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General Section

Adelic geometry on arithmetic surfaces II: Completed adeles and idelic Arakelov intersection theory

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ABSTRACT

We work with completed adelic structures on an arithmetic surface and justify that the construction under consideration is compatible with Arakelov geometry. The ring of completed adeles is algebraically and topologically self-dual and fundamental adelic subspaces are self orthogonal with respect to a natural differential pairing. We show that the Arakelov intersection pairing can be lifted to an idelic intersection pairing.

Contents

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0. Introduction

0.1. Background

Adelic theory for global fields was introduced for the first time by Chevalley in the 1930's as a tool for studying the completions of a number field with respect to all possible absolute values at the same time. This is a great expression of "local-to-global" principles as well as an example of geometric approaches to number theory which have proven to be very powerful. One of the principal applications of adelic theory for number fields was published in John Tate's thesis [23] which presented a proof of meromorphic continuation and functional equation of ζ functions of number fields in clearer and more compact way than the proof given before by Hecke. When C is a curve over a perfect field, one can also define the adelic ring $\mathbf{A}_C$ associated to C as the restricted product of the complete discrete valuation fields K_c for any closed point $c \in C$ with respect to their valuation rings $\mathcal{O}_c$. It is possible to obtain a very elegant proof of the Riemann-Roch theorem for curves by using adeles (see [24, 3.] for a sketch of a proof).

Adelic approach has been generalized for higher dimensions by Beilinson in [3] where he defined adelic structures as functors on the category of quasi-coherent sheaves. An explicit theory of 2-dimensional adelic cohomology and dualities for algebraic surfaces was outlined in [20], where hope for proving adelic Riemann-Roch theorem for a surface over a finite field was expressed. However, the explicit adelic structures introduced in [20] are not equivalent to Beilinson's, since [20] worked with objects that now are called rational adeles. The gap on the definitions was partially fixed in [21], but a complete account of 2-dimensional explicit adelic theory was given by Fesenko in [10], where he also proved an adelic Riemann-Roch theorem for an algebraic surface over a perfect field by using properties of adelic cohomology. In particular, Fesenko showed that the function field of an algebraic surface X can be seen as a discrete subspace inside the ring of 2-dimensional adeles attached to X. Such a result generalizes the classical result of [23] which shows that a global field is a discrete object inside the ring of adeles.

The non-cohomological part of (explicit) adelic theory for algebraic surfaces can be summarized in the following way: fix a nonsingular algebraic surface $(X, \mathscr{O}_X)$ over a perfect field k, then to each "flag" $x \in y$ made of a closed point x inside an integral curve $y \subset X$ we can associate the ring $K_{x,y}$ which will be a 2-dimensional local field if y is nonsingular at x, or a finite product of 2-dimensional local fields if we have a singularity. Note how the geometric dimension of X matches the "dimension" of the ring $K_{x,y}$, and this happens roughly speaking because for a flag $x \in y$ (assuming that x is a nonsingular point of y) we have two distinct levels of discrete valuations: there is the discrete valuation associated to the containment $x \in y$ and the discrete valuation associated to $y \subset X$. $K_{x,y}$ is obtained through a process of successive localizations and completions starting with $\mathscr{O}_{X,x}$ and by the symbol $\mathcal{O}_{x,y}$ we denote the product of valuation rings inside $K_{x,y}$. The step to the global theory is obtained by performing a "double restricted product" of the rings $K_{x,y}$: first over all points ranging on a fixed curve and then over all curves in X, in order to obtain the 2-dimensional adelic ring:

$$\mathbf{A}_X := \prod_{\substack{x \in y \\ y \subset X}}'' K_{x,y} \subset \prod_{\substack{x \in y \\ y \subset X}} K_{x,y} \,.$$

The topology on $K_{x,y}$ can be defined canonically thanks to the construction by completions and localizations, and by starting with the standard $\mathfrak{m}_x$-adic topology on $\mathscr{O}_{X,x}$. The topology on $\mathbf{A}_X$ is obtained after a process of several inductive projective limits by starting from the local topologies on all $K_{x,y}$. In [10] it is shown that $\mathbf{A}_X$ is self-dual as k-vector space. For 2-dimensional local fields with the same structure of $K_{x,y}$ there is a well known theory of differential forms and residues (e.g. [25]); one can globalize the constructions in order to obtain a k-character $\xi^\omega : \mathbf{A}_X \to k$ associated to a rational differential form $\omega \in \Omega^1_{k(X)|k}$ and the differential pairing:

$$\begin{aligned} d_\omega : \mathbf{A}_X \times \mathbf{A}_X &\to k \\ (\alpha, \beta) &\mapsto \xi^\omega(\alpha\beta) \,. \end{aligned}$$

