamma)} \lambda_S(\tau, u)^p \sqrt{\rho_{\tau\tau}}\, d\tau$ and $M := \frac{\kappa_0 A(s+1)}{2\delta(2-p)}$. By (15),

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} g(u)\, du \;\le\; M\,. \tag{16}$$

We claim there exists $u_0 \in]-\delta, \delta[\setminus E$ with $g(u_0) \le M$. If not, then $g(u) > M$ for all $u \in]-\delta, \delta[\setminus E$. Since $g \ge 0$ and E has measure zero:

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} g(u)\, du = \frac{1}{2\delta} \int_{]-\delta,\delta[\setminus E} g(u)\, du > \frac{1}{2\delta} \cdot M \cdot 2\delta = M\,,$$

contradicting (16). Hence such u_0 exists, and since $u_0 \notin E$ we have $\gamma_{u_0} \cap S = \emptyset$ and

$$\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau, u_0)^p \sqrt{\rho_{\tau\tau}(\tau, u_0)}\, d\tau \;=\; g(u_0) \;\le\; M \;=\; \frac{\kappa_0 A(s+1)}{2\delta(2-p)}\,. \tag{17}$$

Substituting Equation (17) into Equation (14) and using $\ell_\rho(\gamma_{u_0}) \le L_0$ we get

$$\ell_S(\gamma_{u_0}) \le \left(\frac{\kappa_0 A}{2\delta}\right)^{1/p} L_0^{1/q} \left(\frac{s+1}{2-p}\right)^{1/p}\,. \tag{18}$$

Since the Fermi coordinate map is a diffeomorphism, the family $\{\gamma_u\}_{u\in[0,u_0]}$ is a free homotopy in $T_\delta(\gamma) \subset B_0$ from $\gamma_0 = \gamma$ to γ_{u_0}. As γ is freely homotopic to α in B_0 (by construction), γ_{u_0} is freely homotopic to α in B_0 by transitivity. Moreover $\gamma_{u_0} \subset B_0 \setminus S$ (since $u_0 \notin E$), so $\gamma_{u_0} \in \Gamma_S^\alpha$.

We now assume $s \ge 1$ (the case $s = 0$ is handled below) and set

$$p := 2 - \frac{1}{\log(s+2)}\,, \quad \text{so that} \quad 2 - p = \frac{1}{\log(s+2)}\,, \quad p > 1\,.$$

Then

$$\frac{s+1}{2-p} = (s+1)\log(s+2)\,, \qquad \frac{1}{p} = \frac{\log(s+2)}{2\log(s+2)-1}\,.$$

We split

$$\left(\frac{s+1}{2-p}\right)^{1/p} = \big((s+1)\log(s+2)\big)^{1/2} \cdot \big((s+1)\log(s+2)\big)^{1/p-1/2}$$

and claim the second factor is $O(1)$. Indeed, $1/p - 1/2 = O(1/\log s)$ and

$$\log\big((s+1)\log(s+2)\big) = \log(s+1) + \log\log(s+2) = O(\log s)\,,$$

so

$$\big((s+1)\log(s+2)\big)^{1/p-1/2} = \exp\!\big(O(1/\log s)\cdot O(\log s)\big) = \exp(O(1)) = O(1)\,.$$

Consider now the first factor of Equation (18), namely $C_0(p) := (\kappa_0 A/(2\delta))^{1/p} L_0^{1/q}$. As $p \to 2^-$ we have $1/p \to 1/2$ and $1/q \to 1/2$, so $C_0(p) \to (\kappa_0 A/(2\delta))^{1/2} L_0^{1/2}$. Since C_0 is continuous on $[2-1/\log 3, 2[$ and has a finite limit at $p=2$, it is bounded there.

Combining, we conclude that, *under the standing assumption that α admits a smooth simple representative $\gamma \Subset B_0$*, there exists $C > 0$, depending only on α, B_0, and ρ, such that for every $S \subset B_0$ with $\#S = s \geq 1$:

$$\ell_S(\Gamma_S^\alpha) \leq \ell_S(\gamma_{u_0}) \leq C\sqrt{(s+1)\log(s+2)}\,.$$

Since this holds for every such S, and since sets with $\#S = s' < s$ satisfy the same bound a fortiori ($s \mapsto \sqrt{(s+1)\log(s+2)}$ being increasing), taking the supremum over all $S \subset B_0$ with $\#S \leq s$ gives $L(\alpha, s) \leq C\sqrt{(s+1)\log(s+2)}$. For $s = 0$, $L(\alpha, 0) = \inf_\beta \ell_{\kappa_{B_0}}(\beta)$ is a fixed positive constant, which is absorbed into C. ☐

For the application to generalized integral points, the Parshin cocycle argument requires loops representing the generators of $\pi_1(B_0, b_0)$ that share a *common base point* and are conjugated to the generators by a *single* path. Theorem 1.5, applied independently to each generator, produces loops with the optimal Kobayashi length bound but with potentially different base points. The next proposition shows that a common base point can be arranged without degrading the asymptotic bound.

Lemma 2.4. *Let M be a smooth orientable Riemannian surface without boundary, let $p \in M$, and let $w \in T_pM$ be a unit tangent vector. Then for every smooth simple loop γ at p with $\gamma \Subset M$, there exists a smooth simple loop $\tilde\gamma$ at p with $\tilde\gamma \Subset M$, homotopic to γ relative to p, and satisfying*

$$\frac{\tilde\gamma'(0)}{|\tilde\gamma'(0)|} = w\,.$$

Proof. Let $w_0 := \gamma'(0)/|\gamma'(0)|$ be the unit tangent vector of γ at p. If $w_0 = w$ there is nothing to prove, so assume $w_0 \neq w$. Let $\theta \in]0, 2\pi[$ be the angle from w_0 to w and for each $t \in [0,1]$ let $R_t \in \mathrm{SO}(2)$ denote the rotation of T_pM by angle $t\theta$, so that $R_0 = \mathrm{Id}$ and $R_1(w_0) = w$.

Choose $\epsilon > 0$ small enough that the exponential map $\exp_p$ is a diffeomorphism from $B(0,\epsilon) \subset T_pM$ onto an open neighborhood U of p in M with $U \Subset M$. Choose a smooth cutoff function $\chi\colon [0,\infty) \to [0,1]$ with $\chi \equiv 1$ on $[0, \epsilon/3]$ and $\chi \equiv 0$ on $[2\epsilon/3, \infty)$. Define the map $\phi\colon M \to M$ by

$$\phi(x) := \begin{cases} \exp_p\!\big(R_{\chi(|y|)}\, y\big), & \text{if } x = \exp_p(y) \in U, \\ x, & \text{if } x \notin U, \end{cases}$$

where $|y|$ denotes the norm in T_pM. Since $\chi \equiv 1$ near $|y| = 0$, the map ϕ equals $\exp_p \circ R_1 \circ \exp_p^{-1}$ in a neighborhood of p, and since $\chi \equiv 0$ for $|y| \geq 2\epsilon/3$, ϕ is the identity outside U. In geodesic polar coordinates (r,ψ) centered at p (via $\exp_p$), the map reads $\phi(r,\psi) = (r, \psi + \chi(r)\theta)$: it preserves the radius r and rotates each geodesic circle $\{r = \text{const}\}$ by the angle $\chi(r)\theta$. On each circle this is a bijection, and distinct circles map to distinct circles, so ϕ is a bijection of U fixing p; together with $\phi = \mathrm{id}$ outside U this makes ϕ a bijection of M. It is smooth with smooth inverse $\phi^{-1}(r,\psi) = (r, \psi - \chi(r)\theta)$ (both expressions are smooth across $r = 0$ because $\chi \equiv 1$ there, where ϕ is the fixed rotation R_1), so ϕ is a diffeomorphism of M. Moreover:

(i) $\phi(p) = p$ and $d_p\phi = R_1$, so $(\phi\circ\gamma)'(0) = R_1(\gamma'(0)) = |\gamma'(0)|\, w$, which has unit tangent direction w;

(ii) ϕ is the identity outside U, so $\phi\circ\gamma$ coincides with γ outside U and in particular $\phi\circ\gamma \Subset M$;

(iii) ϕ is a diffeomorphism, so $\phi\circ\gamma$ is simple;

(iv) the family ϕ_t (defined by replacing χ with $t\chi$ for $t\in[0,1]$) satisfies $\phi_0=\mathrm{Id}$ and $\phi_1=\phi$, and $\phi_t(p)=p$ for all t. Hence $t\mapsto\phi_t\circ\gamma$ is a homotopy of loops based at p from γ to $\tilde\gamma$, so $[\tilde\gamma]=[\gamma]$ in $\pi_1(M,p)$.

Setting $\tilde\gamma:=\phi\circ\gamma$ completes the proof. □

Proposition 2.5. *Let B_0 be as above and fix a simple basis $\alpha_1,\dots,\alpha_k$ of $\pi_1(B_0,b_0)$. There exist constants $C,\delta'>0$, depending only on $\alpha_1,\dots,\alpha_k$, B_0, and ρ, such that for every finite set $S\subset B_0$ with $\#S=s\ge 1$, there exist a point $q\in B_0\setminus S$, a path σ from b_0 to q in B_0 with $\ell_\rho(\sigma)\le\delta'$, and loops $\hat\gamma_1,\dots,\hat\gamma_k\subset B_0\setminus S$ based at q such that*

$$[\sigma^{-1}\circ\hat\gamma_j\circ\sigma]=\alpha_j \quad \text{in } \pi_1(B_0,b_0) \qquad \text{for every } j=1,\dots,k,$$

and

$$\ell_S(\hat\gamma_j)\ \le\ C\sqrt{(s+1)\log(s+2)} \qquad \text{for all } j=1,\dots,k.$$

Moreover, σ and the loops $\hat\gamma_1,\dots,\hat\gamma_k$ are contained in a fixed compact set $\mathcal{K}_0\Subset B_0$ and satisfy $\ell_\rho(\hat\gamma_j)\le L_0$, where $\mathcal{K}_0$ and $L_0>0$ depend only on $\alpha_1,\dots,\alpha_k$, B_0, and ρ (not on S).

Proof. Fix a unit tangent vector $w\in T_{b_0}B_0$ and let η_0 be the rotation of w by $+\pi/2$ with respect to ρ and the orientation of B. By Lemma 2.4, for each $j=1,\dots,k$ we can choose a smooth simple loop $\gamma^{(j)}$ at b_0, compactly contained in B_0, homotopic to α_j relative to b_0, and whose unit tangent vector at b_0 equals w. Since all loops share the tangent direction w at b_0, and since the unit normal field along every loop is the $+\pi/2$-rotation of the unit tangent (the convention fixed in §2.2), their unit normal vectors at b_0 all equal η_0. For each j, construct the Fermi strip $T_\delta(\gamma^{(j)})$ as in the proof of Theorem 1.5, with a common width $\delta>0$ small enough for all k strips. The parallel curve $\gamma_u^{(j)}$ has base point

$$q(u)\ :=\ \exp_{b_0}(u\,\eta_0),$$

which is independent of j. For $1<p<2$, define

$$g_j(u)\ :=\ \int_0^{\ell_\rho(\gamma^{(j)})}\lambda_S(\tau,u)^p\,\sqrt{\rho_{\tau\tau}}\,d\tau$$

and set $g:=\sum_{j=1}^k g_j$. Let $E_j:=\{u\in\,]-\delta,\delta[\,:\ \gamma_u^{(j)}\cap S\neq\emptyset\}$; then $\#E_j\le s$, so the set $E:=\bigcup_{j=1}^k E_j$ has at most ks points and in particular has measure zero. By (8) and Lemma 2.2 applied to each Fermi strip,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}g(u)\,du\ \le\ \sum_{j=1}^k\frac{\kappa_0^{(j)}A_j(s+1)}{2\delta(2-p)}\ =:\ M\,,$$

where $\kappa_0^{(j)}$ and A_j are the constants from (8) and Lemma 2.2 corresponding to the strip $T_\delta(\gamma^{(j)})$. By the mean-value argument used in the proof of Theorem 1.5, there exists $u_0\in\,]-\delta,\delta[\,\setminus E$ with $g(u_0)\le M$. In particular, $g_j(u_0)\le M$ for every j.

Since $u_0\notin E$, the parallel curve $\gamma_{u_0}^{(j)}$ avoids S for every j. Applying Hölder's inequality (14) with $g_j(u_0)\le M$ in place of the individual bound, and the optimisation $p=2-1/\log(s+2)$ exactly as in the proof of Theorem 1.5, we obtain

$$\ell_S(\gamma_{u_0}^{(j)})\ \le\ C_j'\sqrt{(s+1)\log(s+2)}$$

for every $j=1,\dots,k$, where $C_j'>0$ depends only on $\alpha_1,\dots,\alpha_k$, B_0, and ρ.

Set $\hat\gamma_j:=\gamma_{u_0}^{(j)}$ and $q:=q(u_0)=\exp_{b_0}(u_0\eta_0)\in B_0\setminus S$. Define the path $\sigma\colon[0,|u_0|]\to B_0$ by $\sigma(u):=\exp_{b_0}(u\,\eta_0)$; it connects b_0 to q with $\ell_\rho(\sigma)=|u_0|\le\delta$. For each j, the family $\{\gamma_u^{(j)}\}$ for u between 0 and u_0 is a free homotopy in $T_\delta(\gamma^{(j)})\subset B_0$ from $\gamma_0^{(j)}=\gamma^{(j)}$ (based at b_0) to $\gamma_{u_0}^{(j)}=\hat\gamma_j$ (based at q), and this homotopy moves the base point along σ. It follows that $[\sigma^{-1}\circ\hat\gamma_j\circ\sigma]=[\gamma^{(j)}]=\alpha_j$ in $\pi_1(B_0,b_0)$ for every j. Finally, $\hat\gamma_j=\gamma_{u_0}^{(j)}\subset T_\delta(\gamma^{(j)})$ and $\sigma([0,|u_0|])\subset T_\delta(\gamma^{(1)})$, so the last assertion holds with $\mathcal{K}_0:=\bigcup_{i=1}^k T_\delta(\gamma^{(i)})\Subset B_0$ and $L_0:=\max_{1\le i\le k}\sup_{|u|\le\delta}\ell_\rho(\gamma_u^{(i)})<+\infty$. Setting $C:=\max_j C_j'$ and $\delta':=\delta$ completes the proof. □

2.3 Reduction to the simple case

In this subsection we lift the standing simplicity assumption made in §2.2 and complete the proof of Theorem 1.5 for arbitrary nontrivial $\alpha \in \pi_1(B_0, b_0)$.

