Finite translation orbits on double families of abelian varieties (with an appendix by E. Amerik)

Paolo Dolce Francesco Tropeano

Abstract

We study two families of g-dimensional abelian varieties, induced by distinct rational maps defined on a common variety $\overline{\mathcal{A}}$ and mapping to two bases $\overline{S}_1$ and $\overline{S}_2$. Two non-torsion sections induce birational fiberwise translations on $\overline{\mathcal{A}}$. We consider the action of a specific subset of the group generated by these translations. Under the assumption that $\dim \overline{S}_1 (= \dim \overline{S}_2) \leq g$, we prove that the points with finite orbit are contained in a proper Zariski closed subset. This subset is explicitly described to a certain extent. Our results generalize a theorem of Corvaja, Tsimermann, and Zannier to higher dimensions.

0 Introduction

In the context of algebraic dynamics, it is natural to study the distribution of *special points* under the action of the automorphism group of an algebraic variety. Cantat and Dujardin, in [13, Theorem B], establish that if X is a smooth projective surface defined over a number field and $\Gamma \subset \mathrm{Aut}(X)$ is a subgroup satisfying certain properties, then the points of $X(\mathbb{C})$ with finite Γ-orbit are contained in a proper Zariski-closed subset of X. In [15, Theorem 1.1], Corvaja, Tsimerman, and Zannier improve upon this result in the special case of a projective surface endowed with a double elliptic fibration. They demonstrate that if Γ is the group generated by the two translations induced by the elliptic fibrations, then the points with finite orbit under the action of a specific small subset of Γ lie on the union of finitely many fibers of one of the two fibrations. Their proof employs tools from the theory of unlikely intersections, particularly leveraging the properties of the so-called *Betti map*. In this paper we generalize [15, Theorem 1.1] to the case of projective varieties endowed with a double fibration in g-dimensional abelian varieties over bases of dimension at most g.

General notations. We assume that *all* algebraic varieties and morphisms are defined over $\overline{\mathbb{Q}}$. An algebraic point p of a variety X will be denoted simply as $p \in X$ (or, occasionally, more explicitly as $p \in X(\overline{\mathbb{Q}})$). We do not make use of schematic points in this work. Furthermore, we denote by $X(\mathbb{C})$ the analytification of X, which naturally carries the structure of a complex manifold. The dimension of X as a complex manifold is denoted by $\dim X$.

In several proofs, we work with numerous positive real constants, typically denoted by variables such as $C, c_0, c_1, \ldots$ Our convention is that these variables are *local to the paper*, meaning their values and interpretations are valid only within the specific proof in which they appear, unless explicitly stated otherwise.

This paper employs concepts from transcendental Diophantine problems, including o-minimal structures, definable sets, and definable families. For the foundational definitions and properties, we refer the reader to the seminal works [44] and [43].

Additionally, when we write an inequality using the symbol $\gg$, we mean that the left-hand side (LHS) is greater than or equal to the right-hand side (RHS) multiplied by a constant that is independent of the variables involved in the inequality.

Definition 0.1. Let $\overline{S}$ be a non-singular, irreducible variety. A *family of g-dimensional abelian varieties* is a proper flat morphism of finite type $f : \overline{\mathcal{A}} \to \overline{S}$ with a section, where $\overline{\mathcal{A}}$ is a non-singular irreducible variety and the generic fiber is an abelian variety of dimension g over $\overline{\mathbb{Q}}(\overline{S})$ (with a rational point). After removing the singular fibers and their images we obtain a g-dimensional abelian scheme $f : \mathcal{A} \to S$ (the fiberwise group law extends uniquely to a global map that gives the structure of abelian scheme over S, see [39, Theorem 6.14]).

The set of N-torsion points of a family of g-dimensional abelian varieties $\mathcal{A}$ is denoted by $\mathcal{A}[N]$, and moreover we put $\mathcal{A}_{\mathrm{tor}} = \bigcup_{N \geq 1} \mathcal{A}[N]$. We assume the existence of a non-torsion section $\sigma \colon S \to \mathcal{A}$ of f (i.e. the image of σ is not contained in any $\mathcal{A}[N]$) and that $\mathbb{Z}\sigma$ is Zariski dense in $\mathcal{A}$. We define the following automorphism:

$$\begin{aligned} t_\sigma : \mathcal{A}(\mathbb{C}) &\to \mathcal{A}(\mathbb{C}) \\ p &\mapsto p + \sigma(f(p)). \end{aligned}$$

Let Γ_σ be the group generated by t_σ that acts naturally on $\mathcal{A}(\mathbb{C})$, for any $p \in \mathcal{A}(\mathbb{C})$ we are interested in the orbit

$$\Gamma_\sigma(p) := \{t_\sigma^r(p) \colon r \in \mathbb{N}\}.$$

Clearly each orbit is contained in a single fiber of f, but it is important to study whether the locus $\mathfrak{F}^{(1)}$ of points $p \in \mathcal{A}(\mathbb{C})$ such that $\Gamma_\sigma(p)$ is finite can be confined in a subset lying over a proper closed subset of the base. We recall that a torsion value of σ is an element of $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ and obviously $\Gamma_\sigma(p)$ is finite if and only if $f(p)$ is a torsion value. Therefore, such study of $\mathfrak{F}^{(1)}$ can be reduced to the study of the Zariski density of the torsion values of σ. But the last property depends on the values of $\dim S$ and g in the following way: if $\dim S \geq g$ then $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ is Zariski dense in S if and only if the rank of the Betti map β_σ is $2g$ (see [21, Theorem 1.3]). Note that [8, Proposition 2.1.1] shows that $\mathrm{rank}_{\mathbb{R}}\, \beta_\sigma \geq 2g$ implies that $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ is dense in $S(\mathbb{C})$ with respect to the analytic topology. On the other hand if $\dim S < g$ then $\sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ is not Zariski dense in S. This is a special case of the relative Manin-Mumford conjecture that has been recently proved in [21, Theorem 1.1].

We examine a variation of the aforementioned setting.

Definition 0.2. A *double g-dimensional abelian rational fibration* is the datum of two dominant rational maps $f_1 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_1$ and $f_2 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_2$, such that $\overline{\mathcal{A}}$, $\overline{S}_1$ and $\overline{S}_2$ are non-singular and irreducible varieties, and moreover the induced morphisms on the (maximal) loci where f_1 and f_2 are well defined induce families of g-dimensional abelian varieties. In particular, for each of them the generic fiber is an abelian variety over $k_{\overline{S}_1} := \overline{\mathbb{Q}}(\overline{S}_1)$ and $k_{\overline{S}_2} := \overline{\mathbb{Q}}(\overline{S}_2)$ respectively.

Note that $\dim(\overline{S}_1) = \dim(\overline{S}_2)$. Additionally, we usually require that $\overline{\mathcal{A}}$, $\overline{S}_1$ and $\overline{S}_2$ are *projective* and we denote with $\mathrm{Fund}(f_i)$ the fundamental locus of f_i, i.e. the proper closed subset on which the rational map f_i cannot be extended.

Assumptions. *In addition we impose the following rather standard conditions on these families:*

1) *The two abelian families are "distinct", in the sense that their common fibers (if any) lie over a proper Zariski closed subset E either of $\overline{S}_1$ or of $\overline{S}_2$. Let's assume $E \subseteq \overline{S}_1$.*

2) *We consider $i, j \in \{1, 2\}$ with $i \neq j$. We assume that $\mathrm{Fund}(f_j)$ is not horizontal with respect to f_i*[1]*. Hence, the set $\mathrm{Fund}(f_j) \setminus (\mathrm{Fund}(f_1) \cap \mathrm{Fund}(f_2))$ is contained in a closed subset $f_i^{-1}(W)$ where W is a proper Zariski closed subset of $\overline{S}_i$ defined over $\overline{\mathbb{Q}}$. We fix a W as above and we call it Ind_i. As a consequence, after removing from $\overline{S}_i$ and $\overline{\mathcal{A}}$ some suitable closed subset defined over $\overline{\mathbb{Q}}$, the maps f_i induce two families of abelian varieties over a quasi-projective base (we still have bad reduction). Moreover, after removing the respective singular fibers and discriminant loci we obtain two abelian schemes $f_i : \mathcal{A}_i \to S_i$. We assume the existence of non-torsion sections $\sigma_i : S_i \to \mathcal{A}_i$ of f_i.*

3) *$\mathbb{Z}\sigma_i$ is Zariski dense in the generic fiber of $\mathcal{A}_i$. Equivalently, the image of σ_i is not contained in any subgroup-scheme of $\mathcal{A}_i$.*

The fiber of a point $s \in S_i(\mathbb{C})$ with respect to the morphism f_i will be denoted by $\mathcal{A}_{i,s}$ and the discriminant locus of f_i is $\mathrm{Sing}_i = \overline{S}_i \setminus S_i$. Consider the two birational transformations t_i of $\overline{\mathcal{A}}(\mathbb{C})$ acting by translation along the general fiber of f_i and mapping the zero section to σ_i:

$$\begin{array}{rcl} t_i : \overline{\mathcal{A}}(\mathbb{C}) & \dashrightarrow & \overline{\mathcal{A}}(\mathbb{C}) \\ p & \mapsto & p + \sigma_i(f_i(p)). \end{array}$$

We study the action of the subgroup $\Gamma_{\sigma_1,\sigma_2} := \langle t_1, t_2 \rangle$ generated by t_1 and t_2 in the group of birational transformations $\mathrm{Bir}(\overline{\mathcal{A}}(\mathbb{C}))$; in particular we want to confine the points with finite orbits. First of all, since t_1 and t_2 are not defined everywhere on $\overline{\mathcal{A}}(\mathbb{C})$ we have to be careful with the notion of orbit. For $p \in \overline{\mathcal{A}}(\mathbb{C})$ we put:

$$\Gamma_{\sigma_1,\sigma_2}(p) := \{\gamma(p) \colon \gamma \in \Gamma_{\sigma_1,\sigma_2} \text{ and } \gamma \text{ is well defined at } p\}.$$

In fact, we shall focus on a subset of the orbit showing that already the points with finite orbits under the action of a "small subset" of $\Gamma_{\sigma_1,\sigma_2}$ lie in a proper Zariski closed subset of $\overline{\mathcal{A}}(\mathbb{C})$. This small subset of $\Gamma_{\sigma_1,\sigma_2}$ will be precisely the following:

$$O = O_{\sigma_1,\sigma_2} := \{t_1^{r_1} \circ t_2^{r_2} \colon r_1, r_2 \in \mathbb{N}\}.$$

For any $p \in \overline{\mathcal{A}}(\mathbb{C})$ we clearly have $O(p) \subseteq \Gamma_{\sigma_1,\sigma_2}(p)$ and moreover we define

$$\mathfrak{F} = \mathfrak{F}^{(2)} := \{p \in \overline{\mathcal{A}}(\mathbb{C}) : O(p) \text{ is finite}\}.$$

Remark 0.3. Note that if $p \in \mathfrak{F}$ then both $f_1(p)$ and $f_2(p)$ are torsion values for the relative sections, and in particular $p \in \overline{\mathcal{A}}(\overline{\mathbb{Q}})$. In other words $\mathfrak{F}$ is contained in the intersection between the f_1-fibers and the f_2-fibers of the torsion values.

The case $g = 1$ has been already treated in [15, Theorem 1.1] where it is shown that $\mathfrak{F}$ lies over finitely many fibers of f_2. The following theorem is our main result:

Theorem 0.4. *Let $f_1 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_1$ and $f_2 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_2$ be a double g-dimensional abelian rational fibration with $\overline{\mathcal{A}}$, $\overline{S}_1$ and $\overline{S}_2$ projective varieties. Moreover, assume that f_1 and f_2 satisfy the assumptions $1) - 4)$ above. If $\dim \overline{S}_1 = \dim \overline{S}_2 \leq g$, then there exist two proper Zariski closed subsets $Z_1 \subset \overline{S}_1$ and $Z_2 \subset \overline{S}_2$ such that*

$$\mathfrak{F} \setminus (\mathrm{Fund}(f_1) \cap \mathrm{Fund}(f_2)) \subseteq f_1^{-1}(Z_1) \cup f_2^{-1}(Z_2). \tag{1}$$

Our result can be seen as a generalization of the relative Manin-Mumford conjecture for sections in the following way: in the case of a single family of abelian varieties [21, Theorem 1.1] says that the relative locus $\mathfrak{F}^{(1)}$ is not Zariski dense for $\dim S \leq g - 1$. On the other hand, in the case of two families of abelian varieties with same base S, Theorem 0.4 implies that $\mathfrak{F}^{(2)}$ is not Zariski dense for $\dim S \leq g$.

Remark 0.5. If any of the sets $\sigma_i^{-1}(\mathcal{A}_{i,\mathrm{tor}})$ is not Zariski dense then the theorem is obviously true thanks to Remark 0.3. Therefore if $\dim S_1 = \dim S_2 < g$ then Theorem 0.4 follows directly from [21, Theorem 1.1]. For the same reason, thanks to [21, Theorem 1.3] we can restrict ourselves to prove just the case:

$$2 \dim S_1 = 2 \dim S_2 = 2g = \mathrm{rank}_{\mathbb{R}}\, d\beta_1 = \mathrm{rank}_{\mathbb{R}}\, d\beta_2\,, \tag{2}$$

[1] A subset $W \subset \overline{\mathcal{A}}$ is said *horizontal with respect to f_i* if $f_i(W)$ is Zariski-dense in $\overline{S}_i$ for $i = 1, 2$.

where β_i is the Betti map attached to the section σ_i. Observe that Equation (2) is crucial for the application of the so called "height inequality" of [17, Theorem 1.6] that relates the projective height of the base to the fiberwise Neron-Tate height. In our proof this result appears several times, and on different abelian schemes, to ensure that the height of "most of" the torsion values can be uniformly bounded. On the other hand, it is known that the height inequality fails in general without assumptions on the rank of the Betti map. See also [51, Theorem 5.3.5] for a generalization of height inequality which nevertheless requires the same hypotheses in the case of abelian schemes.

Remark 0.6. At first glance it might seem that in the case $1 = \dim S_1 = \dim S_2 = g$, Theorem 0.4 is slightly weaker than [15, Theorem 1.1] where the claim is just $\mathfrak{F} \setminus \mathrm{Fund}(f_2) \subseteq f_2^{-1}(Z)$ for a proper closed subset Z. However, Proposition 2.8 shows that the two statements are actually equivalent.

Remark 0.7. Let Z be a subset of $\overline{\mathcal{A}}$ which is not horizontal with respect to either f_1 or f_2. If Theorem 0.4 holds replacing $\mathfrak{F}$ by $\mathfrak{F} \cap (\overline{\mathcal{A}} \setminus Z)$, it also holds for $\mathfrak{F}$.

Our proof is inspired by the general strategy employed in the low-dimensional setting of [15], which is a variation of the Pila-Zannier method originally introduced in [45]. After some preliminary considerations, we are ultimately reduced to showing that the points of the form $\sigma_2(b)$ for $b \in f_2(\mathfrak{F})$ have uniformly bounded torsion order. Denoting this order by $m := m(b)$, we use the properties of the Betti map to interpret a collection of conjugates of certain torsion values as rational points within a definable family in $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ in the sense of [44].