Fesenko in [10] proves that the subspace $\mathbf{A}_X / k(X)^\perp$ is a linearly compact k-vector space (orthogonal spaces are calculated with respect to d_ω) and the function field $k(X)$ is discrete in $\mathbf{A}_X$. It is possible to define some important subspaces of $\mathbf{A}_X$ denoted as: $k(X) = A_0$, A_1, A_2, A_{01}, A_{02}, A_{12}, $A_{012} = \mathbf{A}_X$ which generate an idelic complex assuming the following form:

$$\mathcal{A}_X^\times : \qquad A_0^\times \oplus A_1^\times \oplus A_2^\times \xrightarrow{d^0_\times} A_{01}^\times \oplus A_{02}^\times \oplus A_{12}^\times \xrightarrow{d^1_\times} A_{012}^\times$$

It can be shown that the space $\ker(d^1_\times)$ is a generalization of the group $\mathrm{Div}(X)$ since there is a surjective map $\ker(d^1_\times) \to \mathrm{Div}(X)$ and the intersection pairing on $\mathrm{Div}(X)$ can be extended to a pairing on $\ker(d^1_\times)$ (cf. [7, 3.]).

The main aim of our work is to obtain a two-dimensional adelic theory, for arithmetic surfaces i.e. objects of the form $\varphi : X \to \operatorname{Spec} O_K$ where K is a number field. The problem is motivated by Fesenko's "analysis on arithmetic schemes programme". The programme develops a two-dimensional generalization of Tate's thesis, i.e. two-dimensional measure, integration and Fourier analysis. Fesenko's work reveals relationships between geometry and analysis not visible without adelic tools (see also [6] for an alternative presentation).

In [16] and [18] Morrow, develops an explicit approach to residues and dualizing sheaves of arithmetic surfaces. In particular he defines the residue map for 2-dimensional local fields arising from an arithmetic surface and he formulates and proves reciprocity laws around a point and along a curve of an arithmetic surface. To have a reciprocity law along a horizontal curve, he completes horizontal curves with points at infinity, i.e. real or complex embeddings of the function field of the horizontal curve.

0.2. Results in this paper

At the center of our considerations there is an adelic object for an arithmetic surface $\varphi : X \to B = \operatorname{Spec} O_K$. One expects that one has to take into account (archimedean) "data at infinity" of the arithmetic surface. Such an adelic space completed by data at infinity was proposed for the first time in [9]. In section 2 we present a simpler and slightly different version of it. Already at the level of local theory, adelic geometry for arithmetic surfaces is quite interesting, in fact the rings $K_{x,y}$ can be equal characteristic or mixed characteristic 2-dimensional local fields depending whether y is horizontal or vertical. Over each point at infinity $\sigma \in B_\infty$, i.e. an embedding $\sigma : K \to \mathbb{C}$, we obtain, by a base change, a Riemann surface X_σ that can be thought as a fiber at infinity. The completed adelic ring $\mathbf{A}_{\widehat{X}}$ will then contain the one dimensional adelic rings $\mathbf{A}_{X_\sigma}$ relative to the fibers at infinity X_σ, but counted twice:

$$\mathbf{A}_{\widehat{X}} = \mathbf{A}_X \oplus \prod_{\sigma \in B_\infty} \left(\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma}\right).$$

The arithmetic counterparts $A_{\widehat{*}}$ of the fundamental subspaces A_* are also defined. There is a specific geometric reason that suggests why we should count adeles at infinity twice, and it involves the interpretation horizontal curves on $\widehat{X}$ in terms of Arakelov geometry i.e. we have to consider their "intersection" with fibers at infinity.

By slightly generalizing the local theory of residues for two dimensional local fields developed in [16], in section 3 we define a global adelic residue

$$\xi^\omega : \mathbf{A}_{\widehat{X}} \to \mathbb{T}$$

(ω is a fixed nonzero rational differential form and $\mathbb{T}$ is the unit complex circle) and we show that ξ^ω is sequentially continuous.

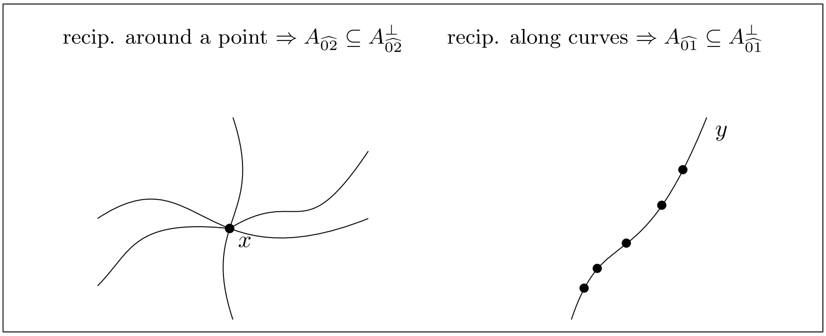

Fig. 1. The sum of local two-dimensional residues is zero when a point is fixed and curves passing through it vary. The sum of local two-dimensional residues is zero when a curve is fixed and the points sitting on it vary.

