Sheet 5: Convex optimization. Implicit function theorem. Inverse function theorem

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1. Consider the following equation in three real variables: $(x^2 + y^2 + 2z^2)^{\frac{1}{2}} = \cos z$. Can it be solved for y in terms of x and z locally around $P = (0, 1, 0)$? Around the same point, can it be solved for z in terms of x and y?

2. Let $a \in]0, 1[$ and consider the following function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} ax + x^2 \sin \frac{1}{x}\,, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$$

Determine whether f is a local diffeomorphism in a neighborhood of 0.

3. Let $F \in \mathcal{C}^2(\mathbb{R}^2; \mathbb{R})$ such that $F(0, 0) = 0$. What conditions of F will guarantee that $F(F(x, y), y) = 0$ can be solved for y as a $\mathcal{C}^1$ function around $(0, 0)$?

4. Let $D \subseteq \mathbb{R}^n$ and consider a strongly convex function[1] $F : D \to \mathbb{R}$. Show that F admits a unique point of minimum.

5. Find an example of a strictly convex function which is not strongly convex.

6. Consider the function $F(x, y) = \frac{1}{2}(x + ay)^2$ with $a \in \mathbb{R}_{>0}$. Verify that it is a strongly convex function and that the values

$$\min\{1, a\}\,, \quad \max\{1, a\}$$

are the tightest choices for the "convexity constants" m and M of the gradient descent algorithm, respectively. Moreover, fix the starting point $p_0 = (a, 1)$ and run the gradient descent algorithm with the exact ray search. Write an explicit formula for the iterates p_k of the algorithm and for $F(p_k)$. Finally verify that the sequence $\{F(p_k)\}$ converges to the minima of F and study the "convergence speed" when a varies.

[1] Recall that F is said strongly convex if D is convex and there exists $m \in \mathbb{R}_{>0}$ such that $\mathcal{H}_f(x) - mI$ is positive semi-definite for any $x \in D$.