We first recall a classical fact. Since B_0 has the homotopy type of a compact orientable surface of genus g with t boundary circles, the fundamental group $\pi_1(B_0, b_0)$ is *free* of rank $k := 2g + t - 1$, and admits a basis $\alpha_1, \ldots, \alpha_k$ of smooth simple loops at b_0. Indeed, such a surface deformation-retracts onto a spine consisting of k smoothly embedded circles meeting only at b_0 (the standard genus loops together with loops encircling all but one of the boundary circles), and each spine circle is a smooth simple loop; for background on curves in surfaces we refer to [4, Chapter 1], and an explicit construction of such loops in the present setting is carried out in [10, Theorem 5.1]. Such a basis is exactly what we have been calling a *simple basis* throughout the paper (cf. Proposition 2.5).

Proof of Theorem 1.5, general case. Fix a nontrivial loop $\alpha \subset B_0$ and a simple basis $\alpha_1, \ldots, \alpha_k$ of $\pi_1(B_0, b_0)$ as above. Write α as a (reduced) word in this basis:

$$\alpha = \alpha_{i_1}^{\epsilon_1} \cdots \alpha_{i_m}^{\epsilon_m} \quad \text{in } \pi_1(B_0, b_0), \qquad \epsilon_j \in \{\pm 1\}, \quad i_j \in \{1, \ldots, k\}.$$

The word length $m = m(\alpha) \geq 1$ is determined by α and the basis.

Fix $S \subset B_0$ with $\#S = s \geq 1$. By Proposition 2.5 applied to the simple basis $\alpha_1, \ldots, \alpha_k$, there exist

- a point $q \in B_0 \setminus S$,
- a path $\sigma\colon [0, 1] \to B_0$ from b_0 to q with $\ell_\rho(\sigma) \leq \delta'$,
- loops $\hat{\gamma}_1, \ldots, \hat{\gamma}_k \subset B_0 \setminus S$ based at q,

such that $[\sigma^{-1} \circ \hat{\gamma}_j \circ \sigma] = \alpha_j$ in $\pi_1(B_0, b_0)$ and

$$\ell_S(\hat{\gamma}_j) \leq C_0 \sqrt{(s+1)\log(s+2)} \quad \text{for all } j = 1, \ldots, k,$$

where $C_0 > 0$ depends only on $\alpha_1, \ldots, \alpha_k$, B_0, and ρ.

Define the concatenation

$$\hat{\gamma}_\alpha := \hat{\gamma}_{i_m}^{\epsilon_m} \circ \hat{\gamma}_{i_{m-1}}^{\epsilon_{m-1}} \circ \cdots \circ \hat{\gamma}_{i_1}^{\epsilon_1},$$

a piecewise smooth loop based at q. We verify that $\hat{\gamma}_\alpha \in \Gamma_S^\alpha$ and bound its Kobayashi length.

Membership in Γ_S^α. Each $\hat{\gamma}_{i_j}$ is contained in $B_0 \setminus S$, hence so is the concatenation $\hat{\gamma}_\alpha$. To see that $\hat{\gamma}_\alpha$ is freely homotopic to α in B_0, insert $\sigma \circ \sigma^{-1}$ between consecutive factors and use distributivity of conjugation:

$$[\sigma^{-1} \circ \hat{\gamma}_\alpha \circ \sigma] = [\sigma^{-1} \circ \hat{\gamma}_{i_1}^{\epsilon_1} \circ \sigma] \cdots [\sigma^{-1} \circ \hat{\gamma}_{i_m}^{\epsilon_m} \circ \sigma] = \alpha_{i_1}^{\epsilon_1} \cdots \alpha_{i_m}^{\epsilon_m} = \alpha \quad \text{in } \pi_1(B_0, b_0).$$

Thus $\hat{\gamma}_\alpha$, viewed as a loop based at q, is conjugate (via the path σ) to a loop based at b_0 representing α; equivalently, the free homotopy class of $\hat{\gamma}_\alpha$ in B_0 coincides with the free homotopy class of α.

Kobayashi length bound. Since κ_S is a Finsler pseudo-metric, its homogeneity gives $\kappa_S(b, -v) = \kappa_S(b, v)$, hence $\ell_S(\hat{\gamma}_{i_j}^{-1}) = \ell_S(\hat{\gamma}_{i_j})$. By additivity of ℓ_S over concatenation,

$$\ell_S(\hat{\gamma}_\alpha) = \sum_{j=1}^m \ell_S\big(\hat{\gamma}_{i_j}^{\epsilon_j}\big) = \sum_{j=1}^m \ell_S(\hat{\gamma}_{i_j}) \leq m \cdot C_0 \sqrt{(s+1)\log(s+2)}.$$

Conclusion. Setting $C := m \cdot C_0$, which depends only on α (through its word length m in the fixed basis), the simple basis $\alpha_1, \ldots, \alpha_k$, B_0, and ρ, we obtain

$$\ell_S(\Gamma_S^\alpha) \leq \ell_S(\hat{\gamma}_\alpha) \leq C\sqrt{(s+1)\log(s+2)}.$$

As $S \subset B_0$ with $\#S = s \geq 1$ was arbitrary, and since sets with $\#S < s$ satisfy the same bound a fortiori (the right-hand side being increasing in s), taking the supremum over all S with $\#S \leq s$ yields

$$L(\alpha, s) \leq C\sqrt{(s+1)\log(s+2)} \quad \text{for all } s \geq 1.$$

For $s = 0$, $L(\alpha, 0) = \inf_\beta \ell_{\kappa_{B_0}}(\beta)$ is a fixed positive constant, absorbed into C. □

We now record two immediate consequences.

Corollary 2.6. *Fix a loop $\alpha \subset B_0$ representing a nontrivial class in $\pi_1(B_0, b_0)$, then*

$$\lim_{s\to+\infty} \frac{\log L(\alpha, s)}{\log s} = \frac{1}{2}.$$

Proof. Combining Theorem 1.5 with Phung's lower bound [10, Theorem C] gives

$$\frac{c\sqrt{s}}{\log(s+2)} \le L(\alpha, s) \le C\sqrt{(s+1)\log(s+2)}$$

so the result follows immediately. ☐

Remark 2.7. Corollary 2.6 gives $L(\alpha, s) = O(s^{1/2+\varepsilon})$ for every $\varepsilon > 0$, but the precise asymptotic behavior of $L(\alpha, s)$ remains open: between Phung's lower bound $c\sqrt{s}/\log(s+2)$ and our upper bound $C\sqrt{(s+1)\log(s+2)}$ there is still a multiplicative gap of order $(\log(s+2))^{3/2}$. Determining the exact asymptotics of $L(\alpha, s)$ is, to our knowledge, an open problem.

3 Bounding generalized integral points

3.1 Phung's bound

In this section, we summarize the proof of Theorem 1.3 of Phung. Our goal is not to reprove any part of it—Phung's argument is already complete—but rather to recall its structure in sufficient detail so that we can later identify precisely where our improvement is applied.