By analyzing the relationships between Weil heights, torsion orders, and conjugates of algebraic points, we establish a lower bound on the number of such rational points and an upper bound on their height. Crucially, these bounds depend on m. On the other hand, the result of Habegger and Pila [23, Corollary 7.2] provides an upper bound on the number of rational points of bounded height in the transcendental part of such a definable family. As a consequence of Gao's weak Ax–Schanuel [20, Theorem 3.5], we prove that the definable family has an empty algebraic part. This allows us to compare the aforementioned bounds on the number of rational points and deduce a uniform upper bound for m.

However, our higher-dimensional setting introduces several subtle complications that do not arise in [15]. We summarize below the new technical ingredients developed in this paper:

(a) The height inequality of Dimitrov, Gao, and Habegger [17] provides a uniform height bound only for torsion values contained in a Zariski open dense subset (see Corollary 1.6). When the base is a curve this poses no difficulty, since a uniform bound on a dense open subset is equivalent to a uniform bound for all torsion values. In higher dimension, however, one must carefully track the excluded closed subset at every step of the proof. Moreover, we apply the height inequality to an abelian scheme with an f_2-fiber as its base, so the open dense subset where heights are bounded is not stable under addition with respect to the base.

(b) We require an upper bound on the order of torsion values (or their images) depending only on the heights and degrees of the points. To this end we prove the following:

Proposition (see Proposition 1.9 in the text)**.** *Let $f : \mathcal{A} \to S$ be a g-dimensional abelian scheme (induced by a morphism of varieties) admitting a non-torsion section $\sigma : S \to \mathcal{A}$. Let K be the field of definition of S, let s be a torsion value for σ, and set $d(s) := [K(s) : \mathbb{Q}]$. Let $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a height on the base. Then there exist constants $c = c(g)$, $C = C(g)$ independent of s, and a Zariski open dense subset $U \subseteq S$ such that*

$$\mathrm{ord}(\sigma(s)) \leq \left((14g)^{64g^2} d(s) \max\left(1,\, c \cdot h(s) + C,\, \log d(s)\right)^2\right)^{\frac{35840g^3}{16}} \qquad \forall s \in U(\overline{\mathbb{Q}}).$$

The proof combines Rémond's bound for abelian varieties [47][2] with modular properties of the Faltings height. When applying this result to f_1, we further require compatibility with the height bound for torsion points relative to f_2, which is achieved by suitable choices of heights.

(c) We establish the following proposition, which is crucial in several steps of the proof of Theorem 0.4:

Proposition (see Proposition 1.12 in the text)**.** *Let X be a projective variety, $B \subseteq X$ a closed subvariety, and K a number field containing the fields of definition of X and B. For any $a > 0$, there exists $\delta = \delta(K, a) > 0$ such that for every $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$, at least $\frac{3}{4}[K(\alpha) : K]$ distinct K-embeddings $\tau \colon K(\alpha) \hookrightarrow \mathbb{C}$ satisfy $\alpha^\tau \in C_\delta$.*

[2] Masser and Zannier obtained a related, though weaker, bound in [35].

Informally, this shows that for a fixed constant C and subvariety B, a positive proportion of the Galois conjugates of a point $\alpha \notin B$ with bounded height avoid a neighborhood of B, with the bound depending only on the degree of α. This generalizes [15, Lemma 2.8], which treated the case where B is a finite union of hypersurfaces. It is a useful tool for Zilber–Pink type arguments, as it allows torsion points to be moved into a "safe region" of the variety where uniform arguments apply.

In this paper, we apply the result *uniformly* with respect to $b \in S_2(\mathbb{C})$, on the basis of the auxiliary families of abelian schemes

$$\mathcal{X} := \mathcal{A}_1 \times_{S_1} F_b \longrightarrow F_b, \qquad s_{\mathcal{X}} = \sigma_1 \circ f_1 \, ,$$

where $F_b \subseteq \mathcal{A}_{2,b}$ is a Zariski open subset. So, we must uniformly bound the heights of points $p \in \mathfrak{F}$, with $b = f_1(p)$, inside a sufficiently large Zariski open subset of $\mathcal{A}$ (see Remark 2.2). This is made possible by a careful choice of height functions, as explained in Section 2.1.1.

(d) In the proof of Theorem 0.4, we must remove a Zariski closed subset from each fiber of f_2, but in a way that does not disrupt the argument. In [15] it is shown that for a point $p \in \mathfrak{F}$ with $f_2(p) = b$ and $m = \mathrm{ord}(\sigma_2(b))$, either "many" $k(b)$-conjugates of p lie outside the bad locus of $\mathcal{A}_{2,b}(\mathbb{C})$, or "many" of its translates do, where "many" depends only on m in a uniform way.

In the elliptic case ($g = 1$) the bad locus of each fiber is a finite set of points, which can be enclosed by small Euclidean disks. In higher dimension, however, the bad locus may have positive dimension, and controlling the number of translates in it becomes problematic. We therefore modify the definable-family construction in the Pila–Zannier method: instead of using translates, we rely exclusively on conjugates. Moreover, the argument must be carried out simultaneously on the fibers of all conjugates of b over the field of definition. This step relies crucially on Proposition 1.9.

(e) A key new ingredient is the following result that outside a closed locus of the base, rules out any nontrivial polynomial relation between a fixed nonconstant Betti coordinate and the remaining ones along a nonconstant real-analytic arc. When the section is non-degenerate in the sense of Definition 1.3, such closed locus is a proper subset and is denoted by S_{deg}: it is defined as the projection on S of one of Gao's deceneracy loci for $X = \sigma(S)$.

Theorem (see Theorem 1.4 in the text)**.** *Let $C \subseteq S(\mathbb{C})$ be an irreducible algebraic curve and consider a simply connected open subset $U \subseteq C$ where periods and $\log_\sigma$ are defined. Denote by $\beta_C = (\beta_1, \dots, \beta_{2g}) : U \to \mathbb{R}^{2g}$ the restricted Betti map. If there exist a non-constant Betti coordinate β_i on U and $2g-1$ polynomials $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_{2g} \in \mathbb{R}[X_1, X_2]$ such that*

$$P_j(\beta_i, \beta_j) \equiv 0 \qquad \textit{for any } j \neq i \, , \tag{3}$$

then $C \subseteq S_{\mathrm{deg}}$.

Although this statement is essentially a consequence of Gao's weak mixed Ax–Schanuel theorem [20, Theorem 3.5], to our knowledge it has not been explicitly recorded in the literature in this form. We include it here because it provides a concrete algebraic independence statement for Betti coordinates of sections, which will be crucial for our later arguments. This should be compared with the elliptic strategy of Corvaja–Tsimerman–Zannier [15], which relies on André's independence theorem [7, Theorem 1.3.1] to control algebraic relations between logarithms and periods. In our higher-dimensional setting, Gao's weak Ax–Schanuel theorem yields the needed transcendence input in a different guise.

Consider the auxiliary family $\mathcal{X} := \mathcal{A}_1 \times_{S_1} F_b$ mentioned in (c), with b varying on $S_2(\mathbb{C})$. Inside F_b we construct a definable set Z_b and we must rule out the existence of algebraic arcs in it. Applying the above reasoning to $f_1 \colon \mathcal{A}_1 \to S_1$ provides a uniform excision: $S_{1,\mathrm{deg}}$ disappears for every $b \in S_2(\mathbb{C})$ (see Equation (31)), and Theorem 1.4 can be applied directly on any F_b to exclude pairwise polynomial relations among Betti coordinates along nonconstant algebraic arcs.

(f) In [15], the definable family is constructed from the two sections $s_{\mathcal{X}}$ and $s'_{\mathcal{X}} := \sigma_1 \circ f_1 \circ t_2$, and the Pila–Wilkie theorem is then applied to count rational points. In our case this approach fails: because of the removed loci discussed above, we cannot control the section $s'_{\mathcal{X}}$, which depends on the translation automorphism t_2. Instead, we propose a different construction of the definable family Z, and apply the Habegger–Pila counting theorem [23, Corollary 7.2] in place of Pila–Wilkie.

Remark 0.8. Let us now explain where the assumptions 1)–3) are used in our proof. Assumptions 1) and 2) ensure that the geometric construction is well-defined and meaningful. Assumption 3) is redundant under Equation (2): in this case, the section is non-degenerate in the sense of Definition 1.3, which already implies that $\mathbb{Z}\sigma$ is dense and ensures both the validity of the height inequality and of Theorem 1.4. By contrast, Assumption 3) is only needed to handle the cases not covered by Equation (2).

Finally, we highlight that the present work raises several natural questions. First, it is meaningful to inquire whether our result is *sharp* with respect to the choice of $O \subset \Gamma_{\sigma_1,\sigma_2}$. Specifically:

Question 0.9. Can we find subsets $G \subset O$ that are as small as possible such that the points with finite G-orbits are confined to a proper Zariski-closed subset?

In this direction, Amerik and Cantat in the case of Lagrangian fibrations demonstrate in [1, Section 6.2] that the points with finite G-orbit become Zariski dense when G is sufficiently small. Furthermore, the following problem is also quite natural:

Question 0.10. What is the generalization of Theorem 0.4 in the case of $n > 2$ abelian rational fibrations $f_i : \overline{\mathcal{A}} \dashrightarrow \overline{S}_i$ for $i = 1, \ldots, n$? In particular, what is the optimal relationship between the dimensions of the bases and g in this setting?

The outline of the paper is the following: in Section 1 we collect the preliminary results. The proof of Theorem 0.4 is carried out in Section 2.1 and Section 2.2. Additionally, in Section 2.3, we make some comments on the shape of the Zariski closed subsets Z_1 and Z_2 that confine the fibers containing the points with finite orbit. Finally, Appendix A by E. Amerik provides explicit constructions of double abelian fibrations. It is worth noting that a well-known example of such fibrations is given in [49] for the case $g = 1$. While examples in higher dimensions can be obtained by considering products of distinct elliptic fibrations on a surface, the appendix presents new constructions for $g > 1$ that are not products.

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1 Auxiliary results

In this section we present all the tools needed for the proof of Theorem 0.4. We describe the results in the most general setting.

1.1 Betti map

Let S be a non-singular, irreducible quasi-projective variety and let $f : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$ with "a zero section" σ_0. Moreover we assume that $\sigma : S \to \mathcal{A}$ is a non-torsion section. Each fiber $\mathcal{A}_s(\mathbb{C})$ is analytically isomorphic to a complex torus $\mathbb{C}^g/\Lambda_s$ and for any subset $T \subseteq S(\mathbb{C})$ we denote $\Lambda_T := \bigsqcup_{s \in T} \Lambda_s$. The space $\mathrm{Lie}(\mathcal{A}) := \bigsqcup_{s \in S(\mathbb{C})} \mathrm{Lie}(\mathcal{A}_s(\mathbb{C}))$ has a natural structure of g-dimensional holomorphic vector bundle $\pi \colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ (it is actually a complex Lie algebra bundle). By using the fiberwise exponential maps one can define a global map $\exp\colon \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}(\mathbb{C})$. Let $\Sigma_0 \subset \mathcal{A}(\mathbb{C})$ be the image of the zero section of the abelian scheme, then obviously $\exp^{-1}(\Sigma_0) = \Lambda_{S(\mathbb{C})}$. Clearly $S(\mathbb{C})$ can be covered by finitely many open simply connected subsets where the holomorphic vector bundle $\pi\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ trivializes. Let $U \subseteq S(\mathbb{C})$ be any of such subsets and consider the induced holomorphic map $\pi\colon \Lambda_U \to U$; it is actually a fiber bundle with structure group $\mathrm{GL}(n, \mathbb{Z})$. Since U is simply connected, by [16, Lemma 4.7] we conclude that $\pi\colon \Lambda_U \to U$ is a topologically trivial fiber bundle. Thus we can find $2g$ continuous sections of π:

$$\omega_i : U \to \Lambda_U\,, \quad i = 1, \ldots 2g \tag{4}$$

such that $\{\omega_1(s), \ldots, \omega_{2g}(s)\}$ is a set of periods for Λ_s for any $s \in U$. Since $\Lambda_U \subseteq \mathrm{Lie}(\mathcal{A})_{|U}$, we can put periods into the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Lie}(\mathcal{A})_{|U} \\ & \overset{\omega_i}{\nearrow} & \big\downarrow \exp_{|U} \\ S(\mathbb{C}) \supset U & \xrightarrow{\ \sigma_0|U\ } & \mathcal{A}_{|U}, \end{array}$$

where σ_0 is the zero section. Since σ_0 is holomorphic and exp is a local biholomorphism, then the period functions defined in Equation (4) are holomorphic. The map $\mathcal{P} = (\omega_1, \dots, \omega_{2g})$ is called a *period map*; roughly speaking it selects a $\mathbb{Z}$-basis for Λ_s which varies holomorphically for $s \in U$. For any $s \in U$ we denote by $\Pi_s \in \mathrm{Mat}(\mathbb{C}, g \times 2g)$ the matrix whose columns are the vectors $\omega_i(s)$, this is called the *period matrix*. The set $U \subseteq S(\mathbb{C})$ is simply connected therefore we can choose a holomorphic lifting $\ell_\sigma : U \to \mathrm{Lie}(\mathcal{A})$ of the restriction $\sigma_{|U}$; ℓ_σ is often called an *abelian logarithm*. Thus for any $s \in U$ we can write uniquely

$$\ell_\sigma(s) = \beta_1(s)\omega_1(s) + \dots + \beta_{2g}\omega_{2g}(s) \tag{5}$$

where $\beta_i : U \to \mathbb{R}$ is a real analytic function for $i = 1, \dots, 2g$. The map $\beta_\sigma : U \to \mathbb{R}^{2g}$ defined as $\beta_\sigma = (\beta_1, \dots, \beta_{2g})$ is called the *Betti map associated to the section* σ, whereas the β_i's are the *Betti coordinates*. The relation between the the logarithm, the periods and the Betti coordinates can be obviously expressed in the following compact and useful way:

$$\ell_\sigma(s) = \Pi_s \beta_\sigma(s)\,, \quad \forall s \in U \tag{6}$$

Observe that the Betti map depends both on the choice of period map $\mathcal{P}$ and on the abelian logarithm ℓ_σ, but this is irrelevant for our applications. The main feature of the Betti map is that $\beta_\sigma(s) \in \mathbb{Q}^{2g}$ if and only if s is a torsion value of σ, so it allows us to treat the study of the torsion values of an abelian scheme as a transcendental Diophantine problem. Note that we need a non-torsion section σ otherwise β_σ would be obviously constant and equal to a rational point. Viceversa, we recall that as a consequence of Manin's "theorem of the kernel" (see [31] or [11]) if β_σ is locally constant then σ is torsion. Moreover, the fibers of β_σ are complex submanifolds of $S(\mathbb{C})$ (see [14, Proposition 2.1] or [8, Section 4.2]).

Remark 1.1. There exists a compact subset $D \subseteq U$ such that the Betti map β_σ restricted to D is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ (using the real charts). This follows for instance by using [41, Fact 4.3] and the fact that for $i = 1, \dots, 2g$ we have $\beta_i = \pi_i \circ \ell_\sigma$, where π_i is the projection on the i-th coordinate with respect to the period map.

The rank, in the sense of real differential geometry, of the Betti map at a point s is denoted by $\mathrm{rank}_\mathbb{R}\, d\beta_\sigma(s)$. It can be shown that it depends only on the point s (see for instance [8, Section 4.2.1] or [19, Section 4]). Moreover we define the *generic rank of the Betti map* by

$$\mathrm{rank}_\mathbb{R}\, d\beta_\sigma = \max_{s \in S(\mathbb{C})} \mathrm{rank}_\mathbb{R}\, d\beta_\sigma(s) \tag{7}$$

and note that it obviously holds that $\mathrm{rank}_\mathbb{R}\, d\beta_\sigma \leq 2\min(g, \dim S)$. When $g = \dim S$ and the generic rank is maximal, the following proposition allows us to have a uniform control on the fibers of the Betti map.

