On the generalisation of Roth's theorem

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Abstract

We present two possible generalisations of Roth's approximation theorem on proper adelic curves, assuming some technical conditions on the behavior of the logarithmic absolute values. We illustrate how tightening such assumptions makes our inequalities stronger. As special cases we recover Corvaja's results [Cor97] for fields admitting a product formula, and Vojta's ones [Voj21] for arithmetic function fields.

0 Introduction

0.1 History

The celebrated Roth's theorem proved in [Rot55] asserts that the irrationality measure of a real algebraic number is 2. An equivalent, but more detailed statement is the following:

Theorem 0.1 (Roth)**.** *Let α be a real algebraic number and let $\varepsilon > 0$ be a real number. Then there exists a real constant $C(\alpha, \varepsilon) > 0$ such that for every pair of coprime integers (p, q) with $q > C(\alpha, \varepsilon)$, it holds that:*

$$\left|\alpha - \frac{p}{q}\right| > q^{-2-\varepsilon}$$

The above statement can be naturally generalised in different directions by considering a number field K instead of $\mathbb{Q}$ and simultaneous approximation of the elements $\alpha_1, \dots, \alpha_n$ algebraic over K by an element of K with respect to different valuations (see [Lan83, Chapter 7]). Actually, the statement proved in [Lan83, Chapter 7] holds for fields that are more general than number fields. At the moment we don't have an *effective* version of Roth's theorem (i.e. a bound for the constant $C(\alpha, \varepsilon)$, see [Cor95]), but we have a *quantitative* version (i.e. bounds, in terms of α and ε, on the number of good approximants see [BvdP88], [BvdP90], [Eve97]).

Let k be a field of characteristic 0 and let $\mathcal{V}_k$ be a set in bijection with a family of absolute values[1] of k which are not pairwise equivalent. The bijection is denoted by $v \mapsto |\cdot|_v$, for any $v \in \mathcal{V}_k$. The couple $(k, \mathcal{V}_k)$ *satisfies the product formula* if for any element $a \in k^\times$ the series $\sum_{v\in\mathcal{V}_k} \log|a|_v$

[1]We recall that as a representative for the complex place it is customary to consider the square of the euclidiean norm, even if this is not properly an absolute value.

converges absolutely and moreover $\sum_{v\in\mathcal{V}_k}\log|a|_v=0$. In this setting one also has a natural notion of *logarithmic height* for $a\in k^\times$:

$$h(a):=\sum_{v\in\mathcal{V}_K}\log^+|a|_v\,,$$

and we set $H(a):=e^{h(a)}$. Roth's theorem was generalised by Corvaja in [Cor97] for any couple $(k,\mathcal{V}_k)$ satisfying the product formula:

Theorem 0.2. *[Cor97, Corollaire 1]. Assume that $(k,\mathcal{V}_k)$ satisfies the product formula. Let $\alpha_1,\dots,\alpha_n$ be distinct elements algebraic on k, and let $|\cdot|_{v_1},\dots,|\cdot|_{v_n}$ be distinct absolute values of k, with $v_1,\dots,v_n\in\mathcal{V}_k$. For any $i=1,\dots,n$ let's fix an appropriately normalised extension of $|\cdot|_{v_i}$ to $k(\alpha_i)$ (and by abuse of notation denote it with the same symbol $|\cdot|_{v_i}$). Then for any $\varepsilon>0$ there exists a constant $C=C(\mathcal{V}_k,\alpha_1,\dots,\alpha_n,v_1,\dots,v_n,\varepsilon)>0$ such that for all $\beta\in k$ with $H(\beta)>C$ it holds that:*

$$\sum_{i=1}^{n}\log|\alpha_i-\beta|_{v_i}>-(2+\varepsilon)h(\beta)\,.$$

Notice that there are no further assumptions on the field k, which might be for instance a function field; therefore, theorem 0.2 is a unifying result, as well as a generalisation of the classical Roth's theorem. Moreover, in the same paper Corvaja obtained also a quantitative version of Theorem 0.2 (see [Cor97, Corollaire 3.7]).

An *arithmetic function field*, is a finitely generated field over $\mathbb{Q}$. The "geometrisation" of these fields in terms of Arakelov geometry is wonderfully explained in [Mor00]. Recently, Vojta in [Voj21] proved a version of Roth's theorem for arithmetic function fields with a big polarisation. For obvious reasons this can be considered as a "higher dimensional" generalisation of Roth's theorem. We conclude this short historic overview by explaining the statement of Vojta's result: a big polarisation of an arithmetic function field K consists of a couple $(X,\mathcal{L})$ where:

- (i) $X\to\operatorname{Spec}\mathbb{Z}$ is a normal arithmetic variety whose function field is isomorphic to K.
- (ii) If we denote with d the relative dimension of X over $\operatorname{Spec}\mathbb{Z}$, then $\mathcal{L}=\{\mathscr{L}_1,\dots\mathscr{L}_d\}$ is a collection of hermitian, arithmetically nef and big line bundles.

Now we fix an arithmetic function field K with a big polarisation; then we can define a geometric height function $h_\mathcal{L}$ for a prime divisor Y as the Arakelov intersection number of the hermitian line bundles $\mathscr{L}_1,\dots,\mathscr{L}_d$ restricted to Y. We can now define a non-archimedean absolute value on K, associated to Y:

$$|a|_Y:=e^{-h_\mathcal{L}(Y)\operatorname{ord}_Y(a)}\quad\forall a\in K\,.$$

Moreover, for any closed point $p\in X_\mathbb{C}=X\times_{\operatorname{Spec}\mathbb{Z}}\operatorname{Spec}\mathbb{C}$ that doesn't come from the base change of a divisor on X we have the following archimedean absolute value:

$$|a|_p:=\sqrt{a(p)\overline{a(p)}}\quad\forall a\in K\,.$$

By putting all together, we get a set of absolute values M_K which turns out to be a measure space with a measure that we denote with μ. The notion of product formula holds true for the couple (K,M_K) in the following form:

$$\int_{M_K}\log|a|_\nu d\mu(\nu)=0\,,\quad\forall a\in K^\times$$

Moreover there is a notion of height for any element of $a\in K^\times$:

$$h_K(a):=\int_{M_K}\log^+|a|_v\,d\mu(v)\,.$$

We set $H_K(a):=e^{h_K(a)}$. One of the many equivalent versions of Vojta's generalisation of Roth's theorem is given below.

Theorem 0.3. *[Voj21, Theorem 4.5] Consider a couple (K,M_K) where K is an arithmetic function field with a big polarisation such that $\mathscr{L}_1=\dots=\mathscr{L}_d$, and M_K is a set of absolute values as explained before. Moreover let S be a subset of M_K of finite measure and let $\alpha_1,\dots,\alpha_n$ be distinct elements of K. Then for any $\varepsilon>0$ there exists a real constant $C>0$ (depending on the fixed data) such that for any $\beta\in\mathbb{K}$ with $H_K(\beta)>C$ it holds that*

$$\int_S\max_{1\le i\le n}\left(\log^-|\beta-\alpha_i|_v\right)d\mu(v)>-(2+\varepsilon)h_K(\beta)$$

0.2 Results in this paper

The goal of this paper is to generalise Roth's theorem in a wider framework which includes Corvaja's and Vojta's settings.

The theory of adelic curves introduced by Chen and Moriwaki in [CM20] provides such natural framework: an adelic curve $\mathbb{X}$ consists of a field $\mathbb{K}$ of characteristic 0 and a measure space $(\Omega, \mathcal{A}, \mu)$ endowed with a function $\phi : \omega \mapsto |\cdot|_\omega$ that maps Ω into the set of places of $\mathbb{K}$ and such that $\omega \mapsto \log|a|_\omega$ is in $L^1(\Omega, \mu)$ for any $a \in \mathbb{K}^\times$. On $\mathbb{X}$ we have a well defined notion of height $h_{\mathbb{X}}$, and moreover a "product formula" which is expressed as an integral over Ω. The adelic curves satisfying this product formula are called *proper*. The fields with a product formula of [Cor97] are trivially proper adelic curves when the set of places is endowed with the counting measure. Moreover in [Voj21, Section 3] it is shown that arithmetic function fields can be endowed with a structure of proper adelic curve.

In this paper we prove two generalisations of Roth's theorem for proper adelic curves. The core of the results will be some inequalities involving the measurable functions $\omega \mapsto \log^- |\beta - \alpha_i|_\omega$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ are the elements we want to approximate by an approximant $\beta \in \mathbb{K}$. Unfortunately, the bare definition of proper adelic curves is too general to give any meaningful result in the direction we want, in other words the functions $\log^- |\beta - \alpha_i|$ can be in principle very "wild". Therefore, we have to impose some analytic conditions on such functions in order to get Roth's theorem. We will see that these assumptions are not too artificial, in fact we recover Theorems 0.2 and 0.3 as special cases.

The first condition we assume on our adelic curves is the *μ-equicontinuity* (see definition 4.1): roughly speaking it means that for any set of finite measure $S \subset \Omega$ one requires for the functions $\omega \mapsto \log|\beta|_\omega$, for any $\beta \in \mathbb{K}^\times$, to be "equicontinuous" on S minus a set of negligible measure. Our first version of Roth's theorem is the following:

Theorem A. *Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ be a proper adelic curve such that for any finite Galois extension $\mathbb{K}'|\mathbb{K}$ the adelic curve $\mathbb{X} = (\mathbb{K}', \Omega', \phi')$ satisfies the μ'-equicontinuity condition. Let $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_n$ be the disjoiunt union of measurable subsets of Ω with finite measure. Fix some distinct elements $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{K}}$, and choose some $\mathbb{K}$-embeddings $\iota_{\omega,i} \colon \mathbb{K}(\alpha_i) \hookrightarrow \overline{\mathbb{K}}_\omega$ for all $i = 1, \ldots, n$ and all $\omega \in S$ in such a way that the function $\omega \mapsto \log^- |\iota_{\omega,i}(\beta - \alpha_i)|_\omega$ is measurable for all i and any $\beta \in K \setminus \{\alpha_1, \ldots, \alpha_n\}$. Assume also that $\iota_{\omega,i}(\alpha_i) \neq \iota_{\omega,i'}(\alpha_{i'})$ for $i \neq i'$. Then for any $\varepsilon > 0$ there exists a real constant $C > 0$ (depending on the fixed data) such that for any $\beta \in \mathbb{K}$ with $h_{\mathbb{X}}(\beta) > C$ it holds that:*

$$\sum_{i=1}^{n} \int_{S_i} \log^- |\iota_{\omega,i}(\beta - \alpha_i)|_\omega d\mu(\omega) > -(2 + \varepsilon) h_{\mathbb{X}}(\beta) \, . \tag{1}$$

In the previous statement $\overline{\mathbb{K}}_\omega$ denotes the algebraic closure of the completion $\mathbb{K}_\omega$ (with respect to the absolute value $|\cdot|_\omega$). It is immediate to show that the μ-equicontinuity trivially holds for the fields considered in [Cor97]. Therefore Theorem A implies Theorem 0.2. On the other hand, despite [Voj21, Proposition 8.9] shows that the μ-equicontinuity holds in the case of arithmetic functions fields, Theorem A is weaker than Theorem 0.3. Furthermore we show that the μ-equicontinuity condition doesn't hold in certain cases, such as for $\overline{\mathbb{Q}}$ (see Example 4.5).

Nevertheless, if in addition we assume that the functions $\omega \to \log^- |\beta|_\omega$ are *uniformly integrable* while β varies (see definition 5.1) then we get the following:

Theorem B. *Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ be a proper adelic curve satisfying: the μ-equicontinuity condition and the uniform integrability condition. Fix some distinct elements $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Let S be a measurable subset of Ω of finite measure. Then for any $\varepsilon > 0$ there exists a real constant $C > 0$ (depending on the fixed data) such that for any $\beta \in \mathbb{K}$ with $h_{\mathbb{X}}(\beta) > C$ it holds that:*

$$\int_S \max_{1 \leq i \leq n} \left(\log^- |\beta - \alpha_i|_\omega \right) d\mu(\omega) > -(2 + \varepsilon) h_{\mathbb{X}}(\beta) \tag{2}$$

For arithmetic function fields Vojta proved the uniform integrability in [Voj21, Proposition 8.8]. Hence, as a special case of Theorem B we recover Theorem 0.3.