Proof of Theorem 1.3. Fix a hermitian metric h on $\mathcal{A}$. Put $k := \mathrm{rk}(\pi_1(B_0))$, where we have implicitly fixed the base point $b_0 \in B_0$ for $\pi_1(B_0)$.
The rank-zero case. If $k = 0$ then $\pi_1(B_0, b_0)$ is trivial, so $H^1(\pi_1(B_0, b_0), \Gamma) = 0$ for the fibre lattice $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \cong \mathbb{Z}^{2n}$. By the Parshin–Phung injectivity ([9, Proposition 1], [10, Proposition 7.2]; cf. Proposition 3.1 below) the map $A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \to H^1(\pi_1(B_0, b_0), \Gamma) = 0$ is the zero homomorphism; hence every two points of $A(K)$ are congruent modulo $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) + A(K)_{\mathrm{tors}}$, so $A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is finite of order $t_{\mathcal{A}} := \#\big(A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big)_{\mathrm{tors}} < \infty$ (Lang–Néron). Consequently $\#\big(I(s, B_0)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big) \le t_{\mathcal{A}}$ for every s, and the bounds of Theorem 1.3 and Theorem 1.4 hold trivially (take $m \ge t_{\mathcal{A}}$). We therefore assume $k \ge 1$ from now on.

Step 1: Reduction to the hyperbolic locus. Consider the non-hyperbolic locus

$$V := \{b \in B : \mathcal{D}_b \text{ is not hyperbolic}\}$$

It is well known that V is a closed subset of B in the analytic (Euclidean) topology (cf. [2, Proposition 1.10]). Moreover, by [10, Lemma B.2], applied via the inclusion $V \subseteq Z(\mathcal{A}, \mathcal{D})$ furnished by [10, Theorem 7.6], and using crucially the assumption (P)(iii) that D contains no translate of a nonzero abelian subvariety, the set V is *at most countable.* Since $V \cup \mathrm{Sing}(f)$ is a countable, analytically closed subset of the compact surface B, [10, Lemma 8.1] produces a finite union of disjoint closed disks containing $V \cup \mathrm{Sing}(f)$, with distinct points of $\mathrm{Sing}(f)$ in distinct disks.

We may therefore enlarge the disks $\overline{\Delta}_1, \dots, \overline{\Delta}_t$ so as to absorb this finite cover: each of the finitely many cover disks V' not already meeting some $\overline{\Delta}_i$ is annexed by setting $\overline{\Delta}'_i := \overline{\Delta}_i \cup \beta \cup V'$, where β is a thin tube running from $\overline{\Delta}_i$ to V', meeting each only in a short boundary arc at its respective end, and disjoint from all other disks and tubes. Each $\overline{\Delta}'_i$ is simply connected with piecewise-C^1 boundary, hence again a closed disk (in general not a metric ball), and the $\overline{\Delta}'_i$ are pairwise disjoint, with distinct points of $\mathrm{Sing}(f)$ still lying in distinct disks. Since every point of $V \cup \mathrm{Sing}(f)$ lies in some annexed disk or original $\overline{\Delta}_i$, and the annexation can be taken so that these points are interior, we have $V \cup \mathrm{Sing}(f) \subseteq \bigsqcup_{i=1}^t \operatorname{int} \overline{\Delta}'_i$. Writing $B'_0 := B \setminus \bigsqcup_{i=1}^t \overline{\Delta}'_i \subseteq B_0$, the surface B'_0 is a deformation retract of B_0 (one retracts each $\overline{\Delta}'_i$ onto $\overline{\Delta}_i$ along the tube β); in particular $\pi_1(B'_0) = \pi_1(B_0)$, so $\mathrm{rk}\,\pi_1(B'_0) = \mathrm{rk}\,\pi_1(B_0) = k$ is unchanged. Since $B'_0 \subseteq B_0$ we have $I(s, B_0) \subseteq I(s, B'_0)$ for every s, so it suffices to prove the bound for B'_0. Replacing B_0 by B'_0, we may thus assume from now on that $V \cup \mathrm{Sing}(f) \subseteq \bigsqcup_{i=1}^t \operatorname{int} \overline{\Delta}_i$; equivalently, the compact set $\overline{B_0}$ is disjoint from $V \cup \mathrm{Sing}(f)$. In particular $\mathcal{D}_b$ is Kobayashi hyperbolic for every $b \in B_0$. By a result of Green ([10, Theorem 7.6]) it follows that $\mathcal{A}_b \setminus \mathcal{D}_b$ is complete and Kobayashi hyperbolic for any $b \in B_0$. This is the point where one has to use the assumption (P)(iii): it is in the hypotheses of Green's theorem.

Finally, fix a compact connected smooth subsurface $N \Subset B_0$ with $b_0 \in N$ (it will be enlarged once and for all in Step 2, so as to contain two further fixed compact sets). Its intrinsic diameter $\delta_N := \operatorname{diam}_\rho(N)$ is finite, so for any two points $x, y \in N$ we can find a path $c_{x,y} \subset N$ joining them and such that:

$$\ell(c_{x,y}) \le \delta_N < +\infty\,.$$

We denote $\mathcal{C}_{b_0} := \{c_{b_0,p}\}_{p\in N}$.

Step 2: Bounding loop lengths in the base. Let $\sigma_P \in I(s, B_0)$ and set $S := f(\sigma_P(B_0) \cap \mathcal{D}) \cap B_0$, so that $\#S \le s$ and $\sigma_P(B_0 \setminus S) \subseteq (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}$; we may assume $S \neq \emptyset$ (otherwise replace S by an arbitrary singleton of B_0, which only increases the Kobayashi lengths below). Let $\alpha_1, \dots, \alpha_k$ be a simple basis of $\pi_1(B_0, b_0)$. By [10, Theorem 5.1] there exists $b \in B_0$ and simple loops $\gamma_j \in \Gamma_S^{\alpha_j}$ such that $\alpha_j = c_{b_0,b}^{-1}\,\gamma_j\, c_{b_0,b}$ in $\pi_1(B_0, b_0)$ where $c_{b_0,b} \in \mathcal{C}_{b_0}$ and the following bound holds:

$$\ell_S(\gamma_j) \le L(s+1) \qquad \forall j = 1, \dots, k \tag{19}$$

for $L \in \mathbb{R}_{>0}$ independent of s, S, b. Two further facts, immediate from the proof of [10, Theorem 5.1], will be needed in Step 5: the loops γ_j lie in a fixed compact set $\mathcal{K}_1 \Subset B_0$ and satisfy $\ell_\rho(\gamma_j) \le L_1$, with $\mathcal{K}_1$ and L_1 independent of s, S, P. Indeed, by [10, Lemmas 3.7 and 4.1] the γ_j are obtained by modifying, within ρ-distance $2a$, finitely many fixed loops of bounded ρ-length, while keeping ρ-distance $\ge 2a$ from ∂B_0; the length bound follows from [10, (5.2)–(5.3)]. Moreover, the base points b lie in the fixed compact set $B_\varepsilon \Subset B_0$ of *loc. cit.* We enlarge N once and for all so that $\mathcal{K}_1 \cup B_\varepsilon \subseteq N$.