Proposition 1.2. *Let $2\dim S = 2g = \mathrm{rank}_\mathbb{R}\, d\beta_\sigma$. There exist a non-empty Zariski open set U of $S(\mathbb{C})$ such that: for any $x \in U$ there is a compact subanalytic set $D \subseteq S(\mathbb{C})$ containing x and a constant $c = c(D)$ such that the Betti map $\beta_\sigma \colon D \to \mathbb{R}^{2g}$ has finite fibers of cardinality at most c.*

Proof. From the condition on the rank of the Betti map it follows immediately that there exists a non-empty Zariski open set $U \subseteq S(\mathbb{C})$ on which β_σ is a submersion. Pick any compact subanalytic D inside U and contained in a chart. Restrict the Betti map on D and identify the latter with an euclidean compact in $\mathbb{R}^{2g}$. Since β_σ is now a submersion, the fibers must have real codimension equal to $2g$ (see for instance [29, Corollary 5.13]), which means that the fibers are discrete, and hence finite (D is compact). It remains to prove the uniform bound on the cardinality. So consider the subanalytic set

$$Z := \{(z, \beta_\sigma(z)) \colon z \in D\} \subset \mathbb{R}^{2g} \times \mathbb{R}^{2g}\,.$$

Let $\pi_2 : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}^{2g}$ the projection on the second factor, then for any $p \in \mathbb{R}^{2g}$ we obviously have

$$Z \cap \pi_2^{-1}(p) = \beta_\sigma^{-1}(p)\,.$$

By Gabrielov's theorem (see [52, Theorem A.4] or [12, Theorem 3.14]) $Z \cap \pi_2^{-1}(p)$ has at most c connected components, hence $\beta_\sigma^{-1}(p)$ has cardinality at most c. □

In the general case, defining $X := \sigma(S)$, the behavior of the generic rank of the Betti map is controlled via the degeneracy loci $\dots X^{\deg}(-1) \subseteq X^{\deg}(0) \subseteq X^{\deg}(1) \subseteq \dots \subseteq X$ introduced in Gao's work. We refer the reader directly to [19, Definition 1.6]. Notice that, by [19, Theorem 1.8], the locus $X^{\deg}(t)$ is Zariski closed in X for every $t \in \mathbb{Z}$.

Definition 1.3. A section $\sigma\colon S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ is *degenerate* if $X^{\deg}(0) = X$.

Note that any section is trivially degenerate if $g < \dim S$. In the case $g = \dim S$, [19, Theorem 1.7] shows that a section $\sigma\colon S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ is degenerate if and only if

$$\operatorname{rank}_{\mathbb{R}}(d\beta_\sigma) < 2\dim S.$$

Furthermore, under the assumption $g = \dim S$, a non-degenerate section σ has the property that $\mathbb{Z}\sigma$ is Zariski dense. Finally, since the structure map $f : \mathcal{A} \to S$ is an isomorphism when restricted to X, we can define the closed locus

$$S_{\deg} := f(X^{\deg}(0)) \subseteq S(\mathbb{C}). \tag{8}$$

Notice that $S_{\deg}$ contains the locus $\{s \in S(\mathbb{C}) : \operatorname{rank}_{\mathbb{R}}(d\beta_\sigma)(s) < 2g\}$ where the rank of the Betti map drops. If σ is non-degenerate, $S_{\deg}$ is a proper closed subset of S.

It is worth noting that Gao's weak Ax–Schanuel theorem (see [20, Theorem 3.5]) plays a crucial role in the proofs of [19, Theorems 1.7 and 1.8].

1.2 Transcendence of Betti coordinates

Let $f : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$ with a section $\sigma : S \to \mathcal{A}$, both defined over a number field K. The following result does not require any additional hypothesis on the abelian scheme nor on the section, altought in our paper we will apply it restricting to the case $2\dim S = 2g = \operatorname{rank}_{\mathbb{R}} d\beta$ where $\beta = (\beta_1, \ldots, \beta_{2g})$ is the Betti map of σ, which ensures that the section σ is non-degenerate.

Theorem 1.4. *Let $C \subseteq S(\mathbb{C})$ be an irreducible algebraic curve and consider a simply connected open subset $U \subseteq C$ where periods and $\log_\sigma$ are defined. Denote by $\beta_C = (\beta_1, \ldots, \beta_{2g}) : U \to \mathbb{R}^{2g}$ the restricted Betti map. If there exist a non-constant Betti coordinate β_i on U and $2g-1$ polynomials $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{2g} \in \mathbb{R}[X_1, X_2]$ such that*

$$P_j(\beta_i, \beta_j) \equiv 0 \qquad \text{for any } j \neq i\,, \tag{9}$$

then $C \subseteq S_{\deg}$.

Proof. Replacing P_j by an irreducible factor if necessary, we may assume that each P_j is irreducible. Define the real algebraic curve

$$\Gamma_j := \{(x,y) \in \mathbb{R}^2 : P_j(x,y) = 0\}$$

and its smooth and singular loci

$$\Gamma_{j,\mathrm{sm}} = \{(x,y) \in \Gamma_j : \nabla P_j(x,y) \neq (0,0)\}\,, \qquad \Gamma_{j,\mathrm{sing}} := \Gamma_j \setminus \Gamma_{j,\mathrm{sm}}\,.$$

Note that $\Gamma_{j,\mathrm{sm}}$ is Zariski open and dense in Γ_j and $\Gamma_{j,\mathrm{sing}}$ is finite. Since β_i is non-constant and β_i, β_j are continuous, the set

$$V := \{s \in U : (\beta_i(s), \beta_j(s)) \in \Gamma_{j,\mathrm{sm}}\,\}$$

is an euclidean open non-empty subset of C. For every $s \in V$ and every $j \neq i$, after differentiating Equation (9) we have a nontrivial linear relation

$$(\partial_1 P_j)\big(\beta_i(s), \beta_j(s)\big)\, d\beta_i(s) \;+\; (\partial_2 P_j)\big(\beta_i(s), \beta_j(s)\big)\, d\beta_j(s) \;=\; 0.$$

In particular, since i is fixed and j varies, this implies $\operatorname{rank}_{\mathbb{R}} d\beta(s) \leq 1$ for any $s \in V$. Note that $\operatorname{rank}_{\mathbb{R}} d\beta(s)$ must be an even number for each s, as each $\omega_i(s)$ is holomorphic. Therefore, since the rank drop locus of the Betti map on C is a Zariski closed subset (see again [19, Theorem 1.8]), we conclude that the generic rank of the Betti map over C is

$$\operatorname{rank}_{\mathbb{R}} d\beta_C = 0.$$

This implies that the map β_C is locally constant. Hence, by Manin's kernel theorem, a multiple of σ_C is contained in the fixed part of the restricted abelian scheme $\mathcal{A}|_C/C$, i.e. the subvariety $\sigma(C)$ is weakly special (we refer the reader to [19, Definition 1.5]). This implies $\sigma(C) \subset X^{\deg}(0)$ where $X := \sigma(S)$, or equivalently $C \subseteq S_{\deg}$. ∎

1.3 Height bounds

In this short subsection we use the same notation of Section 1.1. Let $\mathcal{M}$ be a relative f-ample and symmetric line bundle on $\mathcal{A}$, then we define $\hat{h} : \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ to be the fiberwise Néron-Tate height i.e.

$$\hat{h}(p) = \hat{h}_{\mathcal{M}}(p) := \lim_{n\to\infty} \frac{1}{4^n} h_{\mathcal{M}}(2^n p) .$$

Note that $\hat{h}(p) = \hat{h}_{\mathcal{M}_s}(p)$ with $s = f(p)$. Moreover we consider a height function $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ on the base. The following height inequality proved in [17, Theorem B.1] (see also [51, Theorem 5.3.5] for a more general approach) is a crucial result that relates the values of $\hat{h}$ and h:

Theorem 1.5 (Height inequality for abelian schemes)**.** *Let X be an irreducible and non-degenerate*[3] *subvariety of $\mathcal{A}$ that dominates S. Then there exist two constants $c_1 > 0$ and $c_2 \geq 0$ and a Zariski non-empty open subset $V \subseteq X$ with*

$$\hat{h}(p) \geq c_1 h(f(p)) - c_2 \quad \textit{for all } p \in V\left(\overline{\mathbb{Q}}\right).$$

Proof. See [17, Theorem B.1]. □

Corollary 1.6. *Assume that $f\colon \mathcal{A} \to S$ is endowed with a non-degenerate section $\sigma : S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$. Then there exists a constant $C \geq 0$ and a non-empty Zariski open subset $V \subseteq S$ such that*

$$h(s) \leq C \quad \textit{for all } s \in V(\overline{\mathbb{Q}}) \cap \sigma^{-1}(\mathcal{A}_{\mathrm{tor}}). \tag{10}$$

Remark 1.7. When $\dim(S) = g$ the open set V can be chosen as the complement of the locus defined in Equation (8), i.e. $S \setminus V = S_{\deg}$ (see for instance [22, explanation after Thm. C]). Note that if the abelian scheme $\mathcal{A} \to S$ and the section σ are defined over $\overline{\mathbb{Q}}$, then $S \setminus V$ is a Zariski closed subset defined over $\overline{\mathbb{Q}}$.

1.4 Torsion bounds

Let's quickly recall the definition of the stable Faltings height. Let A be a g-dimensional abelian variety over a number field K. Choose a finite extension $L \supseteq K$ such that $A \otimes L$ has semistable reduction, and let $\mathcal{A} \to S := \operatorname{Spec} O_L$ be the (connected) Néron model of $A \otimes L$ with zero section $\epsilon\colon S \to \mathcal{A}$. Put

$$\omega_{\mathcal{A}/S} := \epsilon^* \Omega^g_{\mathcal{A}/S}.$$

Equip $\omega_{\mathcal{A}/S}$ with the standard *Faltings/Petersson* hermitian metric: at archimedean places, the L^2 metric by integrating translation-invariant differentials on the complex fibres; at finite places, the model metrics coming from $\mathcal{A}/S$. Write $\overline{\omega}_{\mathcal{A}/S}$ for this metrized line bundle. The *stable Faltings height of A* is

$$h_F(A) := \frac{1}{[L : \mathbb{Q}]} \widehat{\deg}(\overline{\omega}_{\mathcal{A}/S}),$$

and it is independent of the choice of L (see [18, §3]). The stable Faltings height controls torsion:

Proposition 1.8. *Let A be an abelian variety of dimension g defined over a number field K. The finite group $A(K)_{\mathrm{tor}}$ has exponent at most $\kappa(A)^{\frac{35}{16}}$ and cardinality at most $\kappa(A)^{4g+1}$, where $d = [K\colon \mathbb{Q}]$ and*

$$\kappa(A) = \left((14g)^{64g^2} d \max(1, h_F(A), \log d)^2\right)^{1024g^3}.$$

Proof. See [47, Proposition 2.9]. □

Now let $f : \mathcal{A} \to S$ be a g-dimensional abelian scheme over a quasi-projective variety S defined over a number field K. Set

$$\lambda_{\mathcal{A}/S} := \det f_*(\Omega^1_{\mathcal{A}/S}).$$

For abelian schemes there is a canonical identification

$$\lambda_{\mathcal{A}/S} \cong \omega_{\mathcal{A}/S}, \tag{11}$$

Equip $\lambda_{\mathcal{A}/S}$ (equivalently $\omega_{\mathcal{A}/S}$) with a semipositive *adelic* metric as follows:

[3]The references [17] and [21] use a slightly different (but equivalent) definition of Betti map and they have a notion of non-degenerate subvariety. A section σ is non-degenerate in our sense if and only if the subvariety $\sigma(S(\mathbb{C}))$ of $\mathcal{A}$ is non-degenerate in the sense of Dimitrov, Gao, Habbegger.

- at archimedean places, the L^2 (Petersson/Faltings) metric;
- at finite places, the model metrics characterized by the property that for any $s \in S(\overline{\mathbb{Q}})$ with field $K(s)$ and any finite extension $L/K(s)$ over which the fibre $\mathcal{A}_s$ is semistable with (connected) Néron model $\mathcal{N} \to \operatorname{Spec} O_L$, the pullback along an integral lift

$$\widetilde{s}: \ \operatorname{Spec} O_L \longrightarrow S$$

(of the L-point $s : \operatorname{Spec} L \to S_L := S \times_{\operatorname{Spec} K} \operatorname{Spec} L$, obtained by the valuative criterion after enlarging L if needed) satisfies the canonical isometry

$$\widetilde{s}^{\,*}\,\overline{\lambda}_{\mathcal{A}/S} \;\cong\; \overline{\epsilon^*\Omega^g_{\mathcal{N}/O_L}}\,. \tag{12}$$

Let $\underline{h}\colon S(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the height attached to this adelically metrized line bundle $\overline{\lambda}_{\mathcal{A}/S}$. Then for every $s \in S(\overline{\mathbb{Q}})$ and any such semistable $L/K(s)$,

$$\underline{h}(s) \;=\; \frac{1}{[L:\mathbb{Q}]}\,\widehat{\deg}\big(\widetilde{s}^{\,*}\,\overline{\lambda}_{\mathcal{A}/S}\big) \;=\; \frac{1}{[L:\mathbb{Q}]}\,\widehat{\deg}\Big(\overline{\epsilon^*\Omega^g_{\mathcal{N}/O_L}}\Big) \;=\; h_F(\mathcal{A}_s), \tag{13}$$

the last equality being exactly the definition of the stable Faltings height. Moreover, by the the discussion at the bottom of [18, p. 39]) and the standard height machine, there exist positive constants C_1, C_2, C_3 and a proper Zariski closed subset $Z \subset S$ such that for all $s \in S(\overline{\mathbb{Q}}) \setminus Z$,

$$h_F(\mathcal{A}_s) = \underline{h}(s) \;\le\; C_1\,h(s) \;+\; C_2\,\log\!\big(1+h(s)\big) \;+\; C_3, \tag{14}$$

where h is any fixed ample height on S. Finally, using $\log(1+t) \le t$ for $t \ge 0$, we absorb the logarithmic term to deduce the linear bound

$$h_F(\mathcal{A}_s) \;\le\; C_3 \;+\; (C_1+C_2)\,h(s) \qquad (s \in S(\overline{\mathbb{Q}}) \setminus Z). \tag{15}$$

Proposition 1.9. *Let $f : \mathcal{A} \to S$ be a g-dimensional abelian scheme (induced by a morphism of varieties) admitting a non-torsion section $\sigma : S \to \mathcal{A}$. Let K be the field of definition of S, let s be a torsion value for σ and put $d(s) := [K(s) : \mathbb{Q}]$. Let $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a height on the base corresponding to an ample line bundle. Then there exist real constants $c = c(g)$, $C = C(g)$ (independent of s) and a Zariski open dense subset $U \subseteq S$ such that*

$$\operatorname{ord}(\sigma(s)) \le \Big((14g)^{64g^2} d(s) \max\left(1,\, c\cdot h(s) + C,\, \log d(s)\right)^2\Big)^{\frac{35840g^3}{16}} \qquad \forall s \in U(\overline{\mathbb{Q}})\,.$$

Proof. Possibly shrinking S we may assume $U := S \setminus Z$ is nonempty. For $s \in U(\overline{\mathbb{Q}})$ with $\sigma(s)$ torsion, the fibre $\mathcal{A}_s$ is an abelian variety over the number field $K(s)$. By Equation (15) we have

$$h_F(\mathcal{A}_s) \;\le\; C_3 + (C_1+C_2)\,h(s).$$

Applying Proposition 1.8 over $K(s)$ (so $d = d(s)$) and inserting this bound for $h_F(\mathcal{A}_s)$ inside $\kappa(\mathcal{A}_s)$ yields the claimed inequality after adjusting absolute constants (depending only on g). This proves the statement for all $s \in U(\overline{\mathbb{Q}})$. □

1.5 Control on conjugate points

Let's fix an affine variety $Y(\mathbb{C}) \subseteq \mathbb{A}^N(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C})$ defined over a number field K. For any point $p \in Y(\mathbb{C})$ we denote by $K(p)$ the field generated by the coordinates of p; this is the same as the residue field of p when the latter is seen as an abstract point of Y. With the letter h we denote both the absolute height on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and $\mathbb{A}^1(\overline{\mathbb{Q}})$, since the formal meaning is clear from the argument of h. Further, we denote by $\|\cdot\|$ the euclidean norm in $\mathbb{A}^N(\mathbb{C})$. We fix a closed subvariety B' of Y and we define

$$W'_\delta := \{x \in Y(\mathbb{C})\colon d(x, B'(\mathbb{C})) < \delta\}, \quad \text{for } \delta \in \mathbb{R}_{>0}$$

where

$$d(x, B'(\mathbb{C})) := \inf_{b \in B'(\mathbb{C})} \|x - b\|\,.$$

Moreover let's consider the set $C'_\delta := Y(\mathbb{C}) \setminus W'_\delta$.

