Sheet 2: differentiable curves

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1) Let $\gamma : [a, b] \to \mathbb{R}^3$ be a regular curve which is also simple (i.e. $\gamma(t_1) \neq \gamma(t_2)$ for any $a < t_1 < t_2 < b$ and $\gamma(a), \gamma(b) \notin \gamma(\,]a, b[\,)$).

(a) Show that there exists an equivalent curve $\widetilde{\gamma} : I \to \mathbb{R}^3$ $s \mapsto \widetilde{\gamma}(s)$ with the following properties: for each point p the support of γ then $p = \widetilde{\gamma}(s)$ where s is the arc length of γ between $\gamma(a)$ and p. Viceversa given $s \in I$ there exists a unique point p in the support of γ such that the arc length between $\gamma(a)$ and p is exactly s.

(b) Show that for the curve $\widetilde{\gamma}$ constructed in (a) it holds that $\left\| \frac{d\widetilde{\gamma}}{ds} \right\| = 1$. (In other words the velocity vector of $\widetilde{\gamma}$ is always unitary).

(c) Compute the curve $\widetilde{\gamma}$ described in (a) for

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{R}^3 \\ t &\mapsto (r \cos t, r \sin t, ct)\,, \quad \text{for } r, c \in \mathbb{R}_{>0}\,. \end{aligned}$$

2) Let $\gamma : [a, b] \to \mathbb{R}^3$ be a regular curve and let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Define the *derivative of f along* γ in the following way:

$$\begin{aligned} df_{|_\gamma} : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \left\langle \nabla f|_{\gamma(t)}, \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \right\rangle \end{aligned}$$

where ∇f is the gradient of f and $\langle\ ,\ \rangle$ denotes the usual scalar product in $\mathbb{R}^3$. Show that if f is constant along the curve γ, then $df_{|_\gamma}(t) = 0$ for all $t \in [a, b]$.

3) Let $\gamma : [a, b] \to \mathbb{R}^2$ be a $C^2[a, b]$ curve. Consider the polar change of coordinates in the real plane defined in the following way:

$$\begin{aligned} \mathbb{R}_{\geq 0} \times [0, 2\pi[&\to \mathbb{R}^2 \\ (\rho, \theta) &\mapsto (\rho \cos\theta, \rho \sin\theta) \end{aligned}$$

and denote by $\widetilde{\gamma} : [a, b] \to \mathbb{R}^2$, $t \mapsto (\rho(t) \cos\theta(t), \rho(t) \sin\theta(t))$ the above curve expressed after such a change of coordinates. Moreover define the following maps $\mathbf{u}, \mathbf{v} : [a, b] \to \mathbb{R}^2$:

$$\mathbf{u}(t) = (\cos\theta(t), \sin\theta(t))$$

$$\mathbf{v}(t) = (-\sin\theta(t), \cos\theta(t))$$

Show that:

$$\dot{\gamma}(t) = \dot{\rho}(t)\mathbf{u}(t) + \rho(t)\dot{\theta}(t)\mathbf{v}(t)\,.$$