Let $g\colon \mathbb{C} \to (\mathcal{A} \setminus \mathcal{D})_{|B_0}$ be a holomorphic map, then the composition $f \circ g\colon \mathbb{C} \to B_0$ is constant, since B_0 is hyperbolic. Then the image of g is in $\mathcal{A}_b \setminus \mathcal{D}_b$ for some $b \in B_0$, so by the previous step g is constant. So $(\mathcal{A} \setminus \mathcal{D})|_{B_0}$ is Brody hyperbolic; the same two-step argument shows that $\mathcal{D}|_{B_0}$ is Brody hyperbolic, using that $\mathcal{D}_b$ is hyperbolic for every $b \in B_0$ (Step 1). Enlarging the topological disks $\overline{\Delta}_i$ slightly once more if necessary (which, as in Step 1, leaves $\pi_1(B_0)$ unchanged and shrinks B_0, so that the new $\overline{B_0}$ is contained in the previous open B_0), we may assume that $\overline{\mathcal{A}|_{B_0} \setminus \mathcal{D}|_{B_0}}$ and $\overline{\mathcal{D}|_{B_0}}$ (closures taken in $\mathcal{A}$) are Brody hyperbolic as well: they are subsets of $(\mathcal{A} \setminus \mathcal{D})$ and $\mathcal{D}$ over the previous open base, just shown to be Brody hyperbolic, and subsets of Brody hyperbolic sets are Brody hyperbolic. These are the hypotheses of Green's theorem for $X = \mathcal{A}|_{B_0}$ and $D = \mathcal{D}|_{B_0}$ inside $\mathcal{A}$. At this point a theorem of Green (see [10, Theorem 7.5]) says that there exists $c \in \mathbb{R}_{>0}$ such that, at the infinitesimal level,

$$\kappa_{(\mathcal{A} \setminus \mathcal{D})|_{B_0}}(x, v) \ge c\,|v|_h \qquad \forall\,(x, v)\,. \tag{20}$$

Now notice that

$$\sigma_P(\gamma_j) \subset \sigma_P(B_0 \setminus S) \subseteq (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}$$

So by Equations (19) to (20) and the decreasing property of the Kobayashi-Royden metric we deduce

$$\ell_h(\sigma_P(\gamma_j)) \le c^{-1} L(s+1)\,. \tag{21}$$

Step 3: The Parshin cocycle. Since $f\colon \mathcal{A}_{B_0} \to B_0$ is a proper smooth submersion, Ehresmann's theorem gives a fiber bundle in the differentiable category. Let σ_O be the zero section and fix $w_0 := \sigma_O(b_0) \in \mathcal{A}_{b_0}$. Since B_0 is a $K(\pi, 1)$-space (i.e. $\pi_i(B_0) = 0$ for $i \ge 2$, which holds because its universal cover is the disc $\mathbb{D}$), the long exact sequence of homotopy groups for the fibration $\mathcal{A}_{b_0} \to \mathcal{A}_{B_0} \to B_0$ yields the short exact sequence

$$1 \longrightarrow \pi_1(\mathcal{A}_{b_0}, w_0) \longrightarrow \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{f_*} \pi_1(B_0, b_0) \longrightarrow 1\,. \tag{22}$$

As $\mathcal{A}_{b_0}$ is a complex torus of dimension n, we have $\pi_1(\mathcal{A}_{b_0}, w_0) = H_1(\mathcal{A}_{b_0}, \mathbb{Z}) =: \Gamma \cong \mathbb{Z}^{2n}$. The zero section σ_O splits (22) via $i_O(\alpha) := [\sigma_O(\gamma)]$ where $\alpha = [\gamma] \in G := \pi_1(B_0, b_0)$. Each rational point $P \in A(K)$ with section $\sigma_P\colon B_0 \to \mathcal{A}_{B_0}$ gives a second splitting

$$i_P(\alpha) := [\eta^{-1}_{w_0,\sigma_P(b_0)} \circ \sigma_P(\gamma) \circ \eta_{w_0,\sigma_P(b_0)}]\,,$$

where $\eta_{w_0,\sigma_P(b_0)}$ is a path in the fiber $\mathcal{A}_{b_0}$ from w_0 to $\sigma_P(b_0)$. Since both i_P and i_O are sections of f_*, for every $\alpha \in G$:

$$f_*\big(i_P(\alpha) \cdot i_O(\alpha)^{-1}\big) = \alpha \cdot \alpha^{-1} = 1\,,$$

so $i_P(\alpha) \cdot i_O(\alpha)^{-1} \in \ker(f_*) = \Gamma$. We define the *Parshin cocycle*

$$c_P\colon G \to \Gamma\,, \quad c_P(\alpha) := i_P(\alpha) \cdot i_O(\alpha)^{-1}\,.$$

One verifies that c_P is a 1-cocycle of G with coefficients in the G-module Γ, where the G-action on Γ is by conjugation

$$\begin{aligned}\varphi\colon G &\to \mathrm{Aut}(\Gamma)\\ \alpha &\mapsto \varphi_\alpha\colon \gamma \mapsto i_O(\alpha)\,\gamma\, i_O(\alpha)^{-1}\end{aligned}$$

Note that the action φ doesn't depend on the choice of the splitting i_O, in fact for another splitting i_P, we have $i_P(\alpha) = c_P(\alpha)\cdot i_O(\alpha)$ with $c_P(\alpha)\in\Gamma$, and

$$i_P(\alpha)\,\gamma\, i_P(\alpha)^{-1} = c_P(\alpha)\cdot\varphi_\alpha(\gamma)\cdot c_P(\alpha)^{-1} = \varphi_\alpha(\gamma)\,,$$

where the last equality uses the commutativity of Γ. The cohomology class $[c_P]\in H^1(G,\Gamma)$ is independent of the choice of path $\eta_{w_0,\sigma_P(b_0)}$ (a different choice changes c_P by a coboundary). The following result proved in [9, Proposition 1] and [10, Proposition 7.2] will be crucial:

Proposition 3.1 (Parshin-Phung)**.** *The homomorphism*

$$\begin{aligned}\Psi\colon A(K) &\to H^1(G,\Gamma)\\ P &\mapsto [c_P]\end{aligned}$$

factors through $A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ and has the following injectivity property: if $\Psi(P)=\Psi(Q)$, then $P-Q\in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) + A(K)_{\mathrm{tors}}$. In particular, each element $[c]\in H^1(G,\Gamma)$ in the image of Ψ arises from at most $t_{\mathcal{A}} := \#\big(A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big)_{\mathrm{tors}} < \infty$ rational points modulo the trace (finite by the Lang–Néron theorem).

Remark 3.2. It would be interesting to understand the relationship between the Parshin cocycle and the "analytic cocycle" induced by the logarithm of the section σ_P, as defined in [3, Equation (13)].

To extract a quantitative bound from the cocycle c_P, we work in the semidirect product coordinates provided by i_O. The splitting i_O exhibits

$$\pi_1(\mathcal{A}_{B_0}, w_0) \;\cong\; \Gamma \rtimes_\varphi G\,. \tag{23}$$

In these coordinates, $i_O(\alpha_j) = (0,\alpha_j)$ and $i_P(\alpha_j) = (\beta_j,\alpha_j)$ for some $\beta_j\in\Gamma$, so the Parshin cocycle reads $c_P(\alpha_j)=\beta_j$. Since G is free, the k-tuple $(\beta_1,\ldots,\beta_k)\in\Gamma^k$ determines c_P completely. The displacement bound in Step 4 will give a norm bound on each β_j.