Lemma 1.10. *Let H be a subset of $Y(\mathbb{C})$ and let C be a compact subset of H. Fixed $p \in Y(\mathbb{C}) \setminus H$, there exists a constant c (uniform with respect to $b \in C$) such that*

$$d(p, H) \geq c \cdot \|p - b\| \quad \textit{for each } b \in C.$$

Proof. For each $b \in C$, let us consider a constant a_b which satisfies $0 < a_b < \frac{d(p,H)}{\|p-b\|}$ (note that it exists since $p \notin H$). Observe that a_b is a constant which depends on b and such that

$$d(p, H) - a_b \cdot \|p - b\| > 0.$$

Then there exists an open (analytic) neighbourhood N_b of b such that

$$d(p, H) - a_b \cdot \|p - b'\| > 0 \qquad \text{for each } b' \in N_b.$$

The family $\{N_b : b \in H\}$ is an open covering of the compact set C. Thus there exists a finite subcovering $\{N_{b_i} : i = 1, \ldots, n\}$. The constant $c := \min_{1 \leq i \leq n}(a_{b_i})$ works uniformly on C. In fact for each $b \in C$ we have

$$c \cdot \|p - b\| \leq a_b \cdot \|p - b\| < d(p, H).$$

□

Proposition 1.11. *Let K be a number field which contains the field of definition of the subvariety B'. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in Y(\overline{\mathbb{Q}}) \setminus B'(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{3}{4}[K(\alpha) : K]$ different K-embeddings $\tau \colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C'_δ.*

Proof. Fix $\beta = (\beta_1, \ldots, \beta_N) \in B'(\overline{\mathbb{Q}})$ such that there exists an index i with $\beta_i \in K(\alpha)$ (observe that such a β always exists); and write $\alpha := (\alpha_1, \ldots, \alpha_N)$. Clearly $h(\alpha) \geq h(\alpha_i)$ and $h(\beta) \geq h(\beta_i)$. This implies

$$h(\alpha_i - \beta_i) \leq h(\alpha_i) + h(\beta_i) + \log(2) \leq h(\alpha) + h(\beta) + \log(2). \tag{16}$$

Fix $\delta > 0$. We define

$$\Sigma := \{\tau : K(\alpha) \hookrightarrow \mathbb{C} \colon \text{ id} = \tau_{|K} \text{ and } \alpha^\tau \notin C'_\delta\}$$

and denote by k the cardinality of Σ. Since τ is a K-embedding we have $\beta^\tau \in B'(\overline{\mathbb{Q}})$. Moreover observe that, given $\tau \in \Sigma$, we have $\alpha^\tau \notin B'(\mathbb{C})$. Thus, by Lemma 1.10 for $p = \alpha^\tau, H = B'(\mathbb{C})$ and $C = \{\beta^\tau \colon \tau \in \Sigma\}$, and since $\alpha^\tau \notin C'_\delta$ (by definition of Σ) there exists a constant c_τ such that

$$\frac{1}{|\alpha_i^\tau - \beta_i^\tau|} \geq \frac{1}{\|\alpha^\tau - \beta^\tau\|} \geq \frac{c_\tau}{d(\alpha^\tau, B(\mathbb{C}))} > \frac{c_\tau}{\delta}.$$

Considering $c := \min_{\tau \in \Sigma}(c_\tau)$ we obtain a constant c such that:

$$\frac{1}{|\alpha_i^\tau - \beta_i^\tau|} \geq \frac{c}{\delta} \quad \text{for fixed } i \text{ and for all } \tau \in \Sigma.$$

Then for δ small enough we obtain

$$\begin{aligned} h(\alpha_i - \beta_i) &\geq \frac{1}{[K(\alpha) : \mathbb{Q}]} \sum_\nu \log \max \left(1, \left|\frac{1}{\alpha_i - \beta_i}\right|_\nu \right) \geq \\ &\geq \frac{1}{[K(\alpha) : \mathbb{Q}]} \sum_{\tau \in \Sigma} \log \max \left(1, \left|\frac{1}{\alpha_i^\tau - \beta_i^\tau}\right| \right) \geq \frac{k}{[K(\alpha) : \mathbb{Q}]} \log \left(\frac{c}{\delta}\right). \end{aligned} \tag{17}$$

By (16), (17) and the fact that α has bounded height we obtain

$$k \leq \frac{(a + h(\beta) + \log(2)) \cdot [K(\alpha) : \mathbb{Q}]}{\log(c/\delta)}.$$

For δ small enough we have

$$\frac{a + h(\beta) + \log(2)}{\log(c/\delta)} \leq \frac{1}{4[K : \mathbb{Q}]}.$$

Therefore

$$k \leq \frac{1}{4}[K(\alpha) : K].$$

□

Now let's fix a projective variety X defined over K and a closed subvariety B of X. For any point $p = (x_0 : \ldots : x_N) \in X(\mathbb{C})$ pick any $x_i \neq 0$ and then put $K(p) := K\left(\frac{x_j}{x_i} : j = 0, \ldots, N\right)$. Note that $K(p)$ doesn't depend on the choice of x_i (i.e. the standard affine chart) and moreover $K(p)$ is the residue field of p when the latter is seen as an abstract point of X. We prove a higher dimensional generalization of a quite useful result already appeared for the projective line in [33, 34, 35, Lemma 8.2] and for hypersurfaces in [15, Lemma 2.8]. Roughly speaking the result claims the following: K is the field of definition of B, $a \in \mathbb{R}$ and $\alpha \in X(\overline{\mathbb{Q}})$ is any point not contained in $B(\mathbb{C})$ with height at most a; then we can give an explicit lower bound, depending only on $[K(\alpha) : K]$, on the number of $K(\alpha)$ conjugates of α that lie in a "big enough" compact not intersecting $B(\mathbb{C})$.

We first construct the compact subset. Denote by $U_0, \ldots, U_N$ the standard affine charts of the projective space. Let's define

$$W_{i,\delta} := \{x \in X(\mathbb{C}) \cap U_i : d(x, B(\mathbb{C}) \cap U_i) < \delta\} \qquad \text{for fixed } \delta \in \mathbb{R}_{>0} \text{ and } i = 1, \ldots, N. \tag{18}$$

Then we put $W_\delta := \bigcup_{i=0}^{N} W_{i,\delta}$ and note that it is an open subset of $X(\mathbb{C})$ containing $B(\mathbb{C})$. Therefore $C_\delta := X(\mathbb{C}) \setminus W_\delta$ is a compact set not intersecting $B(\mathbb{C})$.

Proposition 1.12. *Let K be a number field which contains the field of definition of the subvariety B. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{3}{4}[K(\alpha) : K]$ different K-embeddings $\tau \colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C_δ.*

Proof. Fix $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$ and fix a chart U_i such that $\alpha \in U_i$. Since the chart is invariant under the action of each τ, we can apply Proposition 1.11 for $Y(\mathbb{C}) = X(\mathbb{C}) \cap U_i$, $B'(\mathbb{C}) = Y(\mathbb{C}) \cap B(\mathbb{C})$ and $C'_\delta = C_\delta \cap U_i$. Therefore, we obtain a real number δ_i which only depends on K, a and U_i and which satisfies the statement for $\alpha \in U_i$. We can repeat the argument for any standard chart and after defining $\delta := \min_{0 \leq i \leq N}(\delta_i)$, we can conclude. □

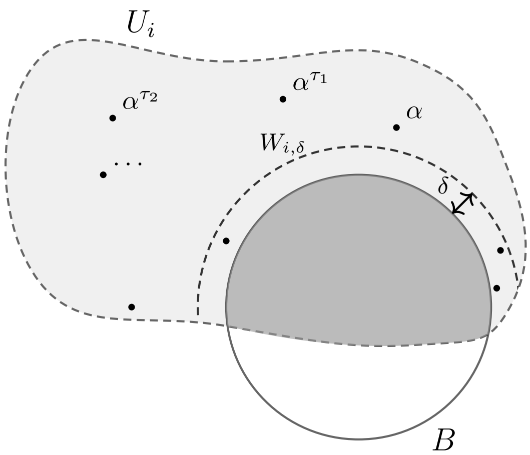

Figure 1: A representation of the portion of conjugates of α that stay away from a euclidean open set $W_{i,\delta}$ that tightly encircles a Zariski closed set B. The set U_i is a selected affine chart.

Remark 1.13. Observe that the the intersection of C_δ with each standard chart U_i is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. In fact, first of all let's identify $U_i \cap X(\mathbb{C})$ with $\mathbb{R}^{2N}$, then the map $\mathbb{R}^{2N} \ni p \mapsto d(p, B(\mathbb{C}) \cap U_i)$ is a globally subanalytic function (see for instance [10, Example 2.10]). At this point we apply [50, §1 Lemma 2.3] to conclude that the set $W_{i,\delta} = U_i \cap W_\delta$ is globally subanalytic for any $\delta > 0$. Finally, note that the intersection $C_\delta \cap U_i$ is the complement set $(U_i \cap X(\mathbb{C})) \setminus (U_i \cap W_{i,\delta})$, so it is also globally subanalytic.

2 The main theorem

In this section we prove Theorem 0.4. The proof is rather long and technical; it will be eventually split in two cases after a common setup. We use the same notations fixed in the introduction and we work under the assumption Equation (2).

2.1 Setup of the proof

Our proof necessitates a considerably intricate preparation, which we delineate as follows.

2.1.1 Construction of the height

To establish an arithmetic relationship between the two fibrations f_1 and f_2, we must construct a specific height on $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$ that will be fixed once and for all.

Let $\mathcal{L}$ be an ample line bundle on $\overline{S}_1$, and let $\mathcal{M}$ be the pullback of $\mathcal{L}$ via $f_1 \colon \overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1) \to \overline{S}_1$. Note that $\mathcal{M}$ is semi-ample on $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$. Since $\overline{\mathcal{A}}$ is nonsingular and $\mathrm{Fund}(f_1)$ has codimension at least 2, the line bundle $\mathcal{M}$ admits a unique extension to the entire $\overline{\mathcal{A}}$. By simplicity we denote this extension by $\mathcal{M}$ but recall that it is semi-ample only on $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$. Define now the line bundles $\mathcal{M}_1 := \mathcal{M} \otimes [-1]^* \mathcal{M}^{-1}$ and $\mathcal{M}_2 := \mathcal{M} \otimes [-1]^* \mathcal{M}$. After restricting to $\mathcal{A}_2$ we construct two canonical heights on $\mathcal{A}_2$:

$$\hat{h}_{\mathcal{M}_i}(p) := \lim_{n\to\infty} \frac{1}{2^{in}} h_{\mathcal{M}_i}\left(2^n p\right) ,$$

where $[-1]$ and the multiplication by 2 are computed with respect to the fibration f_2. Then we define

$$\hat{h}_{\mathcal{M}} := \frac{1}{2}\hat{h}_{\mathcal{M}_1} + \frac{1}{2}\hat{h}_{\mathcal{M}_2} .$$

The height $\hat{h}_{\mathcal{M}}$ has three relevant properties for our aims:

(i) If $x \in \mathcal{A}_{2,\mathrm{tor}}$, then $\hat{h}_{\mathcal{M}}(x) = 0$.

(ii) $\hat{h}_{\mathcal{M}}(x+y) + \hat{h}_{\mathcal{M}}(x-y) = 2\hat{h}_{\mathcal{M}}(x) + \hat{h}_{\mathcal{M}}(y) + \hat{h}_{\mathcal{M}}(-y)$ for any x, y such that $f_2(x) = f_2(y)$.

(iii) $\hat{h}_{\mathcal{M}}(x) - h_{\mathcal{M}}(x) = O\left(h_{\mathcal{L}}(f_1(x))\right)$ for all $x \notin \mathrm{Fund}(f_1)$, by the Silverman-Tate formula (see for instance [48, Exercise 9.A]).

Furthermore, since $\mathcal{M}$ is obtained as pullback of $\mathcal{L}$ via f_1, we also have

(iv) $\hat{h}_{\mathcal{M}}(x) - (h_{\mathcal{L}} \circ f_1)(x) = O\left(h_{\mathcal{L}}(f_1(x))\right)$ for all $x \notin \mathrm{Fund}(f_1)$.

The line bundle $\mathcal{M}$ is semi-ample on $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$, so there exists a power $\mathcal{M}^{\otimes k}$ that is basepoint-free. It means that there exists a morphism (not necessarily an embedding) $\phi_k : \overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1) \to \mathbb{P}^N$, such that $\mathcal{M}^{\otimes k} = \phi_k^*(\mathcal{O}(1))$. By the Weil height machine we have $kh_{\mathcal{M}}(x) - h(\phi_k(x)) = O(1)$ for all $x \in \mathcal{A}_{2,b}(\overline{\mathbb{Q}})$, where the implicit constant is independent from b and h is the standard projective height.

Lemma 2.1. *There exists a projective embedding $\iota \colon \overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1) \hookrightarrow \mathbb{P}^n$ (with n big enough) such that $\phi_k \colon \overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1) \to \mathbb{P}^N$ extends to a rational map $\phi_k \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^N$.*

Proof. We can assume that $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$ comes with a fixed projective embedding $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1) \subset \mathbb{P}^m$. Let $\Gamma \subset \mathbb{P}^m \times \mathbb{P}^N$ be the graph of ϕ_k and consider the Segre embedding:

$$\sigma : \mathbb{P}^m \times \mathbb{P}^N \hookrightarrow \mathbb{P}^n, \quad n = (m+1)(N+1) - 1.$$

Denote $\Sigma = \sigma(\mathbb{P}^m \times \mathbb{P}^N) \subset \mathbb{P}^n$, then $\sigma(\Gamma) \subset \Sigma$ is a closed subvariety. The projection $\pi_2 : \mathbb{P}^m \times \mathbb{P}^N \to \mathbb{P}^N$ induces a rational map $\pi_2 \circ \sigma^{-1} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ well defined on Σ. The desired embedding ι is then given by the composition:

$$\mathcal{A}_{2,b} \hookrightarrow \Gamma \hookrightarrow \mathbb{P}^m \times \mathbb{P}^N \hookrightarrow \mathbb{P}^n.$$

It is clear from the construction that the above rational map $\pi_2 \circ \sigma^{-1}$ extends ϕ_k. □

Now, by [27, Theorem B.2.5(b)] applied to the above rational map $\phi_k \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^N$, there exists an integer ℓ such that $\ell h - h \circ \phi_k = O(1)$ on the whole $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$. Therefore, we can deduce the fifth main property of our height $\hat{h}_{\mathcal{M}}$:

(v) $k\hat{h}_{\mathcal{M}}(x) - \ell h_{\iota}(x) = O(h_{\mathcal{L}}(f_1(x))) + O(1)$, for all $x \notin \mathrm{Fund}(f_1)$. Here h_ι is the projective height induced by the embedding $\iota \colon \overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1) \hookrightarrow \mathbb{P}^n$ introduced in Lemma 2.1.