Section 4 is entirely dedicated to the proof of the self-duality of $\mathbf{A}_{\widehat{X}}$ as topological additive group. In particular we show that $\mathbf{A}_{\widehat{X}} \cong \widehat{\mathbf{A}_{\widehat{X}}}$ as topological groups and moreover that there is a character $\psi : \mathbf{A}_{\widehat{X}} \to \mathbb{T}$ such that any other character of $\mathbf{A}_{\widehat{X}}$ is of the form $\psi(a\cdot)$ for $a \in \mathbf{A}_{\widehat{X}}$.

We define the arithmetic differential pairing

$$\begin{aligned} d_\omega \colon \mathbf{A}_{\widehat{X}} \times \mathbf{A}_{\widehat{X}} &\to \mathbb{T} \\ (\alpha, \beta) &\mapsto \xi^{\omega}(\alpha\beta)\,. \end{aligned}$$

We improve the reciprocity laws proved in [18] by giving a set of "completed" reciprocity laws, i.e. taking into account all flags coming from points at infinity. We show that both $A_{\widehat{01}}$ and $A_{\widehat{02}}$ (adelic subspaces corresponding to curves and points respectively) are self-orthogonal with respect to d_ω i.e. $A_{\widehat{01}} = A_{\widehat{01}}^{\perp}$ and $A_{\widehat{02}} = A_{\widehat{02}}^{\perp}$. The inclusions $A_{\widehat{01}} \subseteq A_{\widehat{01}}^{\perp}$ and $A_{\widehat{02}} \subseteq A_{\widehat{02}}^{\perp}$ are a direct consequence of the completed reciprocity laws, thus the self-orthogonality of $A_{\widehat{01}}$ and $A_{\widehat{02}}$ can be interpreted as "strong reciprocity laws" for arithmetic surfaces. The "strong reciprocity laws" for surfaces over a perfect field were proved in [10]. (See Fig. 1.)

The problems of finding proofs of the discreteness of the function field $K(X)$ inside $\mathbf{A}_{\widehat{X}}$ and of the compactness of the quotient $\mathbf{A}_{\widehat{X}}/K(X)^{\perp}$ are still open, but we plan to publish a solution in a forthcoming paper. Finally, in analogy with the case of algebraic surfaces we show that the Arakelov intersection pairing can be lifted to the idelic group $\mathbf{A}_{\widehat{X}}^{\times}$. The schematic part of the lifting was already proved in [8], so here we solve the problem of the data carried by Green functions on fibers at infinity. It is worth remembering that Arakelov theory is the only known theory that provides consistent intersection theory on arithmetic surfaces, therefore we would expect that a theory of adeles on arithmetic surfaces should resonate with Arakelov geometry.

The text contains also two appendices which are indispensable for the understanding and moreover prerequisites for this paper are [8] and a basic knowledge of the theory of higher local fields (e.g. [17]).

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1. Preliminaries

1.1. Basic notions

General notations All rings are considered commutative and unitary. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M be an A-module, then we put $M^{\text{sep}} := M/\cap_{j\geq 1} \mathfrak{m}^j M$. When we pick a point x in a scheme X we generally mean a *closed point* if not otherwise specified, also all sums $\sum_{x\in X}$ are meant to be "over all closed points of X". The cardinality of a set T is denoted as $\#(T)$. If F is a discrete valuation field, then $\overline{F}$ doesn't denote the algebraic closure but its residue field. In particular if $a \in \mathcal{O}_F$ then $\overline{a}$ is the image of a in $\overline{F}$. For morphisms of schemes $f : X \to S$, the schematic preimage of $s \in S$ is X_s. Sheaves are denoted with the "mathscr" late χ font; in particular the structure sheaf of a scheme X is $\mathscr{O}_X$ (note the difference with the font $\mathcal{O}$). With the symbol $\mathbb{T}$ we denote the unit circle in the complex plane. The superscript $\widehat{\ }$ is used several times in this paper to denote completely different objects: the dual of a topological group, the completion of a local ring or a "completed structure" in the framework of Arakelov geometry. This superposition of notation is harmless because the specific meaning of $\widehat{\ }$ will be clear from the context.

Topological groups If not otherwise specified we assume that any topological group is abelian and Hausdorff. The dual of a topological group G is the group of (unitary) characters:

$$\widehat{G} := \text{Hom}_{\text{cont}}(G, \mathbb{T}).$$

It is a topological group endowed with the compact-to-open topology. Moreover for a compact subset $C \subset G$ and an open $U \subset \mathbb{T}$ neighborhood of 1 we denote

$$\mathcal{W}(C, U) = \left\{ \chi \in \widehat{G} \colon \chi(C) \subset U \right\} \subset \widehat{G}\,.$$

The sets of the type $\mathcal{W}(C, U)$ form an open base at 1 for the compact-to-open topology in $\widehat{G}$.