Step 4: Bounding loop lengths in $\mathcal{A}$. Let us now go back to the special loops (based in b) $\gamma_j\in\Gamma_S^{\alpha_j}$ for $j=1,\ldots,k$ considered in Step 2. Consider the following loop based at w_0:

$$\sigma_P(\gamma_j)^{\#} := (v_b\circ\sigma_O(c_{b_0,b}))^{-1}\circ\sigma_P(\gamma_j)\circ(v_b\circ\sigma_O(c_{b_0,b}))$$

where $v_b\subset\mathcal{A}_b$ is a h-geodesic from $\sigma_O(b)$ to $\sigma_P(b)$. Since $\sigma_P(\gamma_j) = \sigma_P(c_{b_0,b})\circ\sigma_P(\alpha_j)\circ\sigma_P(c_{b_0,b})^{-1}$, $\sigma_P(\gamma_j)^{\#}$ is free homotopic to $\sigma_P(\alpha_j)$ and each of its components satisfies the following uniform bounds:

- the loop $\sigma_P(\gamma_j)$ satisfies:
$$\ell_h(\sigma_P(\gamma_j)) \le c^{-1}L(s+1)$$
by Equation (21);
- the fiber path $v_b\subset\mathcal{A}_b$ satisfies:
$$\ell_h(v_b)\le \delta_0 := \sup_{b\in\overline{B_0}} \mathrm{diam}_h(\mathcal{A}_b) < +\infty\,;$$
(here $\mathrm{diam}_h(\mathcal{A}_b)$ is the *intrinsic* h-diameter of the fibre, which bounds the length of a fibre geodesic; the supremum is finite because $\overline{B_0}\cap\mathrm{Sing}(f)=\varnothing$ by Step 1, so $(\mathcal{A}_b)_{b\in\overline{B_0}}$ is a compact smooth family and $b\mapsto\mathrm{diam}_h(\mathcal{A}_b)$ is continuous)
- since B is compact $\|d\sigma_O\|_\infty := \sup_{b\in B}\|d_b\sigma_O\| < +\infty$, so
$$\ell_h(\sigma_O(c_{b_0,b})) \le \|d\sigma_O\|_\infty\,\ell(c_{b_0,b}) \le \|d\sigma_O\|_\infty\,\delta_N =: \delta_0'$$

In other words for any element of the simple basis α_j we can construct a loop $\left[\sigma_P(\gamma_j)^{\#}\right] \in \pi_1(\mathcal{A}_{B_0}, w_0)$ that is free homotopic to $\sigma_P(\alpha_j)$ and satisfies

$$\ell_h\left(\sigma_P(\gamma_j)^{\#}\right) \leq H(s) := c^{-1}L(s+1) + 2(\delta_0 + \delta_0')$$

where the constants L, δ_0, δ_0' don't depend on S and P.

The loop $\sigma_P(\gamma_j)^{\#}$ is based at w_0, so it defines an element $[\sigma_P(\gamma_j)^{\#}] \in \pi_1(\mathcal{A}_{B_0}, w_0)$. This element need not equal $i_P(\alpha_j) = (\beta_j, \alpha_j)$, because $\sigma_P(\gamma_j)^{\#}$ and the loop defining $i_P(\alpha_j)$ use different conjugating paths from w_0 to the fibre above b: the former uses $v_b \circ \sigma_O(c_{b_0,b})$, while the latter uses $\sigma_P(c_{b_0,b}) \circ \eta_{w_0,\sigma_P(b_0)}$. Both are paths from w_0 to $\sigma_P(b)$, so their concatenation

$$\omega \;:=\; \left(v_b \circ \sigma_O(c_{b_0,b})\right)^{-1} \circ\; \sigma_P(c_{b_0,b}) \circ\; \eta_{w_0,\sigma_P(b_0)}$$

is a loop at w_0, and a direct verification gives

$$[\sigma_P(\gamma_j)^{\#}] \;=\; [\omega]^{-1} \cdot i_P(\alpha_j) \cdot [\omega] \qquad \text{in } \pi_1(\mathcal{A}_{B_0}, w_0)\,.$$

In other words, $[\sigma_P(\gamma_j)^{\#}]$ and $i_P(\alpha_j)$ are *conjugate* in $\pi_1(\mathcal{A}_{B_0}, w_0)$. The map $i_P'\colon G \to \pi_1(\mathcal{A}_{B_0}, w_0)$ defined by $i_P'(\alpha_j) := [\sigma_P(\gamma_j)^{\#}]$ is therefore a section of (22) in the same conjugacy class as i_P, so the associated cocycles are cohomologous: $[c'] = [c_P] \in H^1(G, \Gamma)$. By Proposition 3.1, we may freely replace i_P by i_P' for counting purposes. We do so and, to lighten notation, simply write β_j for the components of the new cocycle.

Step 5: Displacement bound via the universal cover. We now explain how the h-length bound on $\sigma_P(\gamma_j)^{\#}$ constrains the lattice element $\beta_j \in \Gamma$. Let $\pi\colon \widetilde{\mathcal{A}}_0 \to \mathcal{A}_{B_0}$ be the universal cover. Since $\mathcal{A}_{B_0} \to B_0$ is a fibre bundle (by Ehresmann's theorem) with fibre $\mathcal{A}_{b_0}$ (a complex torus of dimension n, so $\widetilde{\mathcal{A}_{b_0}} \cong \mathbb{R}^{2n}$) and base B_0 (a hyperbolic Riemann surface, so $\widetilde{B_0} \cong \mathbb{D}$, the unit disk), pulling back to $\mathbb{D}$ trivialises the bundle and gives $\widetilde{\mathcal{A}}_0 \cong \mathbb{R}^{2n} \times \mathbb{D}$. We use the trivialisation induced by the relative exponential (cf. [10, (7.3)]): the lattice local system $(R^1 f_*\mathbb{Z})^\vee$ is trivial over the contractible $\mathbb{D}$, and with this choice the subgroup $\Gamma \subset \pi_1(\mathcal{A}_{B_0}, w_0)$ acts on $\mathbb{R}^{2n} \times \mathbb{D}$ by $(\tilde{x}, \tilde{y}) \mapsto (\tilde{x} + \gamma, \tilde{y})$.