From now on, in order to make use of the properties (i)-(v), we will fix the following data: the line bundle $\mathcal{L}$ (and consequently $\mathcal{M}$); the height $\hat{h}_{\mathcal{M}}$ on $\overline{\mathcal{A}} \setminus \mathrm{Fund}(f_1)$; the projective embeddings ι.

2.1.2 Removing Zariski closed subsets

By Remark 0.7 it's enough to prove Theorem 0.4 for $\mathfrak{F} \cap \mathcal{A}'$, where $\mathcal{A}'$ is obtained from $\mathcal{A}$ after removing some non-horizontal Zariski closed subsets with respect to f_1 or f_2. Below we describe how to obtain $\mathcal{A}'$.

Let R_1 be the the Zariski closed subset of $\overline{S}_1$ defined as the union of the following proper Zariski closed subsets:

- The locus Sing_1 of singular fibers of the abelian scheme $f_1 : \mathcal{A}_1 \to S_1$.
- The locus E where two fibers $\mathcal{A}_{1,s_1}$ and $\mathcal{A}_{2,s_2}$ are equal.
- The locus Ind_1 containing the f_1-images of points where the rational map f_2 is not defined (see Assumption 2)).
- The locus $\mathcal{C}_{\text{Rém},1}$ where the inequality in Proposition 1.9 does not hold.
- The locus $S_{1,\text{deg}}$ of Theorem 1.4 (applied to $f\colon \mathcal{A}_1 \to S_1$) outside of which the Betti coordinates are algebraically independent along curves. By Remark 1.7, this is also the locus where the height bound in Corollary 1.6 does not hold. Furthermore, note that it also contains the locus of critical points of the Betti map β_1, i.e. where β_1 is not a submersion and Proposition 1.2 fails.

From now on we will work with the abelian scheme $\mathcal{A}_1$ restricted to $\overline{S}_1 \setminus R_1$ and by abuse of notation we keep denoting the base of such abelian scheme as S_1

Let R_2 be the the Zariski closed subset of $\overline{S}_2$ defined as the union of the following proper Zariski closed subsets:

- The locus Sing_2 of singular fibers of the abelian scheme $f_2 : \mathcal{A}_2 \to S_2$.
- The locus Ind_2 containing the f_2-images of points where the rational map f_1 is not defined (see Assumption 2)).
- The locus $\mathcal{C}_{\text{Rém},2}$ where the inequality in Proposition 1.9 does not hold.
- The locus $S_{2,\text{deg}}$ of Theorem 1.4 (applied to $f_2\colon \mathcal{A}_2 \to S_2$), which has the same properties listed above for $S_{1,\text{deg}}$.

We will work with the abelian scheme $\mathcal{A}_2$ restricted to $\overline{S}_2 \setminus R_2$ and by abuse of notation we keep denoting the base of such abelian scheme as S_2

We fix a number field K containing all the fields of definitions of $\overline{\mathcal{A}}, \overline{S}_1, \overline{S}_2, f_1, f_2, \sigma_1, \sigma_2$ and all the proper Zariski closed subset listed above. Let's define

$$\mathcal{A}' := \mathcal{A}_2 \setminus \left(f_1^{-1}(R_1) \cup f_2^{-1}(R_2)\right) . \tag{19}$$

For any f_2-fiber $\mathcal{A}_{2,b} := f_2^{-1}(b)$, we define the Zariski open subset

$$F_b := \mathcal{A}_{2,b} \cap \mathcal{A}'. \tag{20}$$

The restriction to F_b allows to get rid of the 'problematic' Zariski closed subset $\mathcal{A}_{2,b} \setminus \mathcal{A}'$.

Remark 2.2. The height $\hat{h}_{\mathcal{M}}$ satisfies the following crucial condition when restricted to a special open subset: if $p \in \mathfrak{F} \cap \mathcal{A}'$ and $b = f_2(p)$, then the properties (iii)-(v) of Section 2.1.1 hold with a *uniform* $O(1)$ on the right hand side, instead of $O(h_{\mathcal{L}}(f_1(p)))$. In particular, if we restrict to such kind of points $p \in \mathfrak{F} \cap \mathcal{A}'$, there exists a uniform constant C_{height} that bounds from above $\hat{h}_{\mathcal{M}}(p)$, $h(b)$ and $h_{\mathcal{L}}(f_1(p))$.

If $p \in \mathcal{A}'$ and $b = f_2(p)$ we clearly have that $K(b) \subseteq K(p)$. We define the set of complex K-embeddings of the field $K(p)$:

$$\Sigma_p := \{\tau\colon K(p) \hookrightarrow \mathbb{C} \mid \tau_{|K} = \text{id}\}. \tag{21}$$

Given $\tau \in \Sigma_p$ we get $f_2(p^\tau) = b^\tau$, but observe that two conjugates of b might coincide. Each element of Σ_p induces by restriction a complex K-embedding of $K(b)$ in a surjective way. Since we have the uniform bound C_{height} introduced in Remark 2.2, we can apply Proposition 1.9 and we obtain two constants $\eta = \eta(g)$ and $\eta' = \eta'(g)$ depending only on g such that

$$\text{ord}(\sigma_1(s)) \leq C'_{\text{Rém}} \cdot [K(s) : K]^{C_{\text{Rém}}} \qquad \text{for any } s \in S_1, \tag{22}$$

where

$$C_{\text{Rém}} = C_{\text{Rém}}(g) := 3 \cdot \frac{35840 g^3}{16}, \qquad C'_{\text{Rém}} = C'_{\text{Rém}}(g, K) := (14g)^{64g^2} (\eta' \cdot C_{\text{height}} + \eta) \cdot [K : \mathbb{Q}]^{C_{\text{Rém}}}. \tag{23}$$

Analogously, by using again the uniform bound C_{height} for the σ_2-torsion values and Proposition 1.9 we obtain

$$\operatorname{ord}(\sigma_2(b)) \leq C'_{\text{Rém}} \cdot [K(b) : K]^{C_{\text{Rém}}} \qquad \text{for any } b \in S_2, \tag{24}$$

with the same constants defined in Equation (23).

2.1.3 Removing euclidean open subsets

During the proof we need to apply our arguments with enough uniformity after removing the aforementioned Zariski closed subsets on the bases $\overline{S}_1, \overline{S}_2$ and on each fiber $\mathcal{A}_{2,b}$. We want to cut out small euclidean open subsets which encircle the Zariski closed subsets, so that we can work on compact analytic subsets containing enough conjugates of the points that we want to study.

Firstly, we consider the Zariski closed subset R_2 on the base $\overline{S}_2$. By applying Proposition 1.12 with respect to the height bound C_{height} introduced in Remark 2.2, we get an analytic compact set

$$\Delta \subseteq S_2 \tag{25}$$

(in the above notation we have $\Delta = C_\delta$ for some $\delta > 0$ small enough) such that for any $b \in S_2$ with $h(b) \leq C_{\text{height}}$ there are at least $\frac{3}{4}[K(b) : K]$ different K-embeddings $\tau : K(b) \hookrightarrow \mathbb{C}$ satisfying $b^\tau \in \Delta$. By Remark 1.13 the compact set Δ has the property that the intersection $\Delta \cap U_i$ with each standard chart is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.

Analogously, we want to cut out small euclidean open subsets of each f_2-fiber and of the base $\overline{S}_1$ which encircle the sets $\mathcal{A}_{2,b} \setminus F_b$ and R_1 respectively, so that we can work on a compact subsets of each fiber and of the base. We follow the same construction as in Equation (18). Since this construction does not depend on the shape of the Zariski closed subset removed in Equation (19), we explain it for general closed subsets.

Let's embed the fiber $\mathcal{A}_{2,b}(\mathbb{C})$ inside some $\mathbb{P}^N(\mathbb{C})$ and let $U'_0, \ldots, U'_N \subseteq \mathbb{P}^N(\mathbb{C})$ be the standard charts. Let us consider a Zariski closed subset $Y \subseteq \overline{S}_1$ and define

$$X_b = \mathcal{A}_{2,b}(\mathbb{C}) \cap f_1^{-1}(Y(\mathbb{C})). \tag{26}$$

After identifying $\mathcal{A}_{2,b}(\mathbb{C}) \cap U'_i$ with $\mathbb{R}^{2N}$, we can consider the globally subanalytic sets

$$V_{i,\delta} := \{z \in \mathcal{A}_{2,b}(\mathbb{C}) \cap U'_i \colon d(z, X_b \cap U'_i) < \delta\}$$

for any $\delta > 0$ small enough and define

$$V_{b,\delta} := \bigcup_{i=0}^{N} V_{i,\delta}. \tag{27}$$

This shows that the Zariski closed subset X_b is contained in a small enough euclidean open subset $V_{b,\delta} \subseteq \mathcal{A}_{2,b}(\mathbb{C})$ whose intersection $V_{b,\delta} \cap U'_i$ with each standard chart of $\mathbb{P}^N(\mathbb{C})$ is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.

Denote by $U_0, \ldots, U_M$ the standard affine charts on $\overline{S}_1(\mathbb{C})$. Analogously, we can encircle Y with a small enough open set of which we can control the size (chart-by-chart), so let us consider the sets

$$W_{i,\delta} := \{z \in S_1(\mathbb{C}) \cap U_i \colon d(z, Y \cap U_i) < \delta\}$$

for any $\delta > 0$ small enough, and define

$$W_\delta := \bigcup_{i=0}^{M} W_{i,\delta}. \tag{28}$$

We can carry out the construction of $V_{b,\delta}$ and W_δ such that $f_1(V_{b,\delta}) \subseteq W_\delta$, so that their size is controlled via the same δ.

We apply this construction to the Zariski closed sets $\mathcal{A}_{2,b} \setminus F_b$ and R_1. Therefore, in the rest of the proof we denote by $V_{b,\delta} \subset \mathcal{A}_{2,b}(\mathbb{C})$ a euclidean open subset which contains the locus $\mathcal{A}_{2,b} \setminus F_b$ and by W_δ a euclidean open subset which contains the locus R_1 with the property $f_1(V_{b,\delta}) \subseteq W_\delta$. We choose $\delta > 0$ small enough to ensure that Proposition 1.12 can be applied on the compact sets $\mathcal{A}_{2,b} \setminus V_{b,\delta}$ and $\overline{S}_1 \setminus W_\delta$ with respect to the height bound C_{height}. Notice that the intersections $V_{b,\delta} \cap U'_i$ and $W_\delta \cap U_i$ with each standard chart of $\mathbb{P}^N(\mathbb{C})$ and $\mathbb{P}^M(\mathbb{C})$ respectively is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. Define

$$T_{b,\delta} := \mathcal{A}_{2,b}(\mathbb{C}) \setminus V_{b,\delta}, \qquad \Delta' := \overline{S}_1 \setminus W_\delta\,. \tag{29}$$

Remark 2.3. It is essential here to implicitly use property (v) of Section 2.1.1. Indeed, Proposition 1.12 requires the variety to be embedded projectively, and the height must be the one induced by this embedding.

2.1.4 Auxiliary abelian schemes

We need to construct an auxiliary abelian scheme for any $b \in \Delta$ that will play a crucial role in the whole proof. Let us consider the variety F_b introduced in Equation (20) and define the following auxiliary abelian scheme:

$$\mathcal{X} := \mathcal{A}_1 \times_{S_1, f_1} F_b \to F_b\,, \qquad \text{for any } b \in \Delta, \tag{30}$$

endowed with the following non torsion section,

$$s_{\mathcal{X}} := \sigma_1 \circ f_1\,.$$

Let β and ℓ, be the Betti maps and the logarithms of $s_{\mathcal{X}}$ with respect to a period matrix Π. Note that $\mathcal{X}$ depends on the choice of b, but for simplicity of notations we don't write such dependence. The auxiliary scheme $\mathcal{X} \to F_b$ endowed with $s_{\mathcal{X}}$ clearly satisfies the hypotheses needed for Theorem 1.4. In addition, restricting to F_b ensures that

$$F_{b,\mathrm{deg}} = \emptyset\,. \tag{31}$$

The $s_{\mathcal{X}}$-torsion values lying in $\mathcal{A}'$ inherit the height bound C_{height} and the following bound on their order:

$$\mathrm{ord}(s_{\mathcal{X}}(z)) \le C'_{\mathrm{Rém}} \cdot [K(z) : K]^{C_{\mathrm{Rém}}} \qquad \text{for any } z \in F_b. \tag{32}$$

Moreover, when we need we can further restrict to the compact analytic subset $T_{b,\delta}$ constructed in Equation (29), ensuring that each point $z \in T_{b,\delta}$ with height at most C_{height} has enough conjugates in $T_{b,\delta}$.

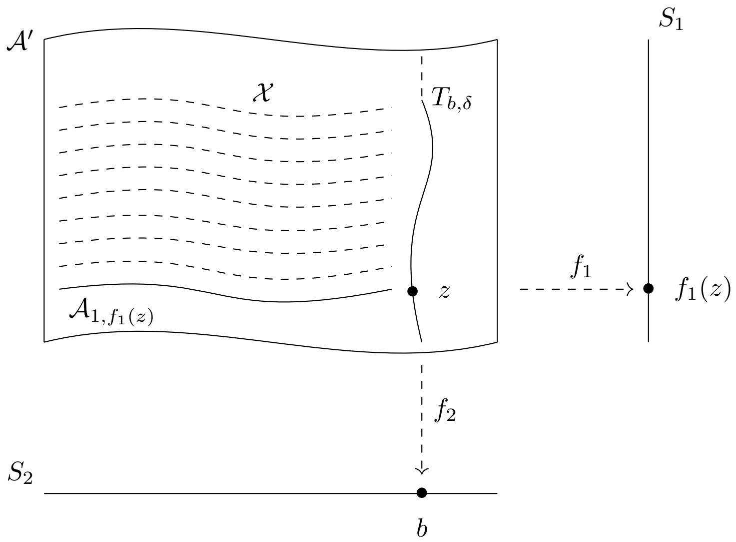

Figure 2: A schematization of the family $\mathcal{X} \to T_{b,\delta}$.

2.1.5 Reduction argument

Let us consider $b \in f_2(\mathcal{A}')$. If b is a σ_2-torsion value it has height bounded by C_{height}, so we can ensure that it has enough conjugates in the compact set Δ constructed in Equation (25). Since the order of $\sigma_2(b)$ and the set S_2 are invariant under the action of any K-embedding $\tau\colon K(b) \hookrightarrow \mathbb{C}$, in our proof we can always replace b by b^τ and consequently assume $b \in \Delta$. Roughly speaking we have just explained that we can assume that b lies in a "big enough" compact set of $\overline{S}_2(\mathbb{C})$ that avoids the bad locus of f_2.