Remark 0.4. If one wants a version of Theorem B for algebraic extensions, then it is enough to assume the uniform integrability conditions for all Galois extensions, similarly to Theorem A. This time we avoided this further technicality in the statement in order to help the quick comparison between B and Theorem 0.3. For more details about possible equivalent statements we refer to [Voj21, Section 4].

The advantage of our approach is that our proofs are rather short and elementary, and in the case of arithmetic function fields (after assuming the technical hypotheses) we don't need any heavy Arakelov geometry. On the other hand we don't have, at the moment, any examples of interesting proper adelic curve different from the ones already known. It is a much harder problem to find new concrete cases in which Roth's theorem holds. Afterall, a highly nontrivial achievement of Vojta's work consists in showing that the μ-equicontinuity and uniform integrability conditions hold for arithmetic function fields. He does this by using the geometry of complex fibres at infinity appearing in Arakelov geometry.

Most of our the ideas are inspired by Corvaja's paper [Cor97], and the very coarse overview of the proofs is the following: Roth's theorem is about the simultaneous approximation of some distinct elements $\alpha_1, \ldots, \alpha_n$ that by simplicity (in fact it will be enough to consider this case) can be fixed in $\mathbb{K}$, with an element $\beta \in \mathbb{K}$. The existence of a very special "interpolating polynomial" δ for such elements (section 2) allows us to write some integral estimates for measurable functions on Ω satisfying some technical properties related to the heights of the α_i's and β (section 3). Then, assuming that Roth's theorem is false leads to the construction of a measurable function $\theta : S \to \mathbb{R}$ that gives the desired contradiction on the integral estimates previously found. The crucial point of the proofs consist in the construction of θ, and this is exactly where we need the additional technical conditions on the adelic curve.

We finally mention that section 1 is a brief review of the theory of adelic curves introduced in [CM20], and moreover that Appendix A sketches the construction of the interpolating polynomial employed in [Cor97].

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1 Adelic curves

We will use the following notations throughout the whole paper:

$$\log^+ x := \max\{0, \log x\}\,, \quad \log^- x := \min\{0, \log x\}\,; \quad \forall x \in \mathbb{R}_{>0}$$

In this section we closely follow [CM20, Chapter 3]. For simplicity we restrict to the characteristic 0 case, but all the definitions work also in positive characteristic.

Definition 1.1. Let $\mathbb{K}$ be a field of characteristic 0, let $M_{\mathbb{K}}$ be the set of all absolute values of $\mathbb{K}$ and let $\Omega = (\Omega, \mathcal{A}, \mu)$ be a measure space endowed with a map

$$\begin{aligned} \phi \colon \Omega &\to M_{\mathbb{K}} \\ \omega &\mapsto |\cdot|_\omega := \phi(\omega)\,. \end{aligned}$$

such that for any $a \in \mathbb{K}^\times$, the real valued function $\omega \mapsto \log |a|_\omega$ lies in $L^1(\Omega, \mu)$. The triple $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ is called an *adelic curve*; Ω and ϕ are respectively the *parameter space* and the *parametrization*. We denote with Ω_∞ the subset of Ω made of all elements ω such that $|\cdot|_\omega$ is archimedean. We set $\Omega_0 := \Omega \setminus \Omega_\infty$.

Remark 1.2. We also recall that a more general notion of adelic curve, with the requirement that $|\cdot|_\omega$ is an absolute value only almost everywhere for $\omega \in \Omega$, had been already given in [Gub97] under the name of M-field.

It is easy to show that the set Ω_∞ is always measurable [CM20, Proposition 3.1.1].

Definition 1.3. An adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ is said to be *proper* if for any $a \in \mathbb{K}^\times$:

$$\int_\Omega \log |a|_\omega \, d\mu(\omega) = 0\,. \tag{3}$$

Let's see examples of adelic curves:

Example 1.4. Any field $(k, \mathcal{V}_k)$ satisfying the product formula in the sense of [Cor97] is a proper adelic curve. In fact in this case $\Omega = \mathcal{V}_k$, ϕ is the identity and μ is the counting measure.

Example 1.5. An arithmetic function field K with a big polarisation is a proper adelic curve. A quick description of this has been already given in section 0.1. For details see [Voj21, Section 3].

Example 1.6. A polarised algebraic function field (in $d \geq 1$ variables) over a field of characteristic 0 can be endowed with a structure of proper adelic curve. For details see [CM20, 3.2.4].

In the remaining part of this section we study the behaviour of adelic curves with respect to field extensions. In particular let's fix an adelic curve $\mathbb{X} = (\mathbb{K}, \Omega_{\mathbb{K}}, \phi_{\mathbb{K}})$, and let $\mathbb{L}$ be a *finite* extension of $\mathbb{K}$, our goal is to endow it with a canonical structure of adelic curve "coming from $\mathbb{K}$". In other words, we have to define a parameter space $\Omega_{\mathbb{L}}$ and a parametrization $\phi_{\mathbb{L}}$ in a canonical way by starting from $\Omega_{\mathbb{K}}$ and $\phi_{\mathbb{K}}$. For any $\omega \in \Omega$ we denote with $M_{\mathbb{L},\omega}$ the set of absolute values of $\mathbb{L}$ extending $|\cdot|_\omega$, so we put:

$$\Omega_{\mathbb{L}} := \bigsqcup_{\omega\in\Omega_{\mathbb{K}}} M_{\mathbb{L},\omega}$$

and we have a natural projection map $\pi_{\mathbb{L}|\mathbb{K}} : \Omega_{\mathbb{L}} \to \Omega_{\mathbb{K}}$ whose fibres are $M_{\mathbb{L},\omega}$, for any ω. The inclusion $M_{\mathbb{L},\omega} \subset M_{\mathbb{L}}$ clearly induces a parametrization $\phi_{\mathbb{L}} \colon \Omega_{\mathbb{L}} \to M_{\mathbb{L}}$ and for any $\nu \in \Omega_{\mathbb{L}}$ we put $|\cdot|_\nu := \phi_{\mathbb{L}}(\nu)$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \Omega_{\mathbb{L}} & \xrightarrow{\pi_{\mathbb{L}|\mathbb{K}}} & \Omega_{\mathbb{K}} \\ \downarrow{\scriptstyle \phi_{\mathbb{L}}} & & \downarrow{\scriptstyle \phi_{\mathbb{K}}} \\ M_{\mathbb{L}} & \longrightarrow & M_{\mathbb{K}} \end{array} \tag{4}$$

where the bottom map is the restriction function. Note that $\Omega_{\mathbb{L}}$ can be abstractly defined as the fibered product in the category of sets. Now on $\Omega_{\mathbb{L}}$ we put the σ-algebra $\mathcal{B}$ generated by $\pi_{\mathbb{L}|\mathbb{K}}$ and all the real maps $\Omega_{\mathbb{L}} \ni \nu \mapsto |a|_\nu$, for any $a \in \mathbb{L}^\times$ (on $\mathbb{R}$ we put the standard Lebesgue measure). We want to define a suitable measure η on the measurable space $(\Omega_{\mathbb{L}}, \mathcal{B})$. This requires a bit of work, since in general there is no straightforward definition of pullback measure through a measurable map. Nevertheless, in the case of our $\pi_{\mathbb{L}|\mathbb{K}}$ we explain how it is possible to define the pullback $\eta = \pi^*_{\mathbb{L}|\mathbb{K}}\mu$ (actually this is a construction from measure theory which works in full generality any time we have a measure fiberwise). Consider a fiber $M_{\mathbb{L},\omega} \subset \Omega_{\mathbb{L}}$, then for any $\nu \in M_{\mathbb{L},\omega}$ we can put

$$P_\omega(\nu) := \frac{[\mathbb{L}_\nu : \mathbb{K}_\omega]}{[\mathbb{L} : \mathbb{K}]} \tag{5}$$

where $\mathbb{L}_\nu$ and $\mathbb{K}_\omega$ denote the completions with respect to $|\cdot|_\nu$ and $|\cdot|_\omega$ respectively. Thanks to the well known equality $\prod_{\nu\in M_{\mathbb{L},\omega}}[\mathbb{L}_\nu : \mathbb{K}_\omega] = [\mathbb{L} : \mathbb{K}]$ (See [Neu99, Corollary 8.4]), we conclude that equation (5) induces a probability measure on the fibre $M_{\mathbb{L},\omega}$, with respect to the power set. Now, for any function $f : \Omega_{\mathbb{L}} \to \mathbb{R}$, by using the fiberwise integral along each probabilised fiber $M_{\mathbb{L},\omega}$, we define the map $I_{\mathbb{L}|\mathbb{K}}(f) : \Omega_{\mathbb{K}} \to \mathbb{R}$ as:

$$I_{\mathbb{L}|\mathbb{K}}(f) : \omega \mapsto \int_{M_{\mathbb{L},\omega}} f \, dP_\omega = \sum_{\nu\in M_{\mathbb{L},\omega}} P_\omega(\nu) f(\nu)$$

Proposition 1.7. *The linear operator $I_{\mathbb{L}|\mathbb{K}}$ sends $\mathcal{B}$-measurable functions to $\mathcal{A}$-measurable functions.*

Proof. See [CM20, Theorem 3.3.4]. □

At this point are ready to define the measure η. For any $E \in \mathcal{B}$ we put:

$$\eta(E) := \int_{\Omega_{\mathbb{K}}} I_{\mathbb{L}|\mathbb{K}}(\chi_E) \, d\mu \tag{6}$$

where χ_E is the characteristic function of E. Note that the integral of equation (6) makes sense because of Proposition 1.7.

Theorem 1.8. *The following statements hold:*

(1) *The map η defined above is a measure on $(\Omega_{\mathbb{L}}, \mathcal{B})$ such that for any $\mathcal{B}$-measurable function f we have*

$$\int_{\Omega_{\mathbb{L}}} f \, d\eta = \int_{\Omega_{\mathbb{K}}} I_{\mathbb{L}|\mathbb{K}}(f) \, d\mu \,.$$

(2) *$f \in L^1(\eta)$ if and only if $I_{\mathbb{L}|\mathbb{K}}(|f|) \in L^1(\mu)$.*

(3) *The pushforward measure of η through $\pi_{\mathbb{L}|\mathbb{K}}$ is μ.*

(4) *With the above constructions the triple $\mathbb{Y} = (\mathbb{L}, \Omega_{\mathbb{L}}, \phi_{\mathbb{L}})$ is an adelic curve. Moreover for any $b \in \mathbb{L}^\times$*

$$[\mathbb{L} : \mathbb{K}] \int_{\Omega_{\mathbb{L}}} \log |b|_\nu d\eta(\nu) = \int_{\Omega_{\mathbb{K}}} \log |N_{\mathbb{L}|\mathbb{K}}(b)|_\omega d\mu(\omega) \tag{7}$$

and in particular if $\mathbb{X}$ is proper, then also $\mathbb{Y}$ is proper.