Fix the point $\tilde{w} = (\tilde{x}_0, \tilde{y}_0) \in \pi^{-1}(w_0)$ with $\tilde{x}_0 = 0$: this choice is possible because $w_0 = \sigma_O(b_0)$ lies on the zero section, whose lift in the chosen trivialisation is $\{0\} \times \mathbb{D}$, and $\tilde{y}_0$ is any point of $\mathbb{D}$ lying over b_0 for the universal covering $v\colon \mathbb{D} \to B_0$. Pull back the hermitian metric h to a Riemannian metric $\tilde{h}$ on $\widetilde{\mathcal{A}}_0$. The group of deck transformations of π is $\pi_1(\mathcal{A}_{B_0}, w_0) \cong \Gamma \rtimes_\varphi G$ (with the composition convention for π_1 already implicit in our loop concatenations). The action of an element $(\beta, \alpha) = \beta \cdot i_O(\alpha) \in \Gamma \rtimes_\varphi G$ on $\mathbb{R}^{2n} \times \mathbb{D}$ is given by (homotopy lifting property, cf. the proof of [10, Theorem A]):

$$(\beta, \alpha) \cdot (\tilde{x}, \tilde{y}) \;=\; \left(\varphi_\alpha(\tilde{x}) + \beta,\;\; \alpha \cdot \tilde{y}\right), \tag{24}$$

where $\alpha \cdot \tilde{y}$ is the deck transformation of $v\colon \mathbb{D} \to B_0$ by $\alpha \in G = \pi_1(B_0, b_0)$ (acting on the base), and φ_α acts on the fibre factor $\mathbb{R}^{2n}$ as the linear extension of the monodromy: the fibre component "twists" as one transports along the base loop α, and the semidirect product structure of $\Gamma \rtimes_\varphi G$ encodes precisely this twisting. In particular the fibre component of the action is a translation only when $\varphi_\alpha = \mathrm{id}$; at the chosen point $\tilde{w} = (\tilde{x}_0, \tilde{y}_0)$, however, thanks to the normalisation $\tilde{x}_0 = 0$, (24) reads

$$(\beta, \alpha) \cdot (\tilde{x}_0, \tilde{y}_0) \;=\; \left(\beta \cdot \tilde{x}_0,\;\; \alpha \cdot \tilde{y}_0\right), \tag{25}$$

where $\beta \cdot \tilde{x}_0$ is the deck transformation of $u\colon \mathbb{R}^{2n} \to \mathcal{A}_{b_0}$ by the element $\beta \in \Gamma$ (acting on the fibre).

The loop $\sigma_P(\gamma_j)^{\#}$ is based at w_0 and has homotopy class (β_j, α_j). Its lift to $\widetilde{\mathcal{A}}_0$ from $\tilde{w}$ is a path from $\tilde{w}$ to $(\beta_j, \alpha_j) \cdot \tilde{w}$. Since π is a local isometry for the metrics $\tilde{h}$ and h, the $\tilde{h}$-length of the lift equals $\ell_h(\sigma_P(\gamma_j)^{\#}) \leq H(s)$, so:

$$d_{\tilde{h}}\left(\tilde{w},\; (\beta_j, \alpha_j) \cdot \tilde{w}\right) \;\leq\; H(s) \;:=\; c^{-1}L(s+1) + 2(\delta_0 + \delta_0') \;=\; O(s)\,, \tag{26}$$

where L, δ_0, δ_0' are independent of S and P. Let d_j be the Γ-invariant geodesic metric on the fibre factor $\mathbb{R}^{2n}$ obtained by restricting $\tilde{h}$ to the slice $\mathbb{R}^{2n} \times \{\alpha_j \cdot \tilde{y}_0\}$; its quotient by the translation action of $\Gamma \cong \mathbb{Z}^{2n}$ is the compact torus $\mathcal{A}_{b_0}$. Let $\mathrm{pr}_1\colon \mathbb{R}^{2n} \times \mathbb{D} \to \mathbb{R}^{2n}$ be the projection; by (25), $\mathrm{pr}_1(\tilde{w}) = \tilde{x}_0$ and $\mathrm{pr}_1((\beta_j, \alpha_j) \cdot \tilde{w}) = \beta_j \cdot \tilde{x}_0$.

The $\mathbb{D}$-component of the lift $\tilde{c}$ of $\sigma_P(\gamma_j)^\#$ from $\tilde{w}$ is the lift, from $\tilde{y}_0$, of the base loop $f\big(\sigma_P(\gamma_j)^\#\big) = c_{b_0,b}^{-1} \circ \gamma_j \circ c_{b_0,b}$, which by Step 2 has image in N and ρ-length at most $L_2 := L_1 + 2\delta_N$, uniformly in s, P. Hence this component is a path of $\tilde{\rho}$-length $\leq L_2$ starting at $\tilde{y}_0$ and contained in $v^{-1}(N)$, so it remains in the closed ball

$$\Omega := \overline{B}_{v^{-1}(N)}(\tilde{y}_0, L_2)$$

of radius L_2 around $\tilde{y}_0$ in $v^{-1}(N)$ for the induced length metric. The space $(v^{-1}(N), \tilde{\rho})$ is a covering of the compact length space N with the lifted length metric, hence complete and locally compact, hence a proper metric space by the Hopf–Rinow theorem for length spaces; therefore Ω is compact. Note that Ω is independent of s and P, and $\tilde{c} \subset \mathbb{R}^{2n} \times \Omega$.

The ratio $|\mathrm{pr}_{1*}\xi|_{d_j}/|\xi|_{\tilde{h}}$, for $0 \neq \xi \in T(\mathbb{R}^{2n} \times \Omega)$, is invariant under Γ: every $\gamma \in \Gamma$ is a $\tilde{h}$-isometry and satisfies $\mathrm{pr}_1 \circ \gamma = \tau_\gamma \circ \mathrm{pr}_1$, with τ_γ the d_j-isometric translation by γ. (The analogous equivariance fails for a general deck element (β, α), whose fibre component has linear part φ_α by (24); this is why we restrict to the Γ-invariant region $\mathbb{R}^{2n} \times \Omega$.) The ratio therefore descends to a continuous function on the unit sphere bundle over the compact set $\mathcal{A}_{b_0} \times \Omega$, hence it is bounded by some $C_{h,j} \geq 1$; we set $C_h := \max_{1 \leq j \leq k} C_{h,j}$, which depends only on $\mathcal{A}$, h, Ω and the basis, uniformly in s, P. Therefore pr_1 is C_h-Lipschitz from $(\mathbb{R}^{2n} \times \Omega, \tilde{h})$ to $(\mathbb{R}^{2n}, d_j)$, so $\mathrm{pr}_1 \circ \tilde{c}$ has d_j-length $\leq C_h\, \ell_{\tilde{h}}(\tilde{c})$, and

$$d_j(\beta_j \cdot \tilde{x}_0,\ \tilde{x}_0) \ \leq\ C_h\, \ell_{\tilde{h}}(\tilde{c}) \ =\ C_h\, \ell_h\big(\sigma_P(\gamma_j)^\#\big) \ \leq\ C_h\, H(s)\,. \tag{27}$$

Remark 3.3. The intermediate bound (26) controls the displacement in the *full* universal cover $\widetilde{\mathcal{A}_0} \cong \mathbb{R}^{2n} \times \mathbb{D}$, involving both fibre and base directions (and the metric $\tilde{h}$ is not a product metric due to the monodromy). The projection step (27) extracts a purely fibre-theoretic bound, which is what we need for the lattice count in the next step.

Step 6: Counting. For each $j = 1, \ldots, k$, we count the elements $\beta \in \Gamma$ compatible with (27):

$$N_j(s) \ :=\ \#\Big\{\beta \in \Gamma :\ d_j\big(\beta \cdot \tilde{x}_0,\ \tilde{x}_0\big) \leq C_h H(s)\Big\}\,. \tag{28}$$

Via the universal covering $u \colon \mathbb{R}^{2n} \to \mathcal{A}_{b_0}$, the torus A_{b_0} becomes a compact geodesic Riemannian manifold with the metric d_j, and $\Gamma \cong \mathbb{Z}^{2n}$ acts on $(\mathbb{R}^{2n}, d_j)$ by deck transformations. The set in (28) is in natural bijection with the set of Γ-translates of $\tilde{x}_0$ lying in the d_j-ball of radius $C_h H(s)$ around $\tilde{x}_0$.