Fix $b \in \Delta$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$. Since $p \in \mathfrak{F}$, then $f_1(p)$ is a σ_1-torsion value and $f_2(p)$ is a σ_2-torsion value. We denote

$$m = m(b) := \mathrm{ord}(\sigma_2(b)) \tag{33}$$

and define

$$\mathfrak{O} := \{\operatorname{ord}(\sigma_2(b)) \colon b \in f_2(\mathfrak{F}) \cap \Delta)\} \subseteq \mathbb{N}, \tag{34}$$

where clearly the order is intended in $\mathcal{A}_{2,b}$. Moreover, for any $r = 0, 1, \ldots, m-1$ we define

$$p_r := t_2^r(p) = p + r\sigma_2(b) \qquad \text{and} \qquad n_r := \operatorname{ord}\sigma_1(f_1(p_r)) \,. \tag{35}$$

Let Σ_p be the set defined in Equation (21). For any $\tau \in \Sigma_p$ we fix the following notation to denote the 'transaltes' of p^τ:

$$a_r = a_r^{(b,p,\tau)} := f_1(p^\tau + r\sigma_2(b^\tau)) \quad \text{ for } r = 0, \ldots, m-1. \tag{36}$$

Further, we can decompose the compact set Δ as a finite union of small definable compact sets Ξ_i. We work in one of those compact sets that contains b and we call it Ξ, in symbols we have

$$\Delta \subseteq \bigcup \Xi_i, \qquad b \in \Xi. \tag{37}$$

Analogously, we can decompose the compact set Δ' on $\overline{S}_1$ (see Equation (29)) as a finite union of small definable compact sets Ξ_i' where the Betti map of the section σ_1 is defined. We work in one of those compact sets that contains $f_1(p)$ and we call it Ξ', in symbols we have

$$\Delta' \subseteq \bigcup \Xi_i', \qquad f_1(p) \in \Xi'. \tag{38}$$

When we want to control the conjugates of p with respect to Ξ and/or Ξ' we will use the following subsets of Σ_p:

$$\Sigma_{p,\Xi} := \{\tau \in \Sigma_p : b^\tau \in \Xi\}, \qquad \Sigma_{p,\Xi,\Xi'} := \{\tau \in \Sigma_p : b^\tau \in \Xi,\ f_1(p)^\tau \in \Xi'\}. \tag{39}$$

Up to replace b, p with b^τ, p^τ and up to change Ξ and Ξ', since the number of Ξ_i's and Ξ_i''s is fixed and by construction of Δ and Δ', we can apply Proposition 1.12 to b and $f_1(p)$ and conclude the following:

$$\#\Sigma_{p,\Xi} \gg [K(p) : K] \qquad \text{and} \qquad \#\Sigma_{p,\Xi,\Xi'} \gg [K(p) : K] \,, \tag{40}$$

where the implicit constants are independent from p and b.

2.1.6 Distribution of f_1-images of conjugates

As a consequence of Remark 2.2 we show in Proposition 2.4 that it is possible to control the distribution of conjugates of p and their images on the two bases S_1 and S_2. Specifically, as explained in Section 2.1.3 we generally work with a subset of the base $S_1(\mathbb{C})$ as defined in Equation (29) and we must ensure that a "good portion" of conjugates is stable with respect to the euclidean coverings defined in Equation (37) and Equation (38).

We use the notations introduced in Equations (36) to (39). Fix $m \in \mathfrak{O}, b \in \Delta$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$ and $\operatorname{ord}(\sigma_2(b)) = m$. Since $K(b) \subseteq K(p)$, by Equation (24) we obtain

$$m = \operatorname{ord}(\sigma_2(b)) \leq C'_{\text{Rém}}[K(p) : K]^{C_{\text{Rém}}} \qquad \text{for any } b \in f_2(\mathcal{A}'). \tag{41}$$

By Remark 2.2, the element $f_1(p)$ has height bounded by C_{height} uniformly. Let us consider conjugation with respect to the set Σ_p defined in Equation (21). As explained before Equation (29) and after Equation (25), we choose $\delta > 0$ small enough such that[4]

$$\#\{a_0^{(b,p,\tau)} : \tau \in \Sigma_p\} \cap \Delta' \geq \frac{3}{4}[K(p) : K] \qquad \text{and} \qquad \#\{b^\tau : \tau \in \Sigma_p\} \cap \Delta \geq \frac{3}{4}[K(p) : K] \,.$$

Therefore, we obtain

$$\#\{a_0^{(b,p,\tau)} : \tau \in \Sigma_p \text{ and } b^\tau \in \Delta\} \cap \Delta' \geq \frac{1}{2}[K(p) : K] \,.$$

We define

$$\mathcal{J}_m^{(b,p)} := \{a_0^{(b,p,\tau)} : \tau \in \Sigma_{p,\Xi,\Xi'}\} \cap \Delta' \,. \tag{42}$$

Since the number of the sets Ξ_i and Ξ_i' is fixed, up to replace b, p with Σ_p-conjugates b^τ, p^τ, we can always choose compact sets Ξ among the Ξ_i and Ξ' among the Ξ_i' such that

$$b \in \Xi,\ f_1(p) \in \Xi' \qquad \text{and} \qquad \#\mathcal{J}_m^{(b,p)} \gg [K(p) : K] \,. \tag{43}$$

[4]We are taking conjugates of the field $K(p)$, which may be larger than $K(b)$ and $K(f_1(p))$: some of these conjugates may coincide but their distribution is preserved.

Proposition 2.4. *Assume that $\mathfrak{O}$ is infinite. Let us consider $m \in \mathfrak{O}$ and $b \in \Delta$ such that* $\operatorname{ord}(\sigma_2(b)) = m$. *Let $p \in \mathfrak{F} \cap \mathcal{A}'$ be such that $f_2(p) = b$. Assume $b \in \Xi$ and $f_1(p) \in \Xi'$ such that Equation (43) holds. For any $m \gg 1$ we have*

$$\#\mathcal{J}_m^{(b,p)} \gg m^{\frac{1}{C_{\text{Rém}}}}, \tag{44}$$

where the implicit constant is independent from m, b and p.

Proof. We proceed by contradiction: after choosing a sequence contained in $\mathfrak{O}$, for any m there exist $b \in \Xi$ and $p \in \mathfrak{F} \cap \mathcal{A}'$ with $f_1(p) \in \Xi'$ such that

$$\frac{\#\mathcal{J}_m^{(b,p)}}{m^{\frac{1}{C_{\text{Rém}}}}} \xrightarrow[m\to\infty]{} 0\,. \tag{45}$$

By Equation (41) and Equation (43) we obtain

$$\#\mathcal{J}_m^{(b,p)} \gg [K(p) : K] \gg m^{\frac{1}{C_{\text{Rém}}}}.$$

Finally we get

$$\frac{\#\mathcal{J}_m^{(b,p)}}{m^{\frac{1}{C_{\text{Rém}}}}} \gg 1,$$

which is a contradiction with Equation (45). □

2.2 Proof

All the notations introduced in Equations (19) to (40) will be fixed in the rest of the paper. In order to get the full proof of Theorem 0.4 it is enough to show that

the set $\mathfrak{O}$ defined in Equation (34) is bounded, i.e. the orders $m \in \mathfrak{O}$ are uniformly bounded.

In fact, if $\mathfrak{O}$ is bounded by a uniform constant C, then

$$\{f_2(p) : p \in \mathfrak{F} \cap \mathcal{A}'\} \subseteq \{b \in f_2(\mathcal{A}') \colon \operatorname{ord}(\sigma_2(b)) \leq C\} \subseteq \sigma_2^{-1}\left(\bigcup_{N \leq C} \mathcal{A}_2[N]\right). \tag{46}$$

Theorem 0.4 follows, since σ_2 is non-torsion. We will partition $\mathfrak{O}$ in two subsets $\mathfrak{O}'$ and $\mathfrak{O}''$ and show that each of them contains a finite number of elements.

2.2.1 First case

For any $m \in \mathfrak{O}$ we consider $b \in \Delta$ such that $\operatorname{ord}(\sigma_2(b)) = m$. Let F_b be the Zariski open subset of the fiber $\mathcal{A}_{2,b}$ introduced in Equation (20) and let $T_{b,\delta}$ be the euclidean compact set defined in Equation (29). Given a point $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$ we use the notation Equation (35) to denote the σ_2-translates of p and their orders with respect to the f_1-group law (namely, $p_r := p + r\sigma_2(b)$ and $n_r := \operatorname{ord}_{f_1}(p_r)$). Let $C_{\text{Rém}}$ be the constant introduced in Equation (23) and let's define

$$\mathfrak{O}' := \left\{m \in \mathfrak{O} \colon \exists b \in \Delta \text{ and } \exists p_r \in F_b \text{ such that } n_r > m^{g(2C_{\text{Rém}}+1)}\right\}.$$

We will prove that the set $\mathfrak{O}'$ is finite, giving a uniform upper bound for $m \in \mathfrak{O}'$. We fix

$$m \in \mathfrak{O}',\ b \in \Delta \text{ with } \operatorname{ord}(\sigma_2(b)) = m,\ p \in \mathfrak{F} \cap \mathcal{A}' \text{ with } f_2(p) = b,$$

and a point

$$\zeta := p_r = p + r\sigma_2(b) \in F_b \qquad \text{such that} \qquad n := n_r > m^{g(2C_{\text{Rém}}+1)}, \tag{47}$$

for some $r \in \{0, \ldots, m-1\}$. Up to choosing $\delta > 0$ small enough, we have $\zeta \in T_{b,\delta}$.

Step 1: construction of the conjugates. Consider the abelian scheme $\mathcal{X} \to F_b$ defined in Equation (30) and fix $z \in F_b(\mathbb{C})$. As explained in Equation (4), there exists a simply connected open set $U'_z \subseteq F_b(\mathbb{C})$ in the complex topology containing z where a period map is defined:

$$\mathcal{P}^{(b)}_{\mathcal{X}} = \left(\omega^{(b)}_{1,\mathcal{X}}, \ldots, \omega^{(b)}_{2g,\mathcal{X}}\right).$$

In other words we have holomorphic functions $\omega^{(b)}_{i,\mathcal{X}} : U'_z \to \mathbb{C}^g$ for $i = 1, \ldots, 2g$ which fix a basis of the corresponding lattice $\Lambda_{z'}$ for each $z' \in U'_z$. Thus, the family of open simply connected sets $\{U'_z : z \in T_{b,\delta}\}$ is a covering of $T_{b,\delta}$. Fixing a standard chart U'_i which contains z, we can consider a simply connected open definable subset $U_z \subseteq U'_z \cap U'_i$ which contains z and whose analytic closure D_z is contained in $U'_z \cap U'_i$. In other words, we can consider an open covering $\{U_z \colon z \in T_{b,\delta}\}$, where each U_z is a simply connected open set with the following properties: its analytic closure D_z in the fixed chart of F_b is a definable compact set in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ and all the period functions $\omega^{(b)}_{i,\mathcal{X}}$ with $i = 1, \ldots, 2g$ are defined as holomorphic functions on D_z. Since $T_{b,\delta}$ is compact, it can be covered with finitely many small compact simply-connected sets of the type D_z.

Since $U'_z \subseteq F_b(\mathbb{C})$ is simply connected, we obtain notions of abelian logarithm $\ell^{(b)}_{\mathcal{X}}$ and Betti map $\beta^{(b)}_{\mathcal{X}} = \left(\beta^{(b)}_{1,\mathcal{X}}, \ldots, \beta^{(b)}_{2g,\mathcal{X}}\right)$ of the section $s_{\mathcal{X}}$ on each U'_z as explained in Equation (5). Note that the abelian logarithm is a holomorphic function on each compact set D_z and the Betti map is described by the equation

$$\ell^{(b)}_{\mathcal{X}}(z) = \beta^{(b)}_{1,\mathcal{X}}(z)\omega^{(b)}_{1,\mathcal{X}}(z) + \ldots + \beta^{(b)}_{2g,\mathcal{X}}(z)\omega^{(b)}_{2g,\mathcal{X}}(z),$$

where the Betti coordinates $\beta^{(b)}_{i,\mathcal{X}}$ are real-analytic functions on each compact set D_z. In addition note that $\beta^{(b)}_{\mathcal{X}}$ doesn't have any critical points on $T_{b,\delta}$ by construction (we have expressly removed them).

Summarizing: we have obtained the existence of finitely many simply connected compact sets D_i with $i = 1, \ldots, N_{\text{comp}}$ which are definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ and where the Betti map $\beta^{(b)}_{\mathcal{X}}$ is $\mathbb{R}_{\text{an,exp}}$-definable and a submersion.

Remark 2.5. Fix $z \in T_{b,\delta}$. Observe that period functions, logarithms and Betti maps of $\mathcal{X} \to F_b$ are uniform with respect to b, since each fiber $\mathcal{X}_z$ only depend on the image $f_1(z)$. Moreover, the number N_{comp} of compact sets D_i's just constructed can be supposed to be uniform, i.e. constant with respect to $b \in \Delta$: in fact the open covering of the $T_{b,\delta}$'s given by the open part of the D_i's can be assumed to be induced (after intersecting with f_2-fibers) by a global open covering of the compact set $f_2^{-1}(\Delta)$ with the same properties.

By Equation (32) we have

$$n^{\frac{1}{C_{\text{Rém}}}} \ll [K(\zeta) : K], \tag{48}$$

where the implicit constant depends only on g and K, which are fixed. Since b lies in the zero locus of $[m] \circ \sigma_2$, the pullback divisor $\big([m] \circ \sigma_2\big)^*(0)$ on S_2 contains b with some multiplicity $e_b \geq 1$. As $\deg([m]) = m^{2g}$, the divisor has total degree $\leq m^{2g}$, and hence

$$e_b\,[K(b) : K] \;\leq\; \deg\big(([m] \circ \sigma_2)^*(0)\big) \;\leq\; m^{2g}, \tag{49}$$

which implies $[K(b) : K] \;\leq\; m^{2g}$.

Combining (48) and (49) we obtain

$$d := [K(\zeta) : K(b)] = \frac{[K(\zeta) : K]}{[K(b) : K]} \;\gg\; \frac{n^{\frac{1}{C_{\text{Rém}}}}}{m^{2g}}.$$

Using (47), namely $n > m^{\,g(2C_{\text{Rém}}+1)}$, we deduce

$$d \;\gg\; n^{\frac{1}{C_{\text{Rém}}} - \frac{2}{2C_{\text{Rém}}+1}} \;=\; n^{\frac{1}{c_0}}, \qquad \text{where } \; c_0 := C_{\text{Rém}}(2C_{\text{Rém}} + 1).$$

Consider the conjugates of ζ over $K(b)$, and call them ζ_j where $j = 1, \ldots, d$; they are torsion values of $s_{\mathcal{X}}$, since the section $s_{\mathcal{X}}$ is defined over K. As explained after Equation (32), up to choose $\delta > 0$ small enough, we can assume that the number of these conjugates lying in a same compact set of the type D_i is $\gg d$, where the implicit constant depends only on the original data (it can be taken for instance equal to $1/(2N_{\text{comp}})$ by Remark 2.5). From now on, we will denote by $\Omega_b \subseteq A_{2,b}(\mathbb{C})$ the compact set (among the D_i's) just described. Hence, we may assume

$$\#\{\zeta_j \in \Omega_b\} \gg n^{\frac{1}{c_0}}. \tag{50}$$

By Equation (37), we decompose the compact set Δ as a finite union of small definable compact sets Ξ_j and we choose a set Ξ among them containing b. We consider the Betti map

$$\beta(z) := \beta^{(b)}(z) := (\beta^{(b)}_{1,\mathcal{X}}(z), \dots, \beta^{(b)}_{2g,\mathcal{X}}(z)). \tag{51}$$

The Betti coordinates $\beta^{(b)}_{i,\mathcal{X}}$ are real-analytic with respect to the variable $z \in \Omega_b$ and also with respect to $b \in \Xi$.