If G is algebraically and topologically isomorphic to $\widehat{G}$, then we say that G *is self-dual*. If G is also a ST ring (here ST means semi-topological, see appendix A for details) and $\xi \in \widehat{G}$ is a nontrivial character, then for any $a \in G$ the map

$$\begin{aligned}\xi_a : G &\to \mathbb{T}\\ x &\mapsto \xi(ax)\end{aligned}$$

is a character. If the map

$$\begin{aligned}\Theta_\xi : G &\to \widehat{G}\\ a &\mapsto \xi_a\end{aligned}$$

is an algebraic and topological isomorphism for any $a \in G$, we say that ξ is a *standard character*. For any subsets $S \subseteq G$ and $R \subseteq \widehat{G}$ we put:

$$S^\perp := \{\chi \in \widehat{G} \colon \chi(S) = 1\} \subseteq \widehat{G}\,,$$
$$R^\perp := \{g \in G \colon \chi(g) = 1,\ \forall \chi \in R\} \subseteq G\,.$$

If H is a subgroup of G, we say that H is *dually closed* if for every element $g \in G \setminus H$, there is a character $\psi \in H^\perp$ such that $\psi(g) \neq 1$.

We will often use the following simple general result:

Proposition 1.1. *Let G be a topological group such that $G = \varinjlim_{i\in\mathbb{Z}} H_i$ where $H_i \subset G$ is a subgroup and $H_i \supset H_{i+1}$ for any $i \in \mathbb{Z}$. Then any compact subset $C \subset G$ is contained in some H_i.*

Proof. Clearly $G = \bigcup_i H_i$. Assume that the claim is false, so we can construct a sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ in G such that $x_i \in C \cap (H_i \setminus H_{i+1})$. Consider now the index $n = -i$ and put $A = \{x_n\}_{n\geq 0}$. If $B \subseteq A$, then $B \cap H_n$ is finite for each n, so since points are closed in H_n, $B \cap H_n$ is closed in H_n. This means that B is closed in G. In particular, A is a closed subset of G, and every subset of A is closed so it has the discrete topology. But a closed subset of a compact space is compact, and a compact discrete space must be finite. This is a contradiction with the construction of A. □

1.2. Geometric setting

Let's fix the main objects and notations that we will use throughout the whole paper. Some of the material contained in this section can be found with more details in [8]. In particular we assume that the reader is familiar with the notion of 2-dimensional local field. Moreover, topological aspects of this section rely on appendix A.

Let K be a number field with ring of integers O_K. Fix the arithmetic surface $\varphi : X \to B = \operatorname{Spec} O_K$ which is a B-scheme with the following properties:

- X is two dimensional, integral, and regular. The generic point of X is η and the function field of X is denoted by $K(X)$.
- φ is proper and flat.

- The generic fiber, denoted by X_K, is a geometrically integral, smooth, projective curve over K. The generic point of B is denoted by ξ.

It is well known that φ is a projective morphism, so in particular also X is projective (see [13, Theorem 8.3.16]). Let's recall a useful result which characterizes all points of dimension 1 on X:

Proposition 1.2. *If x is a closed point of the curve X_K, then $\overline{\{x\}}$ is a horizontal (prime) divisor in X. Vice versa if D is a prime divisor on X, then either $D \subseteq X_b$ for a closed point $b \in B$ or $D = \overline{\{x\}}$ where x is a closed point of X_K.*

Proof. See for example [13, Proposition 8.3.4]. □

Let B_∞ be the set of field embeddings $\sigma\colon K \hookrightarrow \mathbb{C}$ up to conjugation, then $\#B_\infty \leq [K : \mathbb{Q}]$ and the *completion of* B is the set $\widehat{B} := B \cup B_\infty$. For any point (i.e. nonzero prime ideal) $b = \mathfrak{p} \in B$ we put:

- $\mathcal{O}_b := \widehat{\mathscr{O}_{B,b}}$. It is a complete DVR.
- $K_b := \operatorname{Frac} \mathcal{O}_b$. It is a local field with finite residue field. The valuation is denoted by v_b.

From now on, we *always* fix a set of representatives in B_∞. Therefore B_∞ is simply a finite set of embeddings viewed as points at infinity of B. For the non-archimedean place associated to $b = \mathfrak{p} \in B$, on K we choose the absolute value

$$|\cdot|_b := \mathfrak{N}(\mathfrak{p})^{-v_b(\cdot)}$$

where $\mathfrak{N}(\mathfrak{p})$ is the cardinality of $O_K/\mathfrak{p}$. Moreover:

- For any real embedding $\tau : K \to \mathbb{R}$ we consider the absolute value:

$$|\cdot|_\tau := |\tau(\cdot)|$$

  where on the right hand side we mean the usual absolute value on $\mathbb{R}$. In this case we define the real valuation associated to τ as

$$v_\tau(\cdot) := -\log|\cdot|_\tau$$

- For any couple of conjugate embeddings $\sigma, \overline{\sigma} : K \to \mathbb{C}$ we choose:

$$|\cdot|_\sigma := |\sigma(\cdot)|$$

where on the right hand side we have the usual absolute value on $\mathbb{C}$.[1] Note that $|\cdot|_\sigma$ doesn't depend on the choice between σ and $\overline{\sigma}$, since they give the same absolute value. The associated real valuation is

$$v_\sigma(\cdot) := -\log|\cdot|_\sigma \,.$$

For $\sigma \in B_\infty$, K_σ is the completion of K with respect to $|\cdot|_\sigma$, thus $K_\sigma = \mathbb{C}$ or $K_\sigma = \mathbb{R}$. Furthermore, let's introduce a constant, associated to each $\sigma \in B_\infty$:

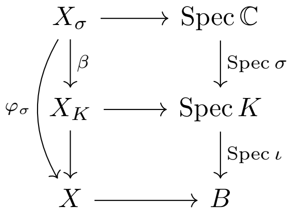
$$\epsilon_\sigma := \begin{cases} 1 & \text{if } \sigma \text{ is real} \\ 2 & \text{if } \sigma \text{ is complex}\,. \end{cases}$$

The adelic ring of $\widehat{B}$ (or equivalently of the number field K) is denoted by $\mathbf{A}_{\widehat{B}}$ or more classically also by $\mathbf{A}_K$, whereas $\mathbf{A}_B := \mathbf{A}_{\widehat{B}} \cap \prod_{b\in B} K_b$ is the ring of finite adeles. Another notation for the ring of finite adeles is $\mathbf{A}_K^f$. For any $\sigma \in B_\infty$ consider the base change diagram:

$$\begin{array}{ccc} X_\sigma := X \times_B \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{C} \\ \downarrow{\scriptstyle \varphi_\sigma} & & \downarrow{\scriptstyle \operatorname{Spec}\sigma} \\ X & \xrightarrow{\ \varphi\ } & B\,. \end{array} \tag{1}$$

By the properties of the fibered product, it turns out that $X_\sigma \to \operatorname{Spec}\mathbb{C}$ is a complex integral (integrality is a consequence of the geometrical integrality of X_K), regular projective curve. We denote the function field of X_σ by the symbol $\mathbb{C}(X_\sigma)$.

Remark 1.3. Diagram (1) arises from the following rather obvious commutative diagram:



where $\iota : O_K \hookrightarrow K$ is the natural embedding and the map β is surjective. In other words φ_σ maps surjectively X_σ onto the curve X_K. Since the morphisms ι and σ are both flat and flatness is preserved after base change, we can conclude that φ_σ is flat.

[1] Many authors in this case take the square of the complex absolute value to keep track of the fact that point at infinity induced by $|\cdot|_\sigma$ is "complex", so roughly speaking "of order two". We will fix this by using the coefficient 2 when necessary.

With the notation $\widehat{X}$, we define the "completed surface"

$$\widehat{X} := X \cup \bigcup_{\sigma \in B_\sigma} X_\sigma .$$

A curve Y on X will always be an integral curve and its unique generic point will be denoted with the letter y. For simplicity we will often identify Y with its generic point y, which means that by an abuse of language and notation we will use sentences like "let $y \subset X$ be a curve on X...". A *flag* on X is a couple (x, y) where x is a closed point sitting on a curve $y \subset X$, it will be denoted simply as $x \in y$.

Definition 1.4. Fix a closed point $x \in X$, then:

- $\mathcal{O}_x := \widehat{\mathscr{O}_{X,x}}$. It is a Noetherian, complete, regular, local, domain of dimension 2 with maximal ideal $\widehat{\mathfrak{m}_x}$.
- $K'_x := \operatorname{Frac} \mathcal{O}_x$.
- $K_x := K(X)\mathcal{O}_x \subseteq K'_x$.

For a curve $y \subset X$ we put:

- $\mathcal{O}_y := \widehat{\mathscr{O}_{X,y}}$. It is a complete DVR with maximal ideal $\widehat{\mathfrak{m}_y}$.
- $K_y := \operatorname{Frac} \mathcal{O}_y$. It is a complete discrete valuation field with valuation ring $\mathcal{O}_y$. The valuation is denoted by v_y.