By the fundamental lemma of the geometry of groups (cf. [10, Proposition A.8 and Lemma A.9]), since $\Gamma \cong \mathbb{Z}^{2n}$ is a lattice of rank $2n$ acting cocompactly on $(\mathbb{R}^{2n}, d_j)$, there exists $m_j > 0$ (depending on $\mathcal{A}$, h, α_j, but not on s) such that

$$N_j(s) \ \leq\ m_j\,(C_h H(s) + 1)^{2n}\,. \tag{29}$$

The exponent $2n$ reflects the rank of $\Gamma \cong \mathbb{Z}^{2n}$: in a ball of radius R in the universal cover of the $2n$-dimensional real torus $\mathcal{A}_{b_0}$, the number of Γ-translates of $\tilde{x}_0$ grows as R^{2n}.

Since G is free on $\alpha_1, \ldots, \alpha_k$, the cohomology class $[c_P] \in H^1(G, \Gamma)$ is completely determined by the k-tuple $(\beta_1, \ldots, \beta_k) \in \Gamma^k$. Bounding each component independently via (29) with $R = C_h H(s)$:

$$\#\{\text{possible cohomology classes } [c_P]\} \ \leq\ \prod_{j=1}^{k} m_j\,(C_h H(s) + 1)^{2n} \ =\ m_0\,(C_h H(s) + 1)^{2nk}\,, \tag{30}$$

where $m_0 := \prod_{j=1}^{k} m_j$.

By Proposition 3.1, each class $[c_P] \in H^1(G, \Gamma)$ arises from at most $t_{\mathcal{A}} := \#\big(A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big)_{\mathrm{tors}} < \infty$ rational points modulo the trace (finite by the Lang–Néron theorem). Therefore:

$$\#\left(I(s, B_0)\big/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) \ \leq\ t_{\mathcal{A}} \cdot m_0\,(C_h H(s) + 1)^{2nk} \ =\ m\,(s+1)^{2nk}\,,$$

where $m := t_{\mathcal{A}} \cdot m_0 \cdot \big(C_h(c^{-1}L + 2(\delta_0 + \delta_0')) + 1\big)^{2nk}$ depends only on B_0, $\mathcal{A}$, D, and h. □

3.2 Improved bound

We now prove a sharper version of Theorem 1.3. The general strategy is unchanged: we follow the same six-step scheme (reduction to the hyperbolic locus, loop construction, Parshin cocycle, lifting to $\mathcal{A}$, displacement in the universal cover, lattice counting). The only modification occurs in Steps 2 and 4, where we replace Phung's linear bound $\ell_S(\gamma_j) = O(s)$ with the sublinear bound $O(\sqrt{(s+1)\log(s+2)})$ provided by Proposition 2.5. This propagates through the counting argument and halves the exponent.

Proof of Theorem 1.4. The proof follows the same six-step scheme as the proof of Theorem 1.3; in particular we may again assume $k \geq 1$, the case $k = 0$ having been settled there. The only modification is in Step 2: we replace the appeal to [10, Theorem 5.1], which provides loops of Kobayashi length $O(s)$, with Proposition 2.5, applied to the set $S := f(\sigma_P(B_0) \cap \mathcal{D}) \cap B_0$ (as in Step 2, we may assume $S \neq \emptyset$; note $\#S \leq s$, and since the bound below is increasing in the cardinality we may state it with s in place of $\#S$), which provides a point $b \in B_0 \setminus S$, a path σ from b_0 to b in B_0 with $\ell_\rho(\sigma) \leq \delta'$, and loops $\gamma_1, \ldots, \gamma_k \subset B_0 \setminus S$ based at b with $[\sigma^{-1} \circ \gamma_j \circ \sigma] = \alpha_j$ in $\pi_1(B_0, b_0)$ and

$$\ell_S(\gamma_j) \;\leq\; C\sqrt{(s+1)\log(s+2)} \qquad \text{for all } j = 1, \ldots, k\,.$$

Since the loops share a common base point b and the conjugation uses the single path σ, Steps 3–5 apply verbatim with the conjugating path $v_b \circ \sigma_O(\sigma)$ in $\mathcal{A}_{B_0}$ (in place of $v_b \circ \sigma_O(c_{b_0,b})$ in Theorem 1.3), whose h-length is bounded by $\delta_0 + \|d\sigma_O\|_\infty \delta'$, independently of s. The compactness input of Step 5 is again available: by the last assertion of Proposition 2.5, the base loop $\sigma^{-1} \circ \gamma_j \circ \sigma$ has image in the fixed compact set $\mathcal{K}_0 \Subset B_0$ and ρ-length at most $L_0 + 2\delta'$, uniformly in s, P; enlarging N so that $\mathcal{K}_0 \subseteq N$, the compact set Ω of Step 5 (with $L_2 := L_0 + 2\delta'$) is independent of s, P. The displacement bound becomes

$$d_j\big(\beta_j \cdot \tilde{x}_0,\ \tilde{x}_0\big) \;\leq\; C_h\, H(s)\,, \qquad H(s) \;:=\; c^{-1} C\sqrt{(s+1)\log(s+2)} + 2(\delta_0 + \|d\sigma_O\|_\infty \delta')\,,$$

with $C_h \geq 1$ the constant of Step 5. The lattice counting in Step 6 then gives

$$\#\left({}^{I(s,B_0)}\!\big/_{\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})} \right) \;\leq\; t_{\mathcal{A}} m_0\, (C_h H(s) + 1)^{2nk}\,.$$

It remains to extract the exponent. Since $C_h \geq 1$, for s large enough

$$C_h H(s) + 1 \;\leq\; C_h\, c^{-1}(C+1)\sqrt{(s+1)\log(s+2)}\,,$$

hence $(C_h H(s)+1)^{2n} \leq \big(C_h c^{-1}(C+1)\big)^{2n} \big((s+1)\log(s+2)\big)^n$. Fix $\varepsilon' > 0$. Since $\log(s+2) \leq (s+1)^{\varepsilon'/n}$ for s sufficiently large, there exists $s_0 = s_0(\varepsilon') \geq 0$ such that for all $s \geq s_0$:

$$\big((s+1)\log(s+2)\big)^n \;\leq\; (s+1)^{n+\varepsilon'}\,.$$

Over k basis elements, $(C_h H(s)+1)^{2nk} \leq \mathrm{const} \cdot (s+1)^{nk+k\varepsilon'}$ for $s \geq s_0$. Setting $\varepsilon := k\varepsilon'$ and absorbing all constants into m, we obtain

$$\#\left({}^{I(s,B_0)}\!\big/_{\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})} \right) \;\leq\; m\,(s+1)^{nk+\varepsilon} \qquad \text{for all } s \geq 0\,,$$

where $m = m(B_0, \mathcal{A}, \mathcal{D}, h, \varepsilon) > 0$ is independent of s. □

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