Proof. See [CM20, Theorem 3.3.7]. □

At this point we study *algebraic* extensions of adelic curves. Let's fix the adelic curve $\mathbb{X} = (\mathbb{K}, \Omega_K, \phi_K)$ and let $\mathbb{L}$ an algebraic extension of $\mathbb{K}$. We denote with $\mathcal{F}_{\mathbb{L}|\mathbb{K}}$ the family of finite field extensions on $\mathbb{K}$ contained in $\mathbb{L}$. Clearly $\mathcal{F}_{\mathbb{L}|\mathbb{K}}$ is a directed set with respect to the inclusion, and for any $\mathbb{K}', \mathbb{K}'' \in \mathcal{F}_{\mathbb{L}|\mathbb{K}}$ such that $\mathbb{K}' \subseteq \mathbb{K}''$ we have a morphism of measurable spaces

$$\pi_{\mathbb{K}''|\mathbb{K}'} \colon (\Omega_{\mathbb{K}''}, \mathcal{B}'') \to (\Omega_{\mathbb{K}'}, \mathcal{B}')$$

and an operator $I_{\mathbb{K}''|\mathbb{K}'}$ sending integrable functions to integrable functions as described above in the case of finite extensions. In other words we obtain an inverse system of measure spaces, and we would like to define the adelic structure on $\mathbb{L}$ as "a projective limit". Unfortunately, in the category of measure spaces we don't have a straightforward notion of projective limit, therefore we need again a bit of extra work. We can define respectively $M_{\mathbb{L},\omega}$, $\Omega_{\mathbb{L}}$ and $\phi_{\mathbb{L}} \colon \Omega_{\mathbb{L}} \to M_{\mathbb{L}}$ exactly as we did before in the case of finite extensions, but we need to construct an adequate structure of measure space on $\Omega_{\mathbb{L}}$. For any $K' \in \mathcal{F}_{\mathbb{L}|\mathbb{K}}$ we have a map $\pi_{\mathbb{L}|\mathbb{K}'} : \Omega_{\mathbb{L}} \to \Omega_{\mathbb{K}'}$ and a square diagram like (4). It turns out that $\pi_{\mathbb{L}|\mathbb{K}'}$ is surjective [CM20, Proposition 3.4.5]. We endow $\Omega_{\mathbb{L}}$ with the σ-algebra Σ generated by the maps $\{\pi_{\mathbb{L}|\mathbb{K}'}\}_{\mathbb{K}' \in \mathcal{F}_{\mathbb{L}|\mathbb{K}}}$, and it can be shown that $(\Omega_{\mathbb{L}}, \Sigma)$ is the projective limit of the inverse system $\{(\Omega_{\mathbb{K}'}, \mathcal{B}')\}_{\mathbb{K}' \in \mathcal{F}_{\mathbb{L}|\mathbb{K}}}$ in the category of measurable spaces. It remains the issue of putting a canonical measure λ on $(\Omega_{\mathbb{L}}, \Sigma)$. This process is quite technical, but similarly to the case of finite extensions, it can be done by using a fiberwise intergration on each $M_{L,\omega}$; all the details are given in [CM20, 3.4.2]. What we really need is the fact that we can construct an adelic curve $(\mathbb{L}, \Omega_{\mathbb{L}}, \lambda)$ which is proper if $\mathbb{X}$ is proper and such that for any $f \in L^1(\mu)$ we have that $f \circ \pi_{\mathbb{L}|\mathbb{K}} \in L^1(\lambda)$ with

$$\int_{\Omega_{\mathbb{L}}} (f \circ \pi_{\mathbb{L}|\mathbb{K}}) d\lambda = \int_{\Omega_{\mathbb{K}}} f d\mu \,. \tag{8}$$

Below we give the notion of height for proper adelic curves:

Definition 1.9. Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ be a proper adelic curve and let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. Then we have a proper adelic curve $\overline{\mathbb{X}} = (\overline{\mathbb{K}}, \overline{\Omega}, \overline{\phi})$ and the *(naive) height* of an element $a \in \overline{\mathbb{K}}^{\times}$ is defined as:

$$h_{\mathbb{X}}(a) := \int_{\overline{\Omega}} \log^+ |a|_\nu \, d\chi(\nu) \,.$$

where ν denotes a generic element of $\overline{\Omega}$ and χ is the measure on $\overline{\Omega}$. Moreover we set $H_{\mathbb{X}} := e^{h_{\mathbb{X}}}$.

From now on we always assume that for an adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ we have fixed algebraic closure of $\mathbb{K}$, therefore also $\overline{\mathbb{X}}$ is fixed and we use the same notations of Definition 1.9. If for $\nu \in \overline{\Omega}$, $|\cdot|_\nu$ is an archimedean absolute value, then by Ostrowski's theorem we know that there exists a real number $\varepsilon(\nu) \in]0,1]$ such that $|\cdot|_\nu = |\cdot|^{\varepsilon(\nu)}$ where on the right we mean the standard euclidean absolute value on $\mathbb{R}$ or $\mathbb{C}$. Thus we have a map $\varepsilon : \overline{\Omega}_\infty \to]0,1]$ which can be extended to $\varepsilon : \overline{\Omega} \to [0,1]$ by putting $\varepsilon_{|\overline{\Omega}_0} := 0$. For instance, for an archimedean $|\cdot|_\nu$ we have $\log|2|_\nu = \varepsilon(\nu)\log 2$, therefore we obtain the explicit expression of the function ε on the whole $\overline{\Omega}$:

$$\varepsilon(\nu) = \frac{\log^+ |2|_\nu}{\log 2} \,.$$

Clearly $\varepsilon(\nu)$ is a measurable function. We can always take a scaling μ' of the measure μ on Ω so that get a new height $h'_{\mathbb{X}}$ that satisfies $h'_{\mathbb{X}}(2) \leq \log 2$. Notice that if $\mathbb{X}$ is proper, then it remains proper after any scaling of the measure μ. From now on, when we are given an adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$, we can always assume that we have performed the above mentioned scaling of the measure μ on Ω so that $h_{\mathbb{X}}(2) \leq \log 2$.

Definition 1.10. Let $P(X_1, \cdots, X_N)$ be a polynomial over $\overline{\mathbb{K}}$, $\alpha \in \overline{\mathbb{K}}^N$ and $\mathbf{i} = (i_1, \cdots, i_N) \in \mathbb{N}^N$. We set:

$$\Delta^{\mathbf{i}} P(\alpha) := \frac{1}{i_1! i_2! \cdots i_N!} \frac{\partial^{i_1+i_2+\ldots+i_N} P}{\partial X_1^{i_1} \partial X_2^{i_2} \ldots \partial X_N^{i_N}}(\alpha).$$

We can define the *local height of P at* $\nu \in \overline{\Omega}$ in the following way:

$$h_\nu(P) := \log\left(\max_{\mathbf{i} \in \mathbb{N}^N} \left\{ \left| \Delta^{\mathbf{i}} P(0, \ldots, 0) \right|_\nu \right\}\right)$$

and then we have also the notion of *global height* of P:

$$h_{\mathbb{X}}(P) := \int_{\overline{\Omega}} h_\nu(P) d\chi(\nu)$$

We put $H_{\mathbb{X}}(P) := e^{h_{\mathbb{X}}(P)}$.

We conclude the section with some rather straightforward results about heights. First of all when we want to calculate the heights of elements lying in $\mathbb{K}^\times$, we don't need to involve the algebraic closure $\overline{\mathbb{K}}$ in the integrals:

Proposition 1.11. *Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ be an adelic curve. If $a \in \mathbb{K}^\times$, then:*

$$h_{\mathbb{X}}(a) = \int_{\Omega_K} \log^+ |a|_\omega d\mu(\omega)\,.$$

Moreover, the same result holds for the heights of polynomials in $\mathbb{K}[X_1, \dots, X_N]$.

Proof. It is an immediate consequence of equation (8). □

In order to simplify the notations, we often omit the subscript $\mathbb{X}$ attached to the heights when the adelic curve is fixed and there is no confusion.

Proposition 1.12. *The height function of a proper adelic curve $(\mathbb{X}, \Omega, \phi)$ satisfies the following properties for any $a, b, a_1, \dots, a_m \in \mathbb{K}$ and any measurable set $S \subseteq \Omega$*

(1) $h(a) = h(a^{-1})$
(2) $-h(a) \leq \int_S \log |a|_\omega d\mu(\omega) \leq h(a)$
(3) $\int_S \log^- |a|_\omega d\mu(\omega) \geq -2h(a)$
(4) $h(a_1 + \dots + a_m) \leq h(m) + h(a_1) + \dots + h(a_m)$
(5) $\int_S \log |a - b|_\omega d\mu(\omega) \geq -\log 2 - h(a) - h(b)$
(6) $\int_S \log^- |a - b|_\omega d\mu(\omega) \geq -\log 4 - 2h(a) - 2h(b)$

Proof. (1) It follows from the product formula and from the fact that $\log |a|_\omega = \log^+ |a|_\omega - \log^+ |\frac{1}{a}|_\omega$.
(2) By definition $\int_S \log |a|_\omega d\mu(\omega) \leq h(a)$, so for the other inequality it is enough to use (1).
(3) By the product formula we have:

$$\int_S \log^- |a|_\omega d\mu(\omega) = -\int_{\Omega \setminus S} \log |a|_\omega d\mu(\omega) - \int_S \log^+ |a|_\omega d\mu(\omega)\,.$$

Then we apply (2) on the right hand side.
(4) It follows from $|a_1 + \dots + a_m|_\omega \leq m \max_i |a_i|_\omega$.
(5) and (6) are direct consequences of (2)-(4) and the fact that $h(2) \leq \log(2)$. □

Here we stress that the entries (5)-(6) of Proposition 1.12 replace the classical Liouville inequality for heights. Finally we recall an important property of heights:

Definition 1.13. A proper adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \mu)$ satisfies the *Northcott property* if for any $C \in \mathbb{R}$ the set $\{\alpha \in \mathbb{K} \colon h_{\mathbb{X}}(a)\} \leq C$ is finite.

Arithmetic function fields satisfy Northcott properties thanks to [Mor00, Theorem 4.3].

2 The interpolating polynomial

We fix a proper adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ and an algebraic closure of $\mathbb{K}$. In this section we recall the existence of an interpolating polynomial $\delta \in \mathbb{K}[X_1, \dots, X_N]$ associated to some elements $\alpha_1, \dots, \alpha_n$ having some explicit bounds on: the degree, the d-index at all the α_j's and the height. The complete construction of δ can be found in [Cor97], and we will recall it in appendix A.

We fix for the whole section the following data: two natural numbers $n, N \geq 2$, and a vector $d = (d_1, \dots, d_N) \in \mathbb{R}_+^N$.

Definition 2.1. The *d-index of $P(X_1, \dots, X_N)$ at $\alpha \in \mathbb{R}^N$* is the real number:

$$\operatorname{Ind}_{\alpha,d}(P) := \min_{\mathbf{i} \in \mathbb{N}^N} \left\{ \sum_{j=1}^N \frac{i_j}{d_j} \in \mathbb{R}_+ \colon \Delta^{\mathbf{i}} P(\alpha) \neq 0 \right\}$$

Let's fix $t \in \mathbb{R}$ such that $0 < t < N$, the following two sets will play a central role in the theory:

$$\mathcal{G}_t := \left\{ \mathbf{i} \in \mathbb{N}^N \colon i_j \leq d_j \ \forall j = 1, \dots, N, \text{ and} \sum_{j=1}^N \frac{i_j}{d_j} \leq t \right\},$$

$$\mathcal{C}_t := \left\{ (x_1, \dots, x_N) \in [0,1]^N \colon \sum_{j=1}^N x_j \leq t \right\}$$

The Lebesgue measure of $\mathcal{C}_t$ will be denoted as $V(t)$, and for simplicity of terminology we will refer to it simply as "volume".

Lemma 2.2. *The cardinality of $\mathcal{G}_t$ is asymptotic to $d_1 d_2 \dots d_N V(t)$.*

Proof. See [BG06, p. 157]. □

Now fix some vectors $\alpha_1, \dots, \alpha_n \in \mathbb{K}^N$ where $\alpha_h = (\alpha_h^{(1)}, \dots, \alpha_h^{(N)})$ for every $h = 1, \dots, n$ and let $X = (X_1, \dots, X_N)$ be a vector of variables. Moreover we say that two vectors $\mathbb{K}^N$ are *componentwise different* if they differ component by component. For any two multi-indices $\mathbf{a} = (a_1, a_2, \dots, a_N)$ and $\mathbf{i} = (i_1, i_2, \dots, i_N)$ of $\mathbb{N}^N$ we use the following notations:

$$\binom{\mathbf{a}}{\mathbf{i}} := \binom{a_1}{i_1}\binom{a_2}{i_2}\dots\binom{a_N}{i_N}$$

$$\alpha_h^{\mathbf{i}} := (\alpha_h^{(1)})^{i_1}(\alpha_h^{(2)})^{i_2}\dots(\alpha_h^{(N)})^{i_N}$$

$$X^{\mathbf{i}} := X_1^{i_1} X_2^{i_2} \dots X_N^{i_N}$$

with the convention $\binom{p}{q} = 0$ if $0 \leq p < q$ for the binomial coefficient. Now, consider $\gamma \in \mathbb{R}$ such that $0 < \gamma < \frac{1}{2nN^2}$; we always assume that $\frac{d_{j+1}}{d_j} \leq \gamma$ for any $j = 1, \dots, N-1$, which means in particular that $d_j = O(d_1)$ for any j. We also put $\eta := 2\gamma n < \frac{1}{N^2}$ and $d := d_1 d_2 \dots d_N$.