For a flag $x \in y \subset X$, we have a surjective local homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x}$, with kernel $\mathfrak{p}_{y,x}$, induced by the closed embedding $y \subset X$ (note that $\mathfrak{p}_{y,x}$ is a prime ideal of height 1). The inclusion $\mathscr{O}_{X,x} \subset \mathcal{O}_x$ induces a morphism of schemes $\varphi\colon \operatorname{Spec} \mathcal{O}_x \to \operatorname{Spec} \mathscr{O}_{X,x}$ and we define the *local branches of y at x* as the elements of the set

$$y(x) := \varphi^{-1}(\mathfrak{p}_{y,x}) = \{\mathfrak{z} \in \operatorname{Spec} \mathcal{O}_x \colon \mathfrak{z} \cap \mathscr{O}_{X,x} = \mathfrak{p}_{y,x}\} .$$

If $y(x)$ contains only an element, we say that y is unbranched at x. Fix a flag $x \in y \subset X$ with $\mathfrak{z} \in y(x)$, then we have the 2-dimensional local field

$$K_{x,\mathfrak{z}} := \operatorname{Frac}\left(\widehat{(\mathcal{O}_x)_{\mathfrak{z}}}\right)$$

explicitly obtained in the following way: we localize $\mathcal{O}_x$ at the prime ideal $\mathfrak{z}$, complete it at its maximal ideal and finally we take the fraction field. The ring of integers of $K_{x,\mathfrak{z}}$ is denoted by $\mathcal{O}_{x,\mathfrak{z}} := \mathcal{O}_{K_{x,\mathfrak{z}}} = \widehat{(\mathcal{O}_x)_{\mathfrak{z}}}$. All the needed material about higher local fields is contained in [8, 1.1], whereas for a deeper study the reader can consult [11]; see also a more recent introduction in [17].

Definition 1.5. Let $x \in y \subset X$ be a flag and let $\mathfrak{z} \in y(x)$, then the first residue field of $K_{x,\mathfrak{z}}$ is $E_{x,\mathfrak{z}} := K^{(1)}_{x,\mathfrak{z}}$ and the second residue field is $k_{\mathfrak{z}}(x) := K^{(2)}_{x,\mathfrak{z}}$. The valuation on $K_{x,\mathfrak{z}}$ is $v_{x,\mathfrak{z}}$ and the valuation on $E_{x,\mathfrak{z}}$ is $v^{(1)}_{x,\mathfrak{z}}$; whereas $\mathcal{O}^{(2)}_{K_{x,\mathfrak{z}}} := \{a \in \mathcal{O}_{x,\mathfrak{z}} : \overline{a} \in \mathcal{O}_{E_{x,\mathfrak{z}}}\}$.

$$\begin{array}{ccccc}
K_{x,\mathfrak{z}} & \supset & \mathcal{O}_{x,\mathfrak{z}} := \mathcal{O}_{K_{x,\mathfrak{z}}} & \supset & \mathcal{O}^{(2)}_{K_{x,\mathfrak{z}}} \\
 & & \downarrow & & \downarrow \\
 & & E_{x,\mathfrak{z}} := K^{(1)}_{x,\mathfrak{z}} & \supset & \mathcal{O}_{E_{x,\mathfrak{z}}} \\
 & & & & \downarrow \\
 & & & & k_{\mathfrak{z}}(x) := K^{(2)}_{x,\mathfrak{z}}
\end{array}$$

Moreover we put:

$$K_{x,y} := \prod_{\mathfrak{z} \in y(x)} K_{x,\mathfrak{z}}\,, \quad \mathcal{O}_{x,y} := \prod_{\mathfrak{z} \in y(x)} \mathcal{O}_{x,\mathfrak{z}}\,,$$
$$E_{x,y} := \prod_{\mathfrak{z} \in y(x)} E_{x,\mathfrak{z}}\,, \quad k_y(x) := \prod_{\mathfrak{z} \in y(x)} k_{\mathfrak{z}}(x)\,.$$

Let's endow $\mathscr{O}_{X,x}$ with the $\mathfrak{m}_x$-adic topology with respect to its maximal ideal, then $K_{x,\mathfrak{z}}$ can be endowed with a canonical topology by using the following steps explained in appendix A.2:

$$\mathscr{O}_{X,x} \overset{(C)}{\rightsquigarrow} \mathcal{O}_x = \widehat{\mathscr{O}_{X,x}} \overset{(L)}{\rightsquigarrow} (\mathcal{O}_x)_{\mathfrak{z}} \overset{(C)}{\rightsquigarrow} \widehat{(\mathcal{O}_x)_{\mathfrak{z}}} \overset{(L)}{\rightsquigarrow} K_{x,\mathfrak{z}} = \mathrm{Frac}\left(\widehat{(\mathcal{O}_x)_{\mathfrak{z}}}\right). \tag{2}$$

Then $K_{x,y}$ is endowed with the product topology and it is a ST ring (see appendix A for an introduction to semi-topological structures). Here it is very important to point out that $K_{x,y}$ is not a topological ring, since it turns out that the multiplication is not continuous as function of two variables.

Remark 1.6. This is one of the several ways to topologise $K_{x,y}$; see for example [5, 1.] for a survey. It is not the most explicit topology for $K_{x,y}$, but it is independent from the choice of the uniformizing parameter since it is obtained by a general process of localizations and completions.