Step 2: the definable family. We are going to introduce now a definable family $Z \subset \mathbb{R}^{4g}$ in terms of the Betti map β on the auxiliary fibration $\mathcal{X} \to F_b$. Consider $t = (t_1, \dots, t_{2g}) \in \mathbb{R}^{2g}$ and the following $\mathbb{R}$-linear map

$$t \mapsto u(t) := \sum_{j=1}^{2g} t_j \omega_j(b) \in \mathbb{C}^g$$

which has full rank. Define

$$[0,1[^{2g} \ni t \mapsto z(t) := \exp_b(u(t)) \in \mathcal{A}_{2,b}$$

where $\exp_b$ here denotes the exponential map of the fiber $\mathcal{A}_{2,b}$. We consider the $\mathbb{R}_{\mathrm{an,exp}}$-definable family Z, whose fibers are the sets

$$Z_b := \left\{(\beta(z(t)), t) \colon\ t \in z^{-1}(\Omega_b)\right\} \subset \mathbb{R}^{2g} \times [0,1[^{2g}, \quad \forall b \in \Xi$$

Recall that $\Omega_b \subseteq \mathcal{A}_{2,b}(\mathbb{C})$ is the precisely compact set containing "many" conjugates of ζ considered above. Notice that Z_b has dimension $2g$ inside $\mathbb{R}^{4g}$, so it has empty euclidean interior. We denote by Z_b^{alg} the algebraic part of Z_b. Following for instance [44, Definition 1.5], we recall that the algebraic part Z_b^{alg}, is the union of all connected semialgebraic subsets of Z_b of positive dimension (with the standard notion of dimension in o-minimal geometry).

In this step, we show that the family Z_b cannot contain any real-analytic arc that gives rise to algebraic relations among the Betti coordinates. In particular, this implies that $Z_b^{\mathrm{alg}} = \emptyset$. Our strategy for proving this claim is a modification of a standard argument based on the algebraic independence of the logarithm coordinates with respect to the periods (see, for instance, [35, Lemma 6.2]).

Proposition 2.6. *A real-analytic arc $\gamma(s) = (x(s), t(s)) \subset Z_b$, with $s \in [0,1]$, with the following properties cannot exist:*

1. *The projection onto the first coordinate $x(s)$ is semi-algebraic.*
2. *The projection onto the second coordinate $t(s)$ is non-constant.*

In particular, Z_b^{alg} is empty.

Proof. Assume by contradiction that such an arc γ does exist. Let $\Pi : \Omega \to \mathrm{Mat}(\mathbb{C}, g \times 2g)$ be the period matrix, and recall that for a chosen branch of the abelian logarithm we have $\ell(w) = \Pi_w \beta(w)$. Here Π and ℓ are defined on the auxiliary scheme $\mathcal{X} \to F_b$, so such periods shouldn't be confused with $\omega_1(b), \dots, \omega_{2g}(b)$ fixed above. By the definition of Z_b we also have the relation

$$\ell(z(t(s))) - \Pi_{z(t(s))} x(s) = 0\,.$$

By hypothesis, the projection $x(s)$ of $\gamma(s)$ to its first component is semi-algebraic of real dimension at most 1. It means that the Betti coordinates β_j restricted to this arc all depend algebraically on any of them, let's call it β_i, via real polynomials P_j. By complexification, and denoting with pr_j the obvious projection, it means that there exist non-trivial polynomials $P_j \in \mathbb{C}[U_1, U_2]$ for any $j = 1, \dots, 2g$ with $j \neq i$ such that

$$P_j(\mathrm{pr}_j(x(s)), \mathrm{pr}_i(x(s))) = 0\,.$$

Define now the set

$$\mathcal{Y} := \{(x,z) \in \mathbb{C}^{2g} \times \Omega_b \colon \ell(z) - \Pi_z x = 0\,,\ P_j(\mathrm{pr}_j(x), \mathrm{pr}_i(x)) = 0\ \forall\, j \neq i\} \subset \mathbb{C}^{2g} \times \mathbb{C}^g.$$

Note that $\mathcal{Y}$ is a complex analytic set. Define a real-analytic arc Γ inside $\mathcal{Y}$ by

$$[0,1] \ni s \longmapsto \Gamma(s) := (x(s), z(t(s))) \in \mathcal{Y} \cap (\mathbb{R}^{2g} \times \Omega_b).$$

Let $\widetilde{\Gamma}(s) := (\mathrm{pr}_2 \circ \Gamma)(s) = z(t(s))$. For $z \in \mathrm{Im}\,\widetilde{\Gamma}$ we have $x(s) = \beta(z)$ by construction, hence

$$P_j(\beta_j(z), \beta_i(z)) = 0 \quad \text{for all } z \in \mathrm{Im}\,\widetilde{\Gamma}. \tag{52}$$

Since Ω_b (and hence each chart D_i) together with the maps Π, ℓ, β are definable in the structure $\mathbb{R}_{\mathrm{an,exp}}$, the holomorphic extension of $\widetilde{\Gamma}$ has definable image. Therefore, the complex-analytic closure of $\widetilde{\Gamma}$ inside F_b is a definable complex-analytic subset. By the definable Chow theorem this closure is algebraic (see [42, Theorem 5.1]); denote it by the (irreducible) curve $C \subset F_b$. Since $\mathcal{Y}$ is complex analytic, the relation Equation (52) extends to a eulidean open subset of C. Now restrict the auxiliary abelian scheme $X := \mathcal{A}_1 \times_{S_1, f_1} F_b \to F_b$ and the section $s_X := \sigma_1 \circ f_1$ to the curve C. If all the Betti coordinates are constant, again by Manin's kernel theorem, the section σ restricted to C is torsion, and hence $C \subseteq S_{\mathrm{deg}}$ contradicting the choice of Ω_b. Therefore, we can assume β_i to be nonconstant and we can apply Theorem 1.4 to the family $X|_C \to C$ with section $s_{\mathcal{X}|C}$: again the curve C would be contained in S_{deg}, which contradicts the choice of Ω_b. This contradiction shows that γ cannot exist. In particular, Z_b^{alg} is empty.

□

Step 3: Habegger-Pila counting. We need the following height function on $\mathbb{Q}^{2g}$:

$$H\left(\frac{x_1}{y_1}, \ldots, \frac{x_{2g}}{y_{2g}}\right) := \max_i\{\max|x_i|, |y_i|\}\,. \tag{53}$$

Moreover, we define

$$Z_b^{\sim}(\mathbb{Q}, T) := \{(x,t) \in Z_b \mid x \in \mathbb{Q}^{2g},\, H(x) \leq T\}\,, \quad T \in \mathbb{R}_{\geq 1}. \tag{54}$$

We consider now the points ζ_j in Equation (50), and we define

$$\Sigma := \{(\beta(\zeta_j), \zeta_j) : \zeta_j \in \Omega_b\}\,.$$

For the properties of the Betti map, each point ζ_j in Equation (50) gives rise to a rational point $\beta(\zeta_j)$ with denominators at most n. Hence, all of these rational points have height $\ll n$, say $\leq c_1 n$. This implies $\Sigma \subseteq Z_b^{\sim}(\mathbb{Q}, c_1 n)$.

Remark 2.7. Let's explain more in detail why c_1 is uniform. Firstly, the denominators of $\beta(\zeta_j)$ are bounded. Moreover we can bound the numerators on each compact set D_z, since the Betti map attains a maximum on each of them. Since the number of compact sets was previously fixed, we can choose analytic continuation of the Betti map such that the numerators of $\beta(\zeta_j)$ are bounded uniformly.

Moreover, by Equation (50), we have $\#\mathrm{pr}_2(\Sigma) \gg n^{\frac{1}{c_0}}$, where the constant depends only on the involved compact sets, which are fixed. We write

$$\#\mathrm{pr}_2(\Sigma) \geq c_2 n^{\frac{1}{c_0}}\,, \qquad \text{for some constant } c_2\,. \tag{55}$$

On the other hand by the rational-point-counting of Habegger and Pila [23, Corollary 7.2] and Proposition 2.6, for any $\varepsilon > 0$ there exists a constant $c(Z, \varepsilon)$ such that

$$\#\mathrm{pr}_2(\Sigma) \leq c(Z, \varepsilon)(c_1 n)^{\varepsilon}, \tag{56}$$

where the constant is independent from $b \in \Xi$. Taking $\varepsilon = 1/(2c_0)$ and combining with Equation (55), we obtain

$$c_2 n^{\frac{1}{c_0}} \leq \#\mathrm{pr}_2(\Sigma) \leq c(Z)(c_1 n)^{\frac{1}{2c_0}}$$

where all constants $c(Z), c_0, c_1, c_2$ are uniform with respect to $b \in \Xi$. This implies $n^{\frac{1}{2c_0}} \leq c_3$, that is $n^{\frac{1}{2C_{\text{Rém}}+1}} \leq c_3^{2C_{\text{Rém}}}$. In particular, by Equation (47) this implies

$$m < n^{\frac{1}{g(2C_{\text{Rém}}+1)}} \leq c_3^{\frac{2C_{\text{Rém}}}{g}}\,.$$

This estimate holds uniformly with respect to $b \in \Xi$. Since we have a finite number of fixed compact sets Ξ_j which cover Δ, we obtain a uniform bound for $m \in \mathfrak{O}'$.

2.2.2 Second case

We keep the same notations used in Section 2.2.1. Define

$$\mathfrak{O}'' := \{m \in \mathfrak{O} \colon \forall b \in \Delta \text{ and } \forall p_r \in F_b \text{ we have } n_r \leq m^{g(2C_{\text{Rém}}+1)}\}\,.$$

We will prove that the set $\mathfrak{O}''$ is finite. Assume by contradiction that it is not finite. We fix

$$m \in \mathfrak{O}'',\ b \in \Delta \text{ with } \operatorname{ord}(\sigma_2(b)) = m,\ p \in \mathfrak{F} \cap \mathcal{A}' \text{ with } f_2(p) = b.$$

Therefore, for any $r \in \{0, \dots, m-1\}$ we have

$$p_r = p + r\sigma_2(b) \in F_b \quad \Longrightarrow \quad n_r \leq m^{g(2C_{\text{Rém}}+1)}. \tag{57}$$

We consider again the abelian scheme $\mathcal{X} \to F_b$ introduced in Equation (20) with the euclidean compact set in Equation (29). We decompose $T_{b,\delta}$ as a finite union of compact subsets $\{D_i\}$ where periods, abelian logarithm and Betti map are defined, as in Section 2.2.1. By Equation (37) we decompose Δ and Δ' as a finite union of definable compact sets and we choose compact sets Ξ and Ξ' among them containing b and $f_1(p)$, respectively. Denote by β_{σ_1} the Betti map of σ_1 on S_1. We consider the $\mathbb{R}_{\text{an,exp}}$-definable family Z with fibers

$$Z'_b := \{(\beta_{\sigma_1}(t), t) : t \in \Xi'\} \subset \mathbb{R}^{2g} \times \mathbb{R}^{2g}\,, \qquad \forall b \in \Xi\,.$$

where by abuse of notation we consider $\Xi' \subseteq \mathbb{R}^{2g}$ via the definable atlas. In the following we use same height of Equation (53) and the same notation of Equation (54).

Let us consider the set $\mathcal{J}_m^{(b,p)}$ introduced in Equation (42), which contains the f_1-images of all the $\Sigma_{p,\Xi,\Xi'}$-conjugates of p, and define

$$\Sigma' := \{(\beta_{\sigma_1}(t), t) : t \in \mathcal{J}_m^{(b,p)}\}\,.$$

By Equation (57), for the properties of the Betti map, the points $\beta_{\sigma_1}(t)$, with $t \in \mathcal{J}_m^{(b,p)}$) are rational with denominators at most $m^{g(2C_{\text{Rém}}+1)}$. By Remark 2.7, these points have height $\ll m^{g(2C_{\text{Rém}}+1)}$, say $\leq c_4 m^{g(2C_{\text{Rém}}+1)}$. This implies $\Sigma' \subseteq Z_b'^{\sim}(\mathbb{Q}, c_4 m^{g(2C_{\text{Rém}}+1)})$. Since we are assuming that $\mathfrak{O}''$ is infinite, by Proposition 2.4, for any $m \gg 1$ we have

$$\#\mathrm{pr}_2(\Sigma') \geq c_5 m^{\frac{1}{C_{\text{Rém}}}} \qquad \text{for some constant } c_5\,, \tag{58}$$

where pr_2 denotes the projection onto the second coordinate and the constant is independent from m, b and p.

By reasoning exactly as in the previous case it is possible to prove the analogous of Proposition 2.6 for Z'_b. Hence, by [23, Corollary 7.2], for any $\varepsilon > 0$ there exists a constant $c(Z, \varepsilon)$ such that

$$\#\mathrm{pr}_2(\Sigma') \leq c(Z, \varepsilon)(c_4 m^{g(2C_{\text{Rém}}+1)})^{\varepsilon}, \tag{59}$$

where the constant is independent from $b \in \Xi$. Taking $\varepsilon < \frac{1}{gC_{\text{Rém}}(2C_{\text{Rém}}+1)}$, and combining with Equation (58), we finally obtain:

$$m \leq \left(\frac{c(Z)c_4^{\varepsilon}}{c_5}\right)^{\frac{C_{\text{Rém}}}{1-\varepsilon g C_{\text{Rém}}(2C_{\text{Rém}}+1)}}.$$

This bound holds uniformly on Ξ and Ξ'. Since $\{\Xi_j\}$ and $\{\Xi'_j\}$ are fixed finite covering of Δ and Δ' respectively, we get a uniform bound for $m \in \mathfrak{O}''$ concluding the proof.

2.3 Some comments on the shape of Z_1 and Z_2

At the beginning of the proof, we removed some proper Zariski closed subset from the total space $\overline{\mathcal{A}}$ (see Section 2.1.2). Consequently, those sets fall inside the Zariski closed sets Z_1 and Z_2 appearing in Theorem 0.4. Thanks to the previous considerations, we get explicit expressions of Z_1 and Z_2 as it follows:

$$Z_1 = \mathrm{Sing}_1 \cup E \cup \mathrm{Ind}_1 \cup \mathcal{C}_{\text{Rém},1} \cup S_{1,\deg},$$

$$Z_2 = \mathrm{Sing}_2 \cup \mathrm{Ind}_2 \cup \mathcal{C}_{\text{Rém},2} \cup S_{2,\deg} \cup \sigma_2^{-1}\left(\bigcup_{N \leq C} \mathcal{A}_2[N]\right),$$

where C is the uniform bound on $\mathfrak{O}$ (see Equation (46)). Unfortunately the constant C is implicit.