We have the following result about the existence of a polynomial $\delta(X_1, \dots, X_N) \in \mathbb{K}[X_1, \dots X_N]$ with some prescribed properties. In the appendix A we will sketch Corvaja's construction of $\delta(X_1, \dots, X_N)$ adapting it to the case of adelic curves:

Proposition 2.3. *Assume that in the adelic curve $\mathbb{X}$ the condition $h_{\mathbb{X}}(2) \leq \log 2$ is satisfied and assume that the number $\eta \in \mathbb{R}$ is chosen as explained above. Let's fix some vectors $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{K}^N$ that are pairwise componentwise different. Moreover let's choose some parameters $s, t_1, \dots, t_n \in \mathbb{R}$, with $0 < s < 1$ and $0 < t_h < \frac{N}{2}$ for $h = 1, \dots, n$ such that the following condition on volumes is verified:*

$$(1+\eta)^N < V(s) + \sum_{h=1}^{n} V(t_h) < 1 + 2N\eta \tag{9}$$

Then there exists a polynomial $\delta(X_1, \dots X_N) \in \mathbb{K}[X_1, \dots, X_N]$ satysfying the following properties:

(1) $\delta(\beta) \neq 0$;

(2) $\deg_{X_j} \delta \leq d d_j V(s) + O(d)$, *for* $1 \leq j \leq N$;

(3) $\mathrm{Ind}_{\alpha_h, d}(\delta) \geq dV(s)\left(t_h - s - \frac{2N^2\eta}{V(s)}\right)$, *for* $1 \leq h \leq n$;

(4) $h_{\mathbb{X}}(\delta) \leq d \sum_{j=1}^{N} d_j \left(\log 2 + \sum_{h=1}^{N} V(t_h) h(\alpha_h^{(j)})\right) + O(\log d)$;

Proof. See [Cor97, Proposition 2.6]. □

3 Integral estimates

This technical section is "the heart" of the proof of our results since here we will prove some integral bounds for very particular measurable functions $\theta : S \subset \Omega \to \mathbb{R}$. We continue with all the notations fixed in section 2 since we want to make full use of Theorem 2.3.

Consider an adelic curve $(\mathbb{K}, \Omega, \phi)$ and a set of vectors $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{K}^N$ which are componentwise different. We construct the following matrices $T := T(\alpha_1, \dots \alpha_n) \in M(n \times N, \mathbb{K})$ and $T(\beta) \in M(n+1 \times N, \mathbb{K})$:

$$T := \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \alpha_1^{(3)} & \dots & \alpha_1^{(N)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \alpha_2^{(3)} & \dots & \alpha_2^{(N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_n^{(1)} & \alpha_n^{(2)} & \alpha_n^{(3)} & \dots & \alpha_n^{(N)} \end{pmatrix}; \quad T(\beta) = \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \alpha_1^{(3)} & \dots & \alpha_1^{(N)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \alpha_2^{(3)} & \dots & \alpha_2^{(N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_n^{(1)} & \alpha_n^{(2)} & \alpha_n^{(3)} & \dots & \alpha_n^{(N)} \\ \beta^{(1)} & \beta^{(2)} & \beta^{(3)} & \dots & \beta^{(N)} \end{pmatrix}$$

We denote by $\alpha^{(j)}$, $j=1,\dots,N$ the columns of T, that is:

$$\alpha^{(j)} = \begin{pmatrix} \alpha_1^{(j)} \\ \alpha_2^{(j)} \\ \vdots \\ \vdots \\ \alpha_n^{(j)} \end{pmatrix}$$

and by α_h, for $h \in \{1,2,\dots,n\}$, the rows:

$$\alpha_h = \left(\alpha_h^{(1)}, \alpha_h^{(2)}, \alpha_h^{(3)}, \dots, \alpha_h^{(N)}\right)$$

Note that we are asking for the matrices T and $T(\beta)$ to have componentwise different rows. Now we need to define a list of properties depending on the aforementioned matrices:

Definition 3.1. For the matrix $T(\beta)$, consider the following real numbers for any $j=1,\dots,N$:

$$\rho_j := 4^{2N!} H(\beta^{(j)}) \prod_{h=1}^{n} H(\alpha_h^{(j)})^{\frac{2N!}{n}} \tag{10}$$

$$\rho_j' := 4^{N!} H(\beta^{(j+1)}) \prod_{h=1}^{n} H(\alpha_h^{(j+1)})^{\frac{N!}{2n}} \tag{11}$$

We say that $T(\beta)$ satisfies the *h-gap condition* is the following inequality is satisfied:

$$\frac{\log \rho_j}{\log \rho_j'} < \frac{1}{4N^2 N!}, \qquad \forall j = 1,\dots,N$$

Definition 3.2. Let's fix a matrix $T(\beta)$. Given some mutually disjoint measurable subsets $S_1,\dots,S_n \subset \Omega$ of finite measure such that $S := S_1 \sqcup S_2 \sqcup \dots \sqcup S_n$, we define:

$$\hat{S} := \left\{\omega \in S \colon \left|\alpha_h^{(j)} - \beta^{(j)}\right|_\omega < 1,\ \forall j = 1,\dots,N,\ \forall h = 1\dots,n\right\}.$$

Moreover we put $\hat{S}_h := \hat{S} \cap S_h$, so that $\hat{S} = \hat{S}_1 \sqcup \hat{S}_2 \sqcup \dots \sqcup \hat{S}_n$.

Note that $\hat{S}$ can be thought as the sublocus of S where it happens that $\left(\beta^{(1)},\dots,\beta^{(N)}\right)$ is a good approximant for all the vectors α_h, moreover we emphasize the fact that $\hat{S}$ depends on $T(\beta)$, whereas S is fixed. The following easy lemma will be useful:

Lemma 3.3. *If $\omega \in \hat{S}_h$, then $\log^+ |\alpha_h^{(j)}|_\omega \le \log^+ |2|_\omega + \log^+ |\beta^{(j)}|_\omega$ for any $j = 1,\dots,N$.*

Proof. The claim holds true for any couple of vectors $\alpha, \beta \in \mathbb{K}^N$, so in order to ease the notations we can just drop any reference to the index h. We distinguish two cases:
$|\cdot|_\omega$ *is ultrametric.* Then

$$|\alpha^{(j)}|_\omega = |\alpha^{(j)} - \beta^{(j)} + \beta^{(j)}|_\omega \le \max\left\{|\alpha^{(j)} - \beta^{(j)}|_\omega, |\beta^{(j)}|_\omega\right\} < \max\left\{1, |\beta^{(j)}|_\omega\right\}.$$

If $|\beta^{(j)}|_\omega \ge 1$, then clearly $\log^+ |\alpha^{(j)}|_\omega \le \log^+ |\beta^{(j)}|_\omega$. If $|\beta^{(j)}|_\omega < 1$, then $|\alpha^{(j)}|_\omega < 1$ which means $0 = \log^+ |\alpha^{(j)}|_\omega \le \log^+ |2|_\omega + \log^+ |\beta^{(j)}|_\omega$.

$|\cdot|_\omega$ *is archimedean.* By Ostrowski's theorem $|\cdot|_\omega = |\cdot|^{\varepsilon(\omega)}$, therefore

$$|\alpha^{(j)}|_\omega^{\frac{1}{\varepsilon(\omega)}} \le |\alpha^{(j)} - \beta^{(j)}|_\omega^{\frac{1}{\varepsilon(\omega)}} + |\beta^{(j)}|_\omega^{\frac{1}{\varepsilon(\omega)}} \le 1 + |\beta^{(j)}|_\omega^{\frac{1}{\varepsilon(\omega)}}.$$

This clearly means $|\alpha^{(j)}|_\omega^{\frac{1}{\varepsilon(\omega)}} \le 2\max\left\{1, |\beta^{(j)}|_\omega^{\frac{1}{\varepsilon(\omega)}}\right\}$. After rising each side to the power of $\varepsilon(\omega)$ and applying $\log^+$ we get the claim. □

Definition 3.4. Fix $S = S_1 \sqcup \dots \sqcup S_n$ of finite measure. We say an the integrable function $\theta \colon S \to \mathbb{R}$ is a *column bounding function for $T(\beta)$* if the following inequality holds:

$$-\frac{1}{\log \rho_j} \log|\alpha_h^{(j)} - \beta^{(j)}|_\omega > \theta(\omega) \qquad \forall j = 1,\dots,N\,,\ \forall h = 1,\dots,n\,,\ \forall \omega \in \hat{S}_h \tag{12}$$

Definition 3.5. Fix $S = S_1 \sqcup \ldots \sqcup S_n$ of finite measure. For any column $\alpha^{(j)} \in \mathbb{K}^N$ of the matrix T and any $b \in \mathbb{K}$ we define the following quantity:

$$\lambda(\alpha^{(j)}, b) := \frac{1}{V(s)}\log 4 + h(b) + \frac{1}{V(s)}\sum_{h=1}^{n} V(t_h) h(\alpha_h^{(j)})$$

Definition 3.6. We say that $T(\beta)$ satisfies the λ-*gap property* of width $\eta \in \mathbb{R}_+$ if

$$\max_{1\le j\le N-1} \frac{\lambda(\alpha^{(j)}, \beta^{(j)})}{\lambda(\alpha^{(j+1)}, \beta^{(j+1)})} < \frac{\eta}{2n}$$

Definition 3.7. Fix a matrix $T(\beta)$. An integrable function $\theta\colon S \to \mathbb{R}$ is λ-*bounding* if:

$$\frac{-1}{\lambda(\alpha^{(j)}, \beta^{(j)})}\log|\alpha_h^{(j)} - \beta^{(j)}|_\omega > \theta(\omega) \qquad \forall j = 1, \ldots, N;\ \forall h = 1, \ldots, n;\ \forall \omega \in \hat{S}_h \tag{13}$$

We will need the following lemma:

Lemma 3.8. *Let $(\mathbb{K}, |\cdot|_\omega)$ be a field with an absolute value. Let $P \in \mathbb{K}[X_1, \ldots X_N]$ such that $N \ge 1$. Then for every $\alpha \in \mathbb{K}^N$*

$$\log|\Delta^{\mathbf{i}} P(\alpha)|_\omega \le h_\omega(P) + \sum_{j=1}^{N}\log(1 + \deg_{X_j} P) + \sum_{j=1}^{N}(\log^+|2|_\omega + \log^+|\alpha^{(j)}|_\omega)\deg_{X_j} P$$

Proof. See [Cor97, Lemme pag 166]. □

Now we prove the analogue of [Cor97, Proposition 3.1].

Proposition 3.9. *Let $(\mathbb{K}, \Omega, \phi)$ be a proper adelic curve. Fix a matrix $T(\beta)$. Let $\eta, s, t_1, \ldots, t_n \in \mathbb{R}$ such that all the hypotheses of Theorem 2.3 are satisfied. Assume that $T(\beta)$ satisfies the λ-gap property of width η (see Definition 3.6). If $\theta\colon S \to \mathbb{R}$ is λ-bounding for $T(\beta)$ (see Definition 3.7), then*

$$\sum_{h=1}^{n}\left(t_h - s - \tfrac{2\eta N^2}{V(s)}\right)\int_{\hat{S}_h}\theta d\mu < N$$

Proof. Fix a real number $D > 0$, and for every $j = 1, \ldots, N$, we put

$$d_j := \frac{D}{\lambda(\alpha^{(j)}, \beta^{(j)})}$$

and $d := \prod_{j=1}^{N} d_j$. Note that here we use the λ-gap property to ensure that $\frac{dj+1}{d_j} < \gamma$. So, the hypotheses of Theorem 2.3 are all satisfied and we have the interpolation polynomial δ such that $\delta(\beta) \neq 0$. We start by distinguishing two cases.