If y is a horizontal curve then $K_{x,\mathfrak{z}}$ is of equal characteristic and isomorphic to $E_{x,\mathfrak{z}}((t))$ where $E_{x,\mathfrak{z}}$ is a finite extension of $\mathbb{Q}_p$ and t is (the image of) a uniformizing parameter. If y is a vertical curve then $K_{x,\mathfrak{z}}$ is of mixed characteristic and isomorphic to a finite extension of $K_p\{\{t\}\}$ where K_p is a finite extension of $\mathbb{Q}_p$ (see [8, example 1.7] for the definition of $K_p\{\{t\}\}$). In this case t it is not (the image of) a uniformizing parameter, but it is (the image of) a uniformizing parameter for $E_{x,\mathfrak{z}} \cong \overline{K_p}((t))$. It is always possible to choose a uniformizing parameter $t = t_y$ of K_y to be also the uniformizing parameter of $K_{x,\mathfrak{z}}$ for all $x \in y$, this will be our canonical choice if not otherwise specified.

If $\varphi(x) = b$ we have an embedding $K_b \hookrightarrow K_{x,\mathfrak{z}}$, and we say that $K_{x,\mathfrak{z}}$ is an arithmetic 2-dimensional local field over K_b. The module of differential forms relative to x and $\mathfrak{z} \in y(x)$ is the $K_{x,\mathfrak{z}}$-vector space:

$$\Omega^1_{x,\mathfrak{z}} := \left(\Omega^1_{\mathcal{O}_{x,\mathfrak{z}}|\mathcal{O}_b}\right)^{\text{sep}} \otimes_{\mathcal{O}_{x,\mathfrak{z}}} K_{x,\mathfrak{z}}\,,$$

where $\Omega^1_{\mathcal{O}_{x,\mathfrak{z}}|\mathcal{O}_b}$ is the usual module of Kähler differential forms and the operator "sep" was defined at the end of section 1.1 in the "General notations" paragraph. Then, $\Omega^1_{x,\mathfrak{z}}$ is endowed with the vector space topology over $K_{x,\mathfrak{z}}$. In [16] and [18] it is defined the residue map:

$$\text{res}_{x,\mathfrak{z}} : \Omega^1_{x,\mathfrak{z}} \to K_b$$

with the following properties:

- It is K_b-linear.
- It is continuous (this is shown in [18, Lemma 2.8, Remark 2.9]).

A more detailed description of $\Omega^1_{x,\mathfrak{z}}$ and $\text{res}_{x,\mathfrak{z}}$ will be given in section 3.

The global adelic theory for the projective scheme X is described in [8, 1.2]. We obtain the adelic ring $\mathbf{A}_X$ as a "double restricted product" of the rings $K_{x,y}$ performed first over closed points ranging on curves, and then over all curves in X. Fix any curve $y \subset X$ and denote by $\mathfrak{J}_{x,y}$ the Jacobson radical of $\mathcal{O}_{x,y}$; we put

$$\mathbb{A}^{(0)}_y := \left\{ \begin{array}{l} (\alpha_{x,y})_{x\in y} \in \prod_{x\in y} \mathcal{O}_{x,y} \colon \forall s > 0,\ \alpha_{x,y} \in \mathcal{O}_x + \mathfrak{J}^s_{x,y} \\ \text{for all but finitely many } x \in y \end{array} \right\} \subset \prod_{x\in y} \mathcal{O}_{x,y}\,.$$

Then for any $r \in \mathbb{Z}$ and for any choice of uniformizing parameter t_y

$$\mathbb{A}^{(r)}_y := \widehat{\mathfrak{m}}^r_y \mathbb{A}^{(0)}_y = t^r_y \mathbb{A}^{(0)}_y \subset \prod_{x\in y} K_{x,y}\,.$$

Clearly $\mathbb{A}^{(r)}_y \supseteq \mathbb{A}^{(r+1)}_y$ and $\bigcap_{r\in\mathbb{Z}} \mathbb{A}^{(r)}_y = 0$; moreover we define

$$\mathbb{A}_y := \bigcup_{r\in\mathbb{Z}} \mathbb{A}^{(r)}_y\,.$$

Definition 1.7. The ring of adeles of X is

$$\mathbf{A}_X := \left\{ (\beta_y)_{y\subset X} \in \prod_{y\subset X} \mathbb{A}_y : \beta_y \in \mathbb{A}_y^{(0)} \text{ for all but finitely many } y \right\} \subset \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y} .$$

Finally we recall the definitions of some important subspaces of $\mathbf{A}_X$. Consider the following diagonal embeddings:

$$K_x \subset \prod_{y\ni x} K_{x,y}, \quad K_y \subset \prod_{x\in y} K_{x,y} ,$$

so we can put:

$$\prod_{x\in X} K_x \subset \prod_{\substack{x\in y\\ y\subset X}} K_{x,y}, \quad \prod_{y\subset X} K_y \subset \prod_{\substack{x\in y\\ y\subset X}} K_{x,y} ,$$