When $\dim \overline{S}_1 = \dim \overline{S}_2 = g = 1$, we have $\overline{S}_1 = \overline{S}_2 = \mathbb{P}^1$. In this case, we denote both bases simply by S. Here, the subsets $S_{i,\mathrm{deg}}$ are empty for obvious reasons, and the locus $f_1^{-1}(E)$ can be equivalently described as a finite union of f_2-fibers. The loci $\mathcal{C}_{\mathrm{Rém},i}$ are empty in this case since we don't need to use Faltings height.

Finally, the following proposition shows that in the case $1 = \dim S = g$ all the points of $(\mathfrak{F} \setminus \mathrm{Fund}(f_2)) \cap f_1^{-1}(\mathrm{Sing}_1)$ are contained in a set of the form $f_2^{-1}(Z)$, where Z is a proper Zariski closed subset of $\overline{S}_2$. In other words we recover the stronger result proved in [15], i.e. $\mathfrak{F} \setminus \mathrm{Fund}(f_2)$ is contained in a finite number of f_2-fibers (see Remark 0.6).

Proposition 2.8. *Let $1 = \dim S = g$, then there exists a proper closed Zariski subset $Z \subset S(\mathbb{C})$ such that:*

$$(\mathfrak{F} \setminus \mathrm{Fund}(f_2)) \cap f_1^{-1}(\mathrm{Sing}_1) \subseteq f_2^{-1}(Z).$$

Proof. Assume that Sing_1 has cardinality n and denote by Z_1 and Z_2 the proper Zariski closed subsets of $\overline{S}_1$ and $\overline{S}_2$ arising from Theorem 0.4, respectively. By Bézout theorem we know that $\#(\mathcal{A}_{2,s}(\mathbb{C}) \cap f_1^{-1}(\mathrm{Sing}_1)) \leq 9n$. Let's put $H = (\mathfrak{F} \setminus \mathrm{Fund}(f_2)) \cap f_1^{-1}(\mathrm{Sing}_1)$ and let's consider the following partition of H:

$$H_1 := \{p \in H : \#(O(p)) \leq 9n\}, \qquad H_2 := \{p \in H : \#(O(p)) > 9n\}.$$

The set $f_2(H_1)$ is finite, since the following containment holds:

$$f_2(H_1) \subseteq \sigma_2^{-1}\left(\bigcup_{N=1}^{9n} \mathcal{A}[N]\right).$$

Fix $p \in H_2$. Observe that there exists $r \in \mathbb{N}$ such that $t_2^r(p) \notin f_1^{-1}(\mathrm{Sing}_1)$: if not, we would have a contradiction by the fact that $O(p) = \{t_2^r(p) : r \in \mathbb{N}\} \subseteq f_1^{-1}(\mathrm{Sing}_1) \cap \mathcal{A}_{2,s}(\mathbb{C})$ and $\#(O(p)) > 9n$. Therefore, for such r we have $f_1(t_2^r(p)) \notin Z_1$. Hence, by Theorem 0.4, we get $f_2(t_2^r(p)) \in Z_2$. Since t_2 acts on the f_2-fibers, we conclude that $f_2(t_2^r(p)) = f_2(p) \in Z_2$. This proves that $f_2(H_2) \subseteq Z_2$. The claim follows if we put $Z = Z_2 \cup f_2(H_1)$.

□

A Construction of double abelian fibrations in the IHS case

by E. Amerik

The purpose of this appendix is to remark that examples of the situation studied in this paper exist in every even dimension, and to provide some explicit constructions, as well as indications how to prove abstract existence results in a case which has been extensively studied by geometers. The general framework is as follows. We consider an **irreducible holomorphically symplectic (IHS) manifold** X, that is, a simply-connected manifold X such that $H^0(X, \Omega^2_X)$ is one-dimensional and generated by a nowhere degenerate form σ. We can take X projective, or more generally compact Kähler (in the situation we are looking for, projectivity shall be automatic). A typical example of such a manifold is a K3 surface S, or, more generally, the n-th punctual Hilbert scheme $S^{[n]}$, parameterizing subschemes of S of finite length n. In all explicit examples, we shall be dealing with $S^{[n]}$, but the general results are valid in the general IHS context.

It is well-known that on the second cohomology $H^2(X, \mathbb{Z})$ there is an integral non-degenerate quadratic form q, called the Beauville-Bogomolov form, which can be seen as an analogue of the intersection form on a surface. If $X \to B$ is a fibration, then the inverse image of an ample line bundle on B is nef and q-isotropic. Conversely, a famous "Lagrangian", or "hyperkähler SYZ", conjecture, checked in all known examples, in particular for $S^{[n]}$, states that if L is a nef line bundle on X with $q(L) = 0$, then some power of L is base-point-free, so that its sections define a fibration $\phi = \phi_L : X \to B$. Matsushita [36] proved that a non-trivial fibration on an IHS manifold is equidimensional, and all smooth fibers are lagrangian tori. In particular, if ϕ has a section, one obtains a family of abelian varieties on an open subset of X, say $\phi^0 : X^0 \to B^0$.

Oguiso ([40]) proved that the Picard number of the generic fiber of such a fibration is always equal to one. In particular, the generic fiber is simple, so that the family does not have a fixed part as soon as it is not isotrivial. In fact it is easy to deduce from [9] or [6] that no finite base-change of ϕ^0 has a fixed part unless the family is isotrivial.

By the same reason, the multiples of any non-torsion section or multisection of a family of abelian varieties arising in this way must be Zariski-dense.

If f is an automorphism of X such that its action on $H^2(X, \mathbb{Z})$ preserves the class of L as above, then a power of f preserves the fibration $\phi_L : X \to B$ ([30]) and acts on the smooth fibers as a translation ([6]). There is a way to say whether an automorphism ψ of the Neron-Severi lattice $NS(X) \subset H^2(X, \mathbb{Z})$ preserving the class of L comes from an actual automorphism $f : X \to X$, see **"Hodge-theoretic Torelli theorem"** by Markman, [32]: it should belong to the (Hodge) monodromy group[5], and it should take some ample class to an ample class. The Hodge monodromy group is of finite index in the automorphism group of $(NS(X), q)$, so replacing any ψ by a power we may assume it is in there. The ample cone is governed by so-called MBM classes, a higher-dimensional analogue of (-2)-classes on K3 surfaces ([2], [3]). These are primitive classes in $H^2(X, \mathbb{Z})$ of bounded negative square ([4]). Inside the cone of classes of positive square in $NS(X) \otimes \mathbb{R}$, the ample cone is a connected component of the complement to the union of the orthogonal hyperplanes to the MBM classes of Hodge type $(1, 1)$. On all known examples of IHS manifolds, in particular on $S^{[n]}$, these classes can be described explicitely. If no MBM class is orthogonal to L in $(NS(X), q)$, then, up to taking a power, an automorphism of the lattice which fixes L lifts to an automorphism of X: indeed the image of an ample class near L in $NS(X) \otimes \mathbb{Q}$ shall be ample, so this is a consequence of Hodge-theoretic Torelli. The automorphisms preserving L, up to a finite index, form a free abelian group of rank $\rho - 2$, where ρ is the Picard number of X (we assume here that $\rho \geq 3$, then the statement is obtained from hyperbolic geometry, see [6]). If there are such MBM classes but not too many, some automorphisms may lift, see e.g. [37]: one has to further subtract from $\rho - 2$ the dimension of the subspace they generate. Such automorphisms are sometimes called **parabolic**.

Let us start with the following explicit example. Let S be a smooth quartic surface in $\mathbb{P}^3$ (it is, of course, a K3 surface). It is well-known and easy to see that S can contain only finitely many (complex) lines, so if S is defined over a number field, then the lines are defined over a (possibly larger) number field too. Assume S contains a line l. Take all planes through l, it is a pencil of planes (they are parameterized by $\mathbb{P}^1$). For each such plane P_t, the intersection with S is $l \cup C_t$, where C_t is a plane cubic. This gives a fibration $\phi : S \to \mathbb{P}^1$ where the smooth fibers are curves of genus 1. The line l induces a multisection: indeed l intersects each C_t in three points. So it is a trisection.

[5] The monodromy group is the group of automorphisms of $H^2(X, \mathbb{Z})$ generated by all parallel transports in families, and the Hodge monodromy group is the image of its Hodge type-preserving subgroup in the group of automorphisms of the Neron-Severi lattice.

If S contains another line l', which does not intersect l (this is possible, e.g. on a Fermat surface, but also on others - in fact over a codimension-two subvariety of the parameter space for quartic surfaces), this gives a section of ϕ, indeed each P_t and hence each C_t intersects l' at one point. In its turn, taking the pencil of planes P'_t through l', we obtain another fibration of S, $\phi' : S \to \mathbb{P}^1$, with genus one fibers C'_t residual to l' in the intersection of S and P'_t, a section induced by l, and a trisection induced by l' itself.

On the resulting abelian schemes, these trisections are non-torsion, see e.g. [24] where it is explained that a torsion multisection of an elliptic fibration of a K3 surface cannot be a rational curve. One can also choose S in such a way that it contains an additional line m skew to both l and l': it shall induce an additional section of both fibrations. Keeping in mind the general theory of automorphisms of IHS manifolds and MBM classes, one may also produce non-torsion sections on S as follows.

Proposition A.1. *If S is general with the above properties, then S admits an automorphism h of infinite order preserving ϕ and acting as a translation along its fibers.*

Proof. For such an S, the lattice $NS(X)$ is of rank 3, generated by the classes H (the hyperplane section class), l and l', and the class L of C_t is $H - l$. The orthogonal complement to L is generated by L itself and $H - 3l'$, which has square -20. Hence there are no MBM classes in the orthogonal complement to L: indeed these have square -2. So the result follows from Hodge-theoretic Torelli. □

We derive in particular that S also has a non-torsion section $h(l')$ of ϕ. The same applies to ϕ' (with $L' = H - l'$) and gives a non-torsion section $h'(l)$.

Consider now the k-th punctual Hilbert scheme $S^{[k]}$ of a K3 surface S: it parameterizes subschemes of S of length k, e. g. k-ples of distinct points, or of not necessarily distinct points with some extra structure. It is often viewed as a resolution of singularities of the k-th symmetric power of S. Any fibration $g : S \to \mathbb{P}^1$ naturally induces the fibration $g^{[k]} : S^{[k]} \to \mathbb{P}^k = Sym^k(\mathbb{P}^1)$. The fiber over a point $t_1 + \cdots + t_k$ (where the t_i are distinct points on the projective line) is just the product $C_{t_1} \times C_{t_2} \times \cdots \times C_{t_k}$. So this is a fibration where the fibers over an open subset of the base are k-dimensional tori. Any section s of g naturally induces a section $s^{[k]}$ of $g^{[k]}$, and non-torsion induces non-torsion.

We are now in a position to give explicit examples of the situation considered in the paper.

Theorem A.2. *For each $k \geq 1$ there exist algebraic varieties X of dimension $2k$ with two fibrations ϕ and ϕ' from X to $\mathbb{P}^k$, such that ϕ resp. ϕ' induces an abelian scheme structure without a fixed part on an open subset U resp. U' of X. Each of these fibrations has an extra non-torsion section. Moreover the multiples of these sections are Zariski-dense in U, U'.*

Proof. Take S a quartic in $\mathbb{P}^3$ containing two skew lines l and l', inducing fibrations ϕ and ϕ', and consider $\phi^{[k]}$ and $\phi'^{[k]}$ on $X = S^{[k]}$. □

Another, maybe slightly less well-known construction is as follows, see [25]. Take S a complete intersection of three quadrics in $\mathbb{P}^5$. This is again a K3 surface. We can arrange for S to contain a rational normal cubic C and to contain no lines. Let H be a hyperplane section divisor, then $(H - C)^2 = 0$, so curves residual to C in a hyperplane section are of square zero and genus one, this gives a fibration of S, and C induces a multisection of degree 5. Lift this fibration to $S^{[2]}$ as before, call it ϕ. Remark that a point of $S^{[2]}$ is either a pair of distinct points of S or a point together with a tangent direction. Through each pair of points of S, or a point with a tangent direction, there is a unique line l, and it does not intersect S at any extra points (indeed, since S is an intersection of quadrics, the line would be contained in S otherwise). The quadrics containing S are parameterized by a projective plane $\mathbb{P}(V)$, and those among them which contain l, by a line in this plane, so we have a natural map from $S^{[2]}$ to the dual projective plane $\mathbb{P}(V^*)$, and a fiber is naturally identified to the set of lines contained in the intersection of two quadrics, known to be an abelian surface generically (when this intersection is smooth), see e.g. [46]. So we have another fibration called ϕ'.

Proposition A.3. *The curve $C^{[2]}$ viewed as a subvariety of $S^{[2]}$ induces a (possibly rational[6]) section of ϕ'.*

Proof. Indeed the intersection of two sufficiently general quadrics from $\mathbb{P}(V)$ and the projective space $\mathbb{P}^3$ generated by C is a union of C and one of its secant lines l, so that $C \cap l$ gives a distinguished point in each fiber of ϕ'. □

[6] By a rational section we mean a section defined over a dense open subset of the base.

Note, though, that the first fibration does not have a natural section arizing from this geometric construction. However one can impose a section, e.g. by requiring S to contain another rational normal cubic C' intersecting C at two points: then C' induces a section of ϕ and $C^{[2]}$ induces a section of $\phi^{[2]}$. One may remark that there is also an abstract existence result, which follows from the Torelli theorem for K3 surfaces and Nikulin's results on lattice embedding: for any nondegenerate even lattice Λ of signature $(1, \rho - 1)$, $\rho \leq 10$, there exists a K3 surface with Neron-Severi group Λ (see [38]).

Once two fibrations are constructed, the existence of parabolic automorphisms preserving each one can be deduced in the same way as in Proposition 1: indeed the description of the Neron-Severi group and of the MBM classes on $S^{[2]}$ is well-known (the latter are the classes of square -2 and those classes of square -10 which have even pairing with all other classes in $H^2(S^{[2]}, \mathbb{Z})$, see [26] for statements, [5] for an easy proof). We check the existence of a parabolic automorphism preserving ϕ on S, and of a parabolic automorphism preserving ϕ' on $S^{[2]}$. The details are left to the reader.

As a final remark, let us mention that many more examples can be constructed in an "abstract" way, by choosing a convenient lattice Λ of low rank (but at least three), so that there is an IHS manifold of one of the four known deformation types (e.g. deformation equivalent to the Hilbert scheme of a K3 surface) X with Neron-Severi lattice Λ. As the description of the MBM classes is available, by choosing the lattice carefully it is possible to arrange for two Beauville-Bogomolov isotropic nef classes with no, or few, orthogonal MBM classes. Since the Lagrangian conjecture is verified, this gives two lagrangian fibrations ϕ, ϕ', and by Hodge-theoretic Torelli, two groups of parabolic automorphisms P resp. P' preserving each. One then may study the locus of points with finite orbit with respect to the group generated by some $f \in P$ and $f' \in P'$.

Note also that IHS manifolds with two transversal lagrangian fibrations have been constructed in [28]; as the ambient space there has Picard rank two, there are no automorphisms which are interesting for us, but a suitable modification of the construction could certainly yield some. The construction of [28] is entirely based on the Torelli theorem, so it is not explicit.

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E. Amerik, Université Paris-Saclay, Laboratoire de Mathématiques d'Orsay, France and National Research University Higher School of Economics, Laboratory of Algebraic Geometry and its Applications, Russia
E-mail address: ekaterina.amerik@gmail.com

P. Dolce, Westlake University, Institute for theoretical sciences, China
E-mail address: dolce@westlake.edu.cn

F. Tropeano, Università degli Studi Roma Tre, Italy
E-mail address: francesco.tropeano@uniroma3.it