First case: $\omega \in \hat{S}_h$. Compose the polynomial δ with the translation of $X \mapsto X - \alpha_h$, then we have

$$\delta(\beta) = \sum_{\mathbf{i}} \Delta^{\mathbf{i}}\delta(\alpha_h)\prod_{j=1}^{N}(\beta^{(j)} - \alpha_h^{(j)})^{i_j} \quad \text{where } \mathbf{i} = (i_1, \ldots, i_n) \tag{14}$$

Now take the absolute value $|\cdot|_\omega$ on both sides and notice that each term

$$\left|\Delta^{\mathbf{i}}\delta(\alpha_h)\prod_{j=1}^{N}(\beta^{(j)} - \alpha_h^{(j)})^{i_j}\right|_\omega$$

is bounded from above by:

$$\max_{\mathbf{i}}\left|\Delta^{\mathbf{i}}\delta(\alpha_h)\right|_\omega \ \overset{*}{\max_{\mathbf{i}}}\left|\prod_{j=1}^{N}(\beta^{(j)} - \alpha_h^{(j)})^{i_j}\right|_\omega,$$

were in order to simplify the notation we put

$$\overset{*}{\max_{\mathbf{i}}} := \max_{\substack{\mathbf{i} \\ \Delta^{\mathbf{i}}(\alpha_h)\neq 0}}$$

In other words from equation (14) we get

$$\log|\delta(\beta)|_\omega \le \sum_{j=1}^{N}\log(1 + \deg_{X_j}\delta) + \max_{\mathbf{i}}\log\left|\Delta^{\mathbf{i}}\delta(\alpha_h)\right|_\omega + \overset{*}{\max_{\mathbf{i}}}\log\left|\prod_{j=1}^{N}(\beta^{(j)} - \alpha_h^{(j)})^{i_j}\right|_\omega \tag{15}$$

The last summand of equation (15) is bounded from above by

$$-\mathrm{Ind}_{\alpha_h,d}(\delta)\,\min\left\{d_1\log\frac{1}{|\beta^{(1)}-\alpha_h^{(1)}|_\omega},\dots,d_N\log\frac{1}{|\beta^{(N)}-\alpha_h^{(N)}|_\omega}\right\}.$$

We use Lemma 3.8 to give an upper bound for $\log\left|\Delta^{\mathbf{i}}\delta(\alpha_h)\right|_\omega$ and by equation (15) we deduce:

$$\int_{\hat S_h}\log|\delta(\beta)|_\omega d\mu(\omega)\le\int_{\hat S_h}\left(h_\omega(\delta)+\sum_{j=1}^N\log(1+\deg_{X_j}\delta)+\sum_{j=1}^N(\log^+|2|_\omega+\log^+|\alpha_h^{(j)}|_\omega)\deg_{X_j}\delta\right)d\mu(\omega)+$$

$$+\int_{\hat S_h}\sum_{j=1}^N\log(1+\deg_{X_j}\delta)d\mu(\omega)-\int_{\hat S_h}(\mathrm{Ind}_{\alpha_h,d}(\delta)\,\min\left\{d_1\log\frac{1}{|\beta^{(1)}-\alpha_h^{(1)}|_\omega},\dots,d_N\log\frac{1}{|\beta^{(N)}-\alpha_h^{(N)}|_\omega}\right\}d\mu(\omega)$$

Now use the fact that $\hat S$ has finite measure, and we put

$$\tau:=\sum_{j=1}^N\log(1+\deg_{X_j}\delta)\mu(\hat S)$$

So, by rearranging the terms and summing over all $h=1,\dots,n$ we get

$$\sum_{h=1}^n\mathrm{Ind}_{\alpha_h,d}(\delta)\int_{\hat S_h}\min\left\{d_1\log\frac{1}{|\beta^{(1)}-\alpha_h^{(1)}|_\omega},\dots,d_N\log\frac{1}{|\beta^{(N)}-\alpha_h^{(N)}|_\omega}\right\}d\mu(\omega)\le-\int_{\hat S}\log|\delta(\beta)|_\omega d\mu(\omega)$$

$$+\int_{\hat S}h_\omega(\delta)d\mu(\omega)+\tau+\sum_{h=1}^n\int_{\hat S_h}\sum_{j=1}^N(\log^+|2|_\omega+\log^+|\alpha_h^{(j)}|_\omega)\deg_{X_j}\delta\, d\mu(\omega)\quad(16)$$

At this point we can use Lemma 3.3 for the following bound:

$$\sum_{h=1}^n\int_{\hat S_h}\sum_{j=1}^N(\log^+|2|_\omega+\log^+|\alpha_h^{(j)}|_\omega)\deg_{X_j}\delta d\mu(\omega)\le\int_{\hat S}\sum_{j=1}^N(\log^+|4|_\omega+\log^+|\beta_h^{(j)}|_\omega)\deg_{X_j}\delta d\mu(\omega)\quad(17)$$

to obtain

$$\sum_{h=1}^n\mathrm{Ind}_{\alpha_h,d}(\delta)\int_{\hat S_h}\min\left\{d_1\log\frac{1}{|\beta^{(1)}-\alpha_h^{(1)}|_\omega},\dots,d_N\log\frac{1}{|\beta^{(N)}-\alpha_h^{(N)}|_\omega}\right\}d\mu(\omega)\le-\int_{\hat S}\log|\delta(\beta)|_\omega d\mu(\omega)$$

$$+\int_{\hat S}h_\omega(\delta)d\mu(\omega)+\tau+\int_{\hat S}\sum_{j=1}^N(\log^+|4|_\omega+\log^+|\beta_h^{(j)}|_\omega)\deg_{X_j}\delta d\mu(\omega)\,.\quad(18)$$

Second case: $\omega\notin\hat S$. Consider the expression:

$$\delta(\beta)=\sum_{\mathbf{i}}\Delta^{\mathbf{i}}\delta(0)\beta^{(1)i_1}\dots\beta^{(N)i_N}\quad\text{where }\mathbf{i}=(i_1,\dots,i_n)$$

then, similarly to equation (15) we have the bound:

$$\log|\delta(\beta)|_\omega\le h_\omega(\delta)+\sum_{j=1}^N(\log^+|\beta^{(j)}|_\omega\deg_{X_j}\delta)+N\log d$$

Hence

$$\int_{\Omega\setminus\hat S}\log|\delta(\beta)|_\omega d\mu(\omega)\le\int_{\Omega\setminus\hat S}h_\omega(\delta)d\mu(\omega)+\sum_{j=1}^N\int_{\Omega\setminus\hat S}(\log^+|\beta^{(j)}|_\omega\deg_{X_j}\delta)d\mu(\omega)+N\log d\quad(19)$$

Since $(\mathbb{K},\Omega,\phi)$ is proper

$$-\int_{\hat S}\log|\delta(\beta)|_\omega d\mu(\omega)=\int_{\Omega\setminus\hat S}\log|\delta(\beta)|_\omega d\mu(\omega)\,,\quad(20)$$

The distinction of the two cases is now finished, so by using equation (20) and estimate (19) inside (18) we get

$$\sum_{h=1}^n\mathrm{Ind}_{\alpha_h,d}(\delta)\int_{\hat S_h}\min\left\{d_1\log\frac{1}{|\beta^{(1)}-\alpha_h^{(1)}|_\omega},\dots,d_N\log\frac{1}{|\beta^{(N)}-\alpha_h^{(N)}|_\omega}\right\}d\mu(\omega)\le N\log d+\tau+$$

$$+\int_\Omega h_\omega(\delta)d\mu(\omega)+\sum_{j=1}^N\int_{\Omega\setminus\hat S}(\log^+|\beta^{(j)}|_\omega\deg_{X_j}\delta)d\mu(\omega)+\int_{\hat S}\sum_{j=1}^N(\log^+|4|_\omega+\log^+|\beta_h^{(j)}|_\omega)\deg_{X_j}\delta d\mu(\omega)\quad(21)$$

Since we can always assume that μ is adequately normalised, we have $\int_\Omega \log^+ |4|_\omega d\mu(\omega) = h(4) \leq \log 4$, therefore:

$$\sum_{j=1}^{N} \int_{\Omega \setminus \hat{S}} (\log^+ |\beta^{(j)}|_\omega \deg_{X_j} \delta) d\mu(\omega) + \int_{\hat{S}} \sum_{j=1}^{N} (\log^+ |4|_\omega + \log^+ |\beta_h^{(j)}|_\omega) \deg_{X_j} \delta d\mu(\omega) \leq$$

$$\leq \sum_{j=1}^{N} \left(h(\beta^{(j)}) + \log 4 \right) \deg_{X_j} \delta$$

By plugging everything inside equation (21) we get

$$\sum_{h=1}^{n} \mathrm{Ind}_{\alpha_h, d}(\delta) \int_{\hat{S}_h} \min \left\{ d_1 \log \frac{1}{|\beta^{(1)} - \alpha_h^{(1)}|_\omega}, \dots, d_N \log \frac{1}{|\beta^{(N)} - \alpha_h^{(N)}|_\omega} \right\} d\mu(\omega) \leq N \log d + \tau +$$

$$+ h(\delta) + \sum_{j=1}^{N} \left(h(\beta^{(j)}) + \log 4 \right) \deg_{X_j} \delta \quad (22)$$

By the λ-bounding hypothesis for $T(\beta)$ we can write

$$-\frac{D}{\lambda(\alpha^{(j)}, \beta^{(j)})} \log |\beta^{(j)} - \alpha_h^{(j)}|_\omega > D\theta(\omega) .$$

Then we use Proposition 2.3(3), so we can conclude

$$D \sum_{h=1}^{n} dV(s) \left(t_h - s - \frac{2N^2\eta}{V(s)} \right) \int_{\hat{S}_h} \theta d\mu \leq$$

$$\leq \sum_{h=1}^{n} \mathrm{Ind}_{\alpha_h, d}(\delta) \int_{\hat{S}_h} \min \left\{ d_1 \log \frac{1}{|\beta^{(1)} - \alpha_h^{(1)}|_\omega}, \dots, d_N \log \frac{1}{|\beta^{(N)} - \alpha_h^{(N)}|_\omega} \right\} d\mu(\omega) \quad (23)$$

Now we are going to use again Proposition 2.3 to find upper bounds for the terms on the right hand side of equation (22). By Proposition 2.3(2):

$$\sum_{j=1}^{N} \left(h(\beta^{(j)}) + \log 4 \right) \deg_{X_j} \delta \leq \sum_{j=1}^{N} \left(h(\beta^{(j)}) + \log 4 \right) (dd_j V(s) + O(d)) . \quad (24)$$

Proposition 2.3(4) says that,

$$h(\delta) \leq d \sum_{j=1}^{N} d_j \left(\log 2 + \sum_{h=1}^{N} V(t_h) h(\alpha_h^{(j)}) \right) + O(d \log d) \quad (25)$$

So now (22) can be written in the following way:

$$D \sum_{h=1}^{n} dV(s) \left(t_h - s - \frac{2N^2\eta}{V(s)} \right) \int_{\hat{S}_h} \theta d\mu \leq N \log d + \tau + d \sum_{j=1}^{N} d_j \left(\log 2 + \sum_{h=1}^{N} V(t_h) h(\alpha_h^{(j)}) \right) + O(d \log d) +$$

$$+ \sum_{j=1}^{N} \left(h(\beta^{(j)}) + \log 4 \right) (dd_j V(s) + O(d)) \quad (26)$$

By inequality (9) of Proposition 2.3 we have: $V(s) \log 4 + \log 2 \leq \log 4$. Then the term

$$d \sum_{j=1}^{N} d_j \left(\log 2 + \sum_{h=1}^{N} V(t_h) h(\alpha_h^{(j)}) \right) + O(d \mathrm{log} d) + \sum_{j=1}^{N} \left(h(\beta^{(j)}) + \log 4 \right) (dd_j V(s) + O(d))$$

is bounded by

$$d \sum_{j=1}^{N} d_j \left(V(s) h(\beta^{(j)}) + \log 4 + \sum_{h=1}^{N} V(t_h) h(\alpha_h^{(j)}) \right) + O(d \log d)$$

But by the definition of the function λ it holds that

$$dV(s) \sum_{j=1}^{N} d_j \left(h(\beta^{(j)}) + \frac{\log 4}{V(s)} + \sum_{h=1}^{N} \frac{V(t_h)}{V(s)} h(\alpha_h^{(j)}) \right) = dV(s) N D . \quad (27)$$

Therefore equation (26) becomes

$$dDV(s)\sum_{h=1}^{n}\left(t_h - s - \frac{2N^2\eta}{V(s)}\right)\int_{\hat{S}_h}\theta d\mu \le dV(s)ND + O(d\mathrm{log}d)$$

since the terms τ, $O(d)$, and $N\log d$ can be put together inside $O(d\log d)$. Finally we divide both sides for $dDV(s)$ and we take the limit for $D\to+\infty$ (i.e. $d\to+\infty$) to conclude the proof. □

Theorem 3.10. *Let $(\mathbb{K},\Omega,\phi)$ be a proper adelic curve. Assume that the matrix $T(\beta)\in M(n\times N,\mathbb{K})$ satisfies the h-gap property. Moreover assume $N>\max\left\{\frac{36}{\log 2n}, 9\log 2n\right\}$. If $\theta\colon S\to\mathbb{R}$ is a column-bounding function for $T(\beta)$, then*

$$\int_{\hat{S}}\theta d\mu < 2 + \frac{6\sqrt{\log 2n}}{\sqrt{N}}$$

Proof. The proof is based on the exact calculation of the volume $V(s)$ of [Sch91, Chap. 2, Lemma 5A, 5B]. It shows that for $N\ge 2$, $s,\rho\in\mathbb{R}_{>0}$ with $0<s\le 1$ and $0<\rho<\frac{N}{2}$, then

$$V(s) = \frac{s^N}{N!} \tag{28}$$

So, in particular

$$V\left(\frac{N}{2}-\rho\right) = \frac{(\frac{N}{2}-\rho)^N}{N!} < e^{\frac{-\rho^2}{N}} \tag{29}$$

We take $t_1=t_2=\dots t_n=\frac{N}{2}-\rho$, $\eta=\frac{1}{2N^2N!}$ where ρ is a number such that:

$$V\left(\frac{N}{2}-\rho\right) = \frac{1-\eta N^2}{n}$$

Note that $\frac{1-\eta N^2}{n} > \frac{1}{2n}$. Now from equation (29) we get

$$\frac{-\rho^2}{N} > \log\left(\frac{1-\eta N^2}{n}\right)$$

which implies easily

$$\rho < \sqrt{N\log 2n}\,.$$

Now let's take s as a solution of:

$$(1+\eta)^N - 1 + \eta N^2 < V(s) < 2N\eta + \eta N^2 \tag{30}$$

Here recall that $1+\eta N^2 = nV(t_h) = \sum_{h=1}^{n} V(t_h)$ (compare with the condition (9)). Note that as a consequence we have $0<s<1$. From equation (30) we get the following two conditions

$$N! < \frac{1}{V(s)} < 2N!$$

$$\frac{N!}{2n} < \frac{V(t_h)}{V(s)} < \frac{2N!}{n}$$

By using such conditions it is easy to see that since $T(\beta)$ satisfies h-gap condition, then it satisfies the λ-gap property of width η. Moreover, since θ is a column-bounding function for $T(\beta)$, then it is λ-bounding for $T(\beta)$. It means that we can apply Proposition 3.9 to get:

$$\int_{\hat{S}_h}\theta d\mu < \frac{N}{n\left(\frac{N}{2}-\rho-s-\frac{2\eta N^2}{V(s)}\right)}$$

which means

$$\int_{\hat{S}}\theta d\mu < \frac{N}{\left(\frac{N}{2}-\rho-s-\frac{2\eta N^2}{V(s)}\right)} \tag{31}$$

Since $s<1$, $\rho<\sqrt{N\log 2n}$ and $N^2\eta < V(s)$, we obtain

$$\frac{N}{\frac{N}{2}-\rho-s-\frac{2\eta N^2}{V(s)}} = \frac{2}{1-\frac{2(\rho+s)}{N}-\frac{4\eta N}{V(s)}} = \frac{2}{1-\frac{2\sqrt{N\log 2n}+2+4}{N}}$$

Now since $N>\frac{36}{\log 2n}$, i.e. $\sqrt{N\log 2n}>6$, we can continue with

$$\frac{2}{1-\frac{2\sqrt{N\log 2n}+2+4}{N}} < \frac{2}{1-\frac{3\sqrt{N\log 2n}}{N}} = \frac{2}{1-\frac{3\sqrt{\log 2n}}{\sqrt{N}}} =$$

$$= \frac{2\sqrt{N}}{\sqrt{N}-3\sqrt{\log 2n}} = \frac{2N+6\sqrt{N\log 2n}}{N-9\log 2n} = \frac{2+\frac{6\sqrt{\log 2n}}{\sqrt{N}}}{1-\frac{9\log 2n}{N}}$$

At this point the assumption $N>9\log 2n$ ensures that the last term is bounded by $2+\frac{6\sqrt{\log 2n}}{\sqrt{N}}$ as we wanted. □

4 Proof of Theorem A

Definition 4.1. We say that a proper adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \mu)$ satisfies the *μ-equicontinity condition* if for any measurable set $S \subset \Omega$ of finite measure and any real numbers $\varepsilon, \delta > 0$ there exists a finite measurable cover $C_1, \dots, C_m$ of S satisfying the following conditions. For all $\beta \in \mathbb{K}^\times$ there exists a measurable set $U \subset \Omega$ such that $\mu(U) < \delta$ and

$$\left| - \log^- |\beta|_\omega + \log^- |\beta|_{\omega'} \right| < \varepsilon h(\beta) \,, \quad \forall \omega, \omega' \in C_j \setminus U \,, \; \forall j = 1, \dots m$$

Note that the definition 4.1 resembles a condition of equicontinuity for the family of maps $\omega \mapsto - \log^- |\beta|_\omega$ on $S \setminus U$ with S compact and U arbitrary small. We stress that in [Voj21, Proposition 8.9] it is shown that the adelic curves arising from arithmetic function fields satisfy the μ-equicontinuity condition.

The following lemma will be crucial in the proof of Theorem A. It is a special case of the deep result [Voj21, Lemma 8.10]. This can be seen as a generalisation of the Arzelà-Ascoli theorem for measure spaces, with the advantage that one doesn't need to provide a uniform bound for the involved family of functions.

Lemma 4.2. *Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ a proper adelic curve satisfying the μ-equicontinuity condition. Fix $\alpha \in \mathbb{K}^\times$ and a measurable subset $S \subset \Omega$ of finite measure. Let $\{\beta_k\}$ be a sequence in $\mathbb{K}^\times$ such that $\beta_k \neq \alpha$, and $h(\beta_k) \to +\infty$. Then for any $\varepsilon > 0$ and any $\delta > 0$ there exists a subsequence $\{\beta_{k_j}\}$ together with measurable sets $U_j \subset \Omega$ satisfying $\mu(U_j) < \delta$ and such that for any $j, j' \in \mathbb{N}$ big enough the following holds:*

$$\left| - \frac{\log^- |\alpha - \beta_{k_j}|_\omega}{h(\beta_{k_j})} + \frac{\log^- |\alpha - \beta_{k_{j'}}|_\omega}{h(\beta_{k_{j'}})} \right| < 10\varepsilon \,, \quad \forall \omega \in S \setminus (U_j \cup U_{j'})$$

Proof. See [Voj21, Lemma 8.10] with the following setting: $\Xi = \{\beta_k\}$, $q = 1$, $\lambda_{\beta_k, 1} = -\frac{1}{2} \log^- |\alpha - \beta_k|$, $c_9 = h(\alpha) + \log 2$ and $\epsilon_{10} = \varepsilon$. Notice that hypothesis (i) of [Voj21, Lemma 8.10] in our case holds automatically true thanks to Proposition 1.12(6). Whereas hypothesis (ii) of [Voj21, Lemma 8.10] is ensured by our Definition 4.1. □

Remember that with $\overline{\mathbb{K}}_\omega$ we denote the algebraic closure the completion $\mathbb{K}_\omega$ (with respect to the absolute value $|\cdot|_\omega$). In this case $|\cdot|_\omega$ extends uniquely to an absolute value of $\overline{\mathbb{K}}_\omega$ and by an abuse of notation we will continue to denote this extension with the symbol $|\cdot|_\omega$. Theorem A will be equivalent to the following theorem:

Theorem 4.3. *Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ a proper adelic curve satisfying the μ-equicontinuity condition. Fix some distinct elements $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Let $S = S_1 \sqcup S_2 \sqcup \dots \sqcup S_n$ be a measurable subset of Ω with finite measure. Then for any $\varepsilon > 0$ there exists a real constant $C > 0$ (depending on the fixed data) such that for any $\beta \in \mathbb{K}$ with $h_{\mathbb{X}}(\beta) > C$ it holds that:*

$$\sum_{i=1}^{n} \int_{S_i} \log^- |\beta - \alpha_i|_\omega d\mu(\omega) > -(2 + \varepsilon) h_{\mathbb{X}}(\beta)$$

Before proving this theorem, let's show its equivalence with Theorem A.

Proposition 4.4. *Theorem 4.3 is equivalent to Theorem A.*

Proof. One direction is obvious, so we only have to show that Theorem 4.3 implies Theorem A. First of all we can assume that the set $\{\alpha_1, \dots, \alpha_n\}$ is $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ invariant, since adding all conjugates to the set of $\alpha_i's$ actually provides a stronger Roth inequality. Moreover put $\mathbb{K}' = \mathbb{K}(\alpha_1, \dots \alpha_n)$ and construct the adelic curve $\mathbb{X}' = (\mathbb{K}', \Omega', \mu')$, but note that $\mathbb{K}'$ is now a finite Galois extension of $\mathbb{K}$. For any index i, the embeddings $\iota_{\omega,i}$ can be then glued together to an embedding $\iota_\omega \colon \mathbb{K}' \hookrightarrow \overline{\mathbb{K}}_\omega$ so that:

$$\sum_{i=1}^{n} \log^- |\iota_{\omega,i}(\beta - \alpha_i)|_\omega = \sum_{i=1}^{n} \log^- |\iota_\omega(\beta - \alpha_i)|_\omega$$

Now fix $i_0 \in \{1, \dots, n\}$let $S'_{i_0} := \pi^{-1}_{\mathbb{K}'|\mathbb{K}}(S_{i_0})$ and assume that Theorem 4.3 holds for the adelic curve $\mathbb{X}'$ and the set S', then the following chain of equalities show that Theorem A holds true. Notice that at a certain point we will use equation (7) and the fact that, since $\mathbb{K}'|\mathbb{K}$ is finite and Galois, we have: $N_{\mathbb{K}'|\mathbb{K}}(\beta - \alpha_{i_0}) = \prod_{i=1}^{n} (\beta - \alpha_i)$:

$$\int_{S'_{i_0}} \log^- |\beta - \alpha_{i_0}|_\nu d\mu'(\nu) =^{\text{eq.}(7)} \frac{1}{[\mathbb{K}' : \mathbb{K}]} \int_{S_{i_0}} \log^- |N_{\mathbb{K}'|\mathbb{K}}(\beta - \alpha_{i_0})|_\omega d\mu(\omega) =$$

$$= \frac{1}{[\mathbb{K}':\mathbb{K}]}\int_{S_{i_0}} \log^- \left|\prod_{i=1}^n (\beta-\alpha_i)\right|_\omega d\mu(\omega) = \frac{1}{[\mathbb{K}':\mathbb{K}]}\int_{S_{i_0}} \left(\sum_{i=1}^n \log^- |(\iota_\omega(\beta-\alpha_i)|_\omega\right) d\mu(\omega)\,.$$

□

Proof of theorem 4.3. When $n=1$, then Proposition 1.12(6) says that

$$\int_S \log^- |\beta-\alpha|_\omega d\mu(\omega) \geq -\log 4 - 2h(\beta) - 2h(\alpha)\,,$$

hence any β such that $h(\beta) > \frac{\log 4 + 2h(\alpha)}{\varepsilon}$ satisfies Roth's inequality.

Assume $n \geq 2$. We consider the following matrix of dimension $n \times N$, where $N > \max\left\{\frac{36}{\log 2n}, 9\log 2n\right\}$ will be a big enough fixed integer:

$$T = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \dots & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_n & \alpha_n & \alpha_n & \dots & \alpha_n \end{pmatrix}$$

notice that we have repeated N times the same column vector. Assume by contradiction that the theorem is false. Namely that there exists an $\varepsilon_0 > 0$ such that the inequality

$$\sum_{i=1}^n \int_{S_i} \log^- |\beta_k - \alpha_i|_\omega d\mu(\omega) \leq -(2+\varepsilon_0)h_{\mathbb{K}}(\beta_k) \tag{32}$$

is satisfied by a sequence $\{\beta_k\}_{k\in\mathbb{N}}$ in $\mathbb{K}$ with the properties that $h(\beta_k) \to +\infty$. Pick a constant:

$$L > \log\left(4^{2N!}\prod_{i=1}^n H(\alpha_i)^{\frac{2N!}{n}}\right)$$

with N big enough such that $\varepsilon_0 > \frac{7\sqrt{\log 2n}}{\sqrt{N}}$. Since $h(\beta_k) \to +\infty$, by eventually passing to a subsequence, we can assume that $\{h(\beta_k)\}$ is increasing and bounded from below by a very big value. Therefore we can assume

$$\varepsilon_0 > \frac{2L}{h(\beta_k)} + \frac{L+h(\beta_k)}{h(\beta_k)}\frac{7\sqrt{\log 2n}}{\sqrt{N}}\,, \qquad \forall k \in \mathbb{N} \tag{33}$$

Choose any N elements from the sequence $\{\beta_k\}$, call them $\beta^{(1)}, \dots \beta^{(N)}$ and consider the following matrix $T(\beta)$:

$$T(\beta) = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \dots & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_n & \alpha_n & \alpha_n & \dots & \alpha_n \\ \beta^{(1)} & \beta^{(2)} & \beta^{(3)} & \dots & \beta^{(N)} \end{pmatrix}$$

Let $\hat{S}$ be the subset of S associated to the above matrix $T(\beta)$ as in Definition 3.2.

We are ready to construct some functions $\theta_k \in L^1(S,\mu)$ which will give the desired contradiction. We define them as piecewise functions by putting for any $\omega \in \hat{S}_i$ and any $k \in \mathbb{N}$:

$$\theta_k(\omega) = \frac{-\log^- |\beta_k - \alpha_i|_\omega}{L + h(\beta_k)}$$

We extend θ_k on the whole S by setting $\theta_k(S \setminus \hat{S}) = 0$. Now we can write

$$\int_{\hat{S}} \theta_k d\mu = \int_S \theta_k d\mu = \sum_{i=1}^n \int_{S_i} \theta_k d\mu = \frac{-1}{L+h(\beta_k)}\sum_{i=1}^n \int_{S_i} \log^- |\beta_k - \alpha_i|_\omega d\mu(\omega) \geq^{\text{eq.}(32)} \frac{(2+\varepsilon_0)h(\beta_k)}{L+h(\beta_k)} \tag{34}$$

By plugging inequality (33) inside (34) and simplifying the expressions, we finally get:

$$\int_S \theta_k d\mu = \int_{\hat{S}} \theta_k d\mu > 2 + \frac{7\sqrt{\log 2n}}{\sqrt{N}} \qquad \forall k \in \mathbb{N} \tag{35}$$

Thanks to Proposition 1.12(6):

$$\int_{S_i} \theta_k < \frac{B_i + 2h(\beta_k)}{L + h(\beta_k)}$$

for $B_i \in \mathbb{R}_{>0}$. So, if $B = \max_i\{B_i\}$ we obtain:

$$\int_S \theta_k d\mu \leq \frac{nB + 2nh(\beta_k)}{L + h(\beta_k)} \leq 2n$$

Hence, after possibly passing to a subsequence of $\{\theta_k\}$, we can assume that $\int_S \theta_k d\mu$ admits limit and by inequality (35):

$$\lim_{k\to\infty} \int_{\hat{S}} \theta_k d\mu \geq 2 + \frac{7\sqrt{\log 2n}}{\sqrt{N}} > 2 + \frac{6\sqrt{\log 2n}}{\sqrt{N}} \,. \tag{36}$$

Put by simplicity $\lambda_{i,k}(\omega) := -\log^- |\beta_k - \alpha_i|_\omega$ so that

$$\theta_k(\omega) = \frac{\lambda_{i,k}(\omega)}{L + h(\beta_k)} \,.$$

Then we have:

$$\theta_k(\omega) = \frac{\lambda_{i,k}(\omega)}{h(\beta_k)} A_k$$

where $A_k := 1 - \frac{L}{L+h(\beta_k)} < 1$ and $A_k \to 1$. We want to show that from the sequence $\left\{\frac{1}{A_k}\theta_k\right\}$ we can extract a subsequence uniformly convergent to a function θ on $S \setminus U$ where U is a measurable set of arbitrary small measure. We apply Lemma 4.2 on S_i, for the fixed value α_i, with $0 < \varepsilon < 1$ and $\delta = \delta_1 = \frac{\varepsilon}{2}$. We denote the extracted subsequence by $\mathcal{R}_1 := \left\{\frac{1}{A_j}\theta_j\right\}$. For $\omega \in S_i$, Now pick two natural numbers $j > j' > N_1$, for any $\omega \in S_i \setminus U_1$ where $\mu(U_1) \leq \varepsilon$:

$$\left| \frac{1}{A_j}\theta_j(\omega) - \frac{1}{A_{j'}}\theta_{j'}(\omega) \right| = \left| \frac{\lambda_{i,j}(\omega)}{h(\beta_j)} - \frac{\lambda_{i,j'}(\omega)}{h(\beta_{j'})} \right| < 10\varepsilon \,.$$

Now we use again Lemma 4.2 on the sequence $\mathcal{R}_1$ with $\delta = \delta_2 = \frac{\varepsilon}{4}$, to find a subsequence $\mathcal{R}_2 \subseteq \mathcal{R}_1$ such that again for $j > j' > N_2$ we get

$$\left| \frac{1}{A_j}\theta_j(\omega) - \frac{1}{A_{j'}}\theta_{j'}(\omega) \right| < 10\varepsilon \tag{37}$$

but this time on $S_i \setminus U_2$ with $\mu(U_2) \leq \frac{\varepsilon}{2}$. We can iterate this process of extracting subsequences thanks to Lemma 4.2 and we get a family of subsequences $\mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \ldots \supseteq \mathcal{R}_m \supseteq \ldots$ such that equation (37) holds for elements in $\mathcal{R}_m$ for two indices $j, j' > N_m$ and outside from a set U_m of measure $\mu(U_m) \leq \frac{\varepsilon}{2^{m-1}}$. At this point, by properly selecting one element $\frac{1}{A_m}\theta_m$ for each $\{\mathcal{R}_m\}$, we construct a sequence $\left\{\frac{1}{A_m}\theta_m\right\}$, which is in fact a subsequence of the initial $\left\{\frac{1}{A_k}\theta_k\right\}$, converging uniformly to a function $\theta^{(i)}$ on a measurable set $S_i \setminus U^{(i)}$ where $U^{(i)} = \cup_{m\geq 1} U_m$ and

$$\mu(U^{(i)}) \leq \sum_{m=1}^{\infty} \mu(U_m) \leq 2\varepsilon \,.$$

Since $S_i \setminus U^{(i)}$ has finite measure, by a basic argument of measure theory we know that $\theta^{(i)} \in L^1(S_i \setminus U^{(i)}, \mu)$ and

$$\int_{S_i\setminus U^{(i)}} \theta^{(i)} d\mu = \lim_{m\to\infty} \int_{S_i\setminus U^{(i)}} \frac{1}{A_m}\theta_m d\mu = \lim_{m\to\infty} \int_{S_i\setminus U^{(i)}} \theta_m d\mu \,. \tag{38}$$

Put $U = \cup_{i=1}^n U^{(i)}$ and note that

$$\mu(U) < 2n\varepsilon \,.$$

We finally define $\theta \in L^1(S, \mu)$ in the following way: for any $i = 1, \ldots, n$ and any $\omega \in S_i \setminus U^{(i)}$ we set $\theta(\omega) := \theta^{(i)}(\omega)$ and $\theta(U) := 0$. Therefore we get:

$$\int_{\hat{S}} \theta d\mu = \int_S \theta d\mu = \sum_{i=1}^n \int_{S_i\setminus U^{(i)}} \theta^{(i)} d\mu = \sum_{i=1}^n \lim_{m\to\infty} \int_{S_i\setminus U^{(i)}} \theta_m d\mu =$$

$$= \sum_{i=1}^n \lim_{m\to\infty} \int_{S_i} \theta_m d\mu - \sum_{i=1}^n \lim_{m\to\infty} \int_{U^{(i)}} \theta_m d\mu = \lim_{m\to\infty} \int_S \theta_m d\mu - \lim_{m\to\infty} \int_U \theta_m d\mu$$

where the rightmost term is finte. Now, thanks to equation (36) and the fact that $\mu(U)$ can be taken arbitrarily small by the choice of ε, we conclude that

$$\int_{\hat{S}} \theta d\mu > 2 + \frac{6\sqrt{\log 2n}}{\sqrt{N}}\,. \tag{39}$$

Now we show that since $h(\beta_k) \to +\infty$, the elements $\beta^{(1)}, \ldots, \beta^{(N)}$ can be chosen, along the above subsequence $\{\beta_m\}$, in a way that the following two conditions are satisfied.

(i) $T(\beta)$ satisfies the h-gap condition. This is obvious by choosing the $\beta^{(j)}$ separated by big enough gaps.

(ii) The function θ is column bounding for $T(\beta)$. In fact for $\varepsilon > 0$ small enough and for infinitely many big enough indexes m, thanks to the choice of the constant L, we have the following inequalities for any $i = 1, \ldots, n$ and any $\omega \in \hat{S}_i$:

$$\theta(\omega) < \theta_m(\omega) + \varepsilon < \frac{-\log^- |\alpha_i - \beta_m|_\omega}{h(\beta_m) + \log\left(4^{2N!} \prod_{i=1}^n H(\alpha_i)^{\frac{2N!}{n}}\right)}$$

Therefore we can apply Theorem 3.10 to conclude that:

$$\int_{\hat{S}} \theta d\mu < 2 + \frac{6\sqrt{\log 2n}}{\sqrt{N}}$$

which contradicts inequality (39). □

Example 4.5. We give an example of a proper adelic curve that doesn't satisfy the μ-equicontinuity condition of definition 4.1; for such adelic curve one cannot apply our generalisation of Roth's theorem. Consider $\mathbb{X} = (\mathbb{Q}, \Omega, \mathrm{id})$ naturally obtained from the field $\mathbb{K} = \mathbb{Q}$ as in Example (1.4), we show that the adelic curve $\overline{\mathbb{X}} = (\overline{\mathbb{Q}}, \overline{\Omega}, \mathrm{id})$ doesn't satisfy the μ-equicontinuity condition. The measure on $\overline{\Omega}$ is denoted by χ, whereas the measure on Ω is μ. Consider $S = \overline{\Omega}_\infty$, and since $\mu = (\pi_{\overline{\mathbb{Q}}|\mathbb{Q}})_* \chi$ we have that $\chi(S) = \mu(\Omega_\infty) < +\infty$. Any element $\nu \in S$ is a complex absolute value $|\sigma(\cdot)|$ where $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is a field embedding considered up to complex conjugation. One can be more precise and say that S as measure space (with the restricted structure inherited by $\overline{\Omega}$) can be identified with the following profinite group endowed with a Haar measure:

$$G = \frac{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}{\sim} = \varprojlim_K \frac{\mathrm{Gal}(K/\mathbb{Q})}{\sim}\,,$$

where the limit is taken over all finite Galois extensions $K \supset \mathbb{Q}$, and $\sim$ is the relation of complex conjugation. Take any measurable cover $C_1, \ldots C_m$ of S; by rearranging the terms we can always suppose that the C_i's are pairwise disjoint and that $\mu(C_i) > 0$ for any $i = 1, \ldots, m$. By definition of Haar measure, since $\mu(C_i) > 0$ there exists a normal compact (so closed and of finite index) subgroup $H_i \subseteq G$ such that $\mu(H_i) > 0$ and $g_i H_i \subseteq C_i$ for an element $g_i \in G$. Consider $H = \cap_{i=1}^m H_i$, it has positive measure and $g_i H \subset C_i$ for any $i = 1, \ldots, m$. By the fundamental theorem of Galois theory, there exists a finite Galois extension $K \supset \mathbb{Q}$ such that $H = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$; moreover we can assume that $1 \in C_1$ so that $g_1 = 1$. Fix a prime p such that $\sqrt{p} \notin K$ and let $a, b \in \mathbb{Z}$ such that $a^2 - pb^2 = 1$ and $a - b\sqrt{p} < 1$. At this point inside $H \subseteq C_1$ we consider the place ω' corresponding to the identity automorphism and another place ω corresponding to an automorphism that exchanges $\sqrt{p}$ and $-\sqrt{p}$. Therefore we have:

$$\left|-\log^- |a + b\sqrt{p}|_\omega + \log^- |a + b\sqrt{p}|_{\omega'}\right| = -\log(a - b\sqrt{p})$$

This shows that the μ-equicontinuity condition doesn't hold.

5 Proof of Theorem B

Definition 5.1. A proper adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \mu)$ satisfies the *uniform integrability property* if for any $\beta \in \mathbb{K}^\times$ and any $\varepsilon > 0$ there exist $\delta > 0$ such that if $T \subset \Omega_\infty$ is a measurable subset satisfying $\mu(T) < \delta$, then

$$\int_T -\log^- |\beta| d\mu < \varepsilon h_{\mathbb{X}}(\beta)$$

Vojta showed that arithmetic function fields satisfy the uniform integrability property in [Voj21, Proposition 8.8].

Lemma 5.2. *Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ a proper adelic curve satisfying the μ-equicontinuity condition, and the uniform integrability condition. Assume that Theorem B doesn't hold for certain S, $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and $\varepsilon_0 \in \mathbb{R}_{>0}$. Let $N > 0$ an integer, $\varepsilon \in]0, \varepsilon_0[$, and $r_0, r_1 > 1$ two real numbers. Then there exist $\beta_1, \ldots, \beta_N \in \mathbb{K}$ satisfying the following conditions:*

(i) $h(\beta_1) > r_0$

(ii) $\frac{h(\beta_k)}{h(\beta_{k-1})} > r_1$ *for any* $k = 2, \ldots, N$.

(iii) There is a partition $S = S_1 \sqcup \ldots \sqcup S_n$ *such that:*

$$\sum_{i=1}^{n} \int_{S_i} \min_{1 \le k \le N} \left(\frac{-\log^- |\beta_k - \alpha|_\omega}{h(\beta_k)} \right) d\mu(\omega) > 2 + \varepsilon + \frac{1}{h(\beta_1)}$$

Proof. The reader can follow line by line [Voj21, Proposition 8.12]. The proof uses the previous [Voj21, Propositions 8.10, 8.11]. Note that in the statement of [Voj21, Proposition 8.12] the condition (i) is not explicitly mentioned, but it can be easily deduced from the proof. □

The above lemma will be the main ingredient for the construction of the desired contradiction in the following proof.

Proof of Theorem B. For $n = 1$ the theorem follows trivially from the integral Liouville inequality, i.e. Proposition 1.12(6); so let's assume that $n \ge 2$. Assume by contradiction that the theorem is false for some $\varepsilon_0 > 0$. Hence we apply Lemma 5.2; for any $r_0, r_1 > 1$ and $\varepsilon < \varepsilon_0$ there exist $\beta_1, \ldots \beta_N \in \mathbb{K}$ such that:

(i) $h(\beta_1) > r_0$

(ii) $\frac{h(\beta_k)}{h(\beta_{k-1})} > r_1$ for any $k = 2, \ldots, N$.

(iii) There is a partition $S = S_1 \sqcup \ldots \sqcup S_n$ such that:

$$\sum_{i=1}^{n} \int_{S_i} \min_{1 \le k \le N} \left(\frac{-\log^- |\beta_k - \alpha|_\omega}{h(\beta_k)} \right) d\mu(\omega) > 2 + \varepsilon + \frac{1}{h(\beta_1)} \tag{40}$$

Consider the matrix constructed with the vector $\beta = (\beta_1, \ldots, \beta_N)$:

$$T(\beta) = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \ldots & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \ldots & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_n & \alpha_n & \alpha_n & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_N \end{pmatrix}$$

And let $\hat{S}$ be the subset of S associated to $T(\beta)$ as in definition 3.2. Pick a constant:

$$L > \log \left(4^{2N!} \prod_{i=1}^{n} H(\alpha_i)^{\frac{2N!}{n}} \right)$$

We are ready to construct a function $\theta \in L^1(S, \mu)$ which will give the desired contradiction. We define it as a piecewise function by putting for any $\omega \in S_i$:

$$\theta|_{S_i}(\omega) = \min_k \frac{-\log^- |\beta_k - \alpha_i|_\omega}{L + h(\beta_k)} = \min_k \frac{-\log^- |\beta_k - \alpha_i|_\omega}{h(\beta_k)} A_k$$

where $A_k = 1 - \frac{L}{L + h(\beta_k)}$. We can choose $h(\beta_1) >> L >> N >> 1$ in a way that we can assume:

$$\varepsilon > \frac{2}{A_1} + \frac{6\sqrt{\log 2n}}{A_1 \sqrt{N}} - 2 - \frac{1}{h(\beta_1)} > 0 \tag{41}$$

Clearly $\int_S \theta d\mu = \int_{\hat{S}} \theta d\mu$, so by using equations (40) and (41) we get:

$$\int_{\hat{S}} \theta d\mu = \sum_{i=1}^{n} \int_{S_i} \theta|_{S_i} d\mu > A_1 \sum_{i=1}^{n} \int_{S_i} \min_k \frac{-\log^- |\beta_k - \alpha_i|_\omega}{h(\beta_k)} > A_1 \left(2 + \varepsilon + \frac{1}{h(\beta_1)} \right) > 2 + \frac{6\sqrt{\log 2n}}{\sqrt{N}}. \tag{42}$$

Now notice that

- $T(\beta)$ satisfies the h-gap condition since we can choose the β_k such that the heights $h(\beta_k)$ are separated enough by (ii).
- The function θ is column bounding for $T(\beta)$. Indeed for any $k = 1, \ldots, N$ we have that:

$$\theta|_{S_i}(\omega) < \frac{-\log^-|\beta_k - \alpha_i|_\omega}{L + h(\beta_k)} < \frac{-\log^-|\alpha_i - \beta_k|_\omega}{\log\left(4^{2N!}\prod_{i=1}^n H(\alpha_i)^{\frac{2N!}{n}}\right) + h(\beta_k)}$$

Therefore we can apply theorem 31 to conclude that:

$$\int_{\hat{S}} \theta d\mu < 2 + \frac{6\sqrt{\log 2n}}{\sqrt{N}}$$

which contradicts inequality (42).

□

Note that in the proof of Theorems A and B we didn't assume Northcott property for our adelic curve.

Appendices

A Construction of the interpolating polynomial

In this appendix we will sketch the construction of the interpolating polynomial δ of section 2. For all the deails the reader can check [Cor97].

We employ the same notations of section 2 and for simplicity of notations X denotes the vector of variables $(X_1, \ldots X_N)$. We are going to construct a complicated matrix $A(X)$ depending on the following parameters: $s, t_1, \ldots, t_n \in \mathbb{R}$, with $0 < s < 1$ and $0 < t_h < \frac{N}{2}$ for $h = 1, \ldots, n$. The columns of the matrix are indicized by $\mathbf{a} \in \mathcal{G}_N$; the rows are indicized by $\mathbf{i}_h \in \mathcal{G}_{t_h}$ (for any $h = 1, \ldots, n$) and moreover we put $\mathbf{i}_{n+1} \in \mathcal{G}_s$. The order on multi-indices is the lexicographic one.

$$A(X) := \begin{pmatrix} \binom{\mathbf{a}}{\mathbf{i}_h}\alpha_h^{\mathbf{a}-\mathbf{i}_h} \\ \\ \binom{\mathbf{a}}{\mathbf{i}_{n+1}}X^{\mathbf{a}-\mathbf{i}_{n+1}} \end{pmatrix}_{\mathbf{i}_h, \mathbf{a}} \tag{43}$$

Note that $A(X)$ has $\#(\mathcal{G}_s) + \sum_h \#(\mathcal{G}_{t_h})$ rows and $\#(\mathcal{G}_N) = \prod_{h=1}^n \lfloor d_h + 1 \rfloor$ columns, and moreover the last $\#(\mathcal{G}_s)$ rows are monomials. Clearly we can always choose the paramenters $s, t_1, \ldots, t_n \in \mathbb{R}$ in order to obtain a matrix $A(X)$ with more rows than columns. Let's see an explicit condition that tells us when this can be achieved: the number of rows is greater than the number of columns if

$$\#(\mathcal{G}_s) + \sum_{h=1}^n \#(\mathcal{G}_{t_h}) > \prod_{h=1}^n \lfloor d_h + 1 \rfloor,$$

therefore thanks to Lemma 2.2, for $d_1, \ldots, d_N$ very big, it is enough to have the following conditions on volumes:

$$V(s) + \sum_{h=1}^n V(t_h) > 1. \tag{44}$$

The volumes $V(s), V(t_1), \ldots V(t_n)$ heavily determine the algebraic properties of the matrix $A(X)$, in fact we will now present a stronger condition on the quantity $V(s) + \sum_{h=1}^n V(t_h)$ ensuring that $A(X)$ has maximal rank.

Proposition A.1. *Let $d_1, \ldots, d_N$ be big enough and let $d_1 > d_2 > \ldots > d_N$. Moreover assume that*

$$\alpha_h^{(j)} \neq \alpha_k^{(j)}, \ \forall j = 1, \ldots N, \ \forall h, k = 1, \ldots, n, \ h \neq k.$$

If for $s, t_1, \ldots, t_n \in \mathbb{R}$, with $0 < s < 1$ and $0 < t_h < \frac{N}{2}$ we have that

$$V(s) + \sum_{h=1}^n V(t_h) > \prod_{j=1}^{N-1}\left(1 + (n-1)\sum_{i=j+1}^N \frac{d_j}{d_{j+1}}\right) \tag{45}$$

then, for any $\beta = (\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(N)}) \in \mathbb{K}^N$ such that

$$\beta^{(j)} \neq \alpha_h^{(j)}, \ \forall j = 1, \ldots N, \ \forall h = 1, \ldots, n, ,$$

the rank of $A(\beta)$ is maximal and equal to the number of columns.

Proof. See [Cor97, Proposition 2.1] and notice that it uses a version of Dyson's lemma for polynomials in many variables proved in [EV84]. □

One can always assume that the parameter $s, t_1, \ldots, t_n$ are chosen in a way that we always get

$$(1+\eta)^N < V(s) + \sum_{h=1}^{n} V(t_h) < 1 + 2N\eta \tag{46}$$

It is not difficult to see (check [Cor97, page 159]) that equation (46) implies (45). Therefore thanks to Proposition A.1 we can extract from $A(X)$ a square submatrix $M(X)$ of dimension

$$r := \#(\mathcal{G}_N) = \prod_{h=1}^{n} \lfloor d_h + 1 \rfloor ,$$

which is the number of columns of $A(X)$, such that $M(X)$ has maximal rank for any β componentwise different from any α_h. Moreover one can choose $M(X)$ in a way that it is contained in the last $\#(\mathcal{G}_s)$ rows of $A(X)$, since they are linearly independent for any choice of $\beta \in \mathbb{R}^N$. The polynomial $\delta(X)$ is the determinant of the matrix $M(X)$. At this point all the arguments used in [Cor97] to prove the bounds about δ can be applied verbatim in our setting with the only difference that in order to estimate $h_{\mathbb{X}}(\delta)$ one has to take the integral over Ω instead of the summation over all places.

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