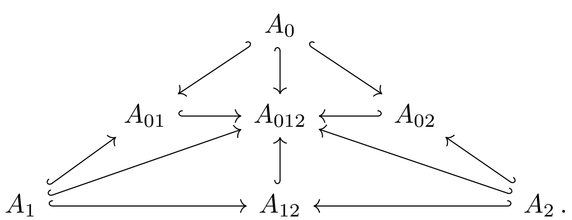
then we define

$$A_{012} := \mathbf{A}_X ; \quad A_{12} := \mathbf{A}_X \cap \prod_{\substack{x\in y\\ y\subset X}} \mathcal{O}_{x,y} = \prod_{y\subset X} \mathbb{A}_y^{(0)};$$

$$A_{02} := \mathbf{A}_X \cap \prod_{x\in X} K_x ; \quad A_2 := \mathbf{A}_X \cap \prod_{x\in X} \mathcal{O}_x ; \quad A_{01} := \mathbf{A}_X \cap \prod_{y\subset X} K_y ;$$

$$A_1 := \mathbf{A}_X \cap \prod_{y\subset X} \mathcal{O}_y ; \quad A_0 := K(X) .$$

The subspaces satisfy a series of inclusion relations depicted in the following diagram:



When X is an algebraic surface over a perfect field k, the algebraic and topological properties of the subspaces A_* were studied in [10].

1.3. Topology on adelic structures

In this crucial subsection we explain how to put a topology on all adelic structures introduced so far. We point out that all categorical limit considered here are in the category linear topological groups (so linear direct/inverse limits). For more details see appendix A.

- For any $s > 0$ let's put:

$$\mathbb{A}_y^{(0)}\{s\} := \{(a_{x,y})_{x\in y} \in \prod_{x\in y} \mathcal{O}_{x,y} \colon a_{x,y} \in \mathcal{O}_x + \mathfrak{J}_{x,y}^s \text{ for all but fin. many } x \in y\}.$$

  Endow $\mathbb{A}_y^{(0)}\{s\}$ with the restricted product topology (i.e. linear direct limit).
- $\mathbb{A}_y^{(0)} = \bigcap_{s\geq 0} A_y^{(0)}\{s\}$, so we put on $\mathbb{A}_y^{(0)}$ the linear inverse limit topology.
- The topology is transferred from $\mathbb{A}_y^{(0)}$ to $\mathbb{A}_y^{(r)}$ for any $r \in \mathbb{Z}$, by the multiplication by t_y^r.
- Each $\mathbb{A}_y^{(r)}/\mathbb{A}_y^{(r+j)}$, for $j > 0$, is endowed with the quotient topology.
- We endow $\mathbb{A}_y = \varinjlim \mathbb{A}_y^{(r)} = \bigcup_r \mathbb{A}_y^{(r)}$ with the linear direct limit topology.
- $\mathbf{A}_X$ is the restricted product (seen as linear direct limit) of the topological groups $\mathbb{A}_y$ with respect to $\mathbb{A}_y^{(0)}$.

Since $\mathcal{O}_x + \mathfrak{J}_{x,y}$ surjects onto $E_{x,y}$, it is easy to see that the natural projection (which is continuous and open)

$$p_y : \mathbb{A}_y^{(0)} \to \mathbf{A}_{k(y)}^f$$
$$(a_{x,y})_{x\in y} \mapsto (\overline{a_{x,y}})_{x\in y}$$

induces an algebraic and topologic isomorphism between $\mathbb{A}_y^{(0)}/\mathbb{A}_y^{(1)}$ and the ring of the one dimensional finite adeles $\mathbf{A}_{k(y)}^f$. Consider the exact sequence:

$$0 \to \mathbb{A}_y^{(1)}/\mathbb{A}_y^{(2)} \to \mathbb{A}_y^{(0)}/\mathbb{A}_y^{(2)} \to \mathbb{A}_y^{(0)}/\mathbb{A}_y^{(1)} \to 0$$

Since $\mathbb{A}_y^{(1)}/\mathbb{A}_y^{(2)}$ and $\mathbb{A}_y^{(0)}/\mathbb{A}_y^{(1)}$ are locally compact and self-dual, then $\mathbb{A}_y^{(0)}/\mathbb{A}_y^{(2)}$ is locally compact. We conclude that for any $j > 0$ the quotient $\mathbb{A}_y^{(r)}/\mathbb{A}_y^{(r+j)}$ is a locally compact topological group (hence complete).

Proposition 1.8. *The following two fundamental topological properties hold:*

(i) $\mathbb{A}_y^{(r)}$ *is complete for any* $r \in \mathbb{Z}$ *(but in general is not locally compact).*
(ii) *For each open neighborhood* $U \subset \mathbb{A}_y^{(r)}$ *of* 0 *there is* $s > r$ *such that* $\mathbb{A}_y^{(s)} \subset U$.

Proof. (i) is true since $\mathcal{O}_{x,y}$ is complete and $\mathcal{O}_x + \mathfrak{J}^s_{x,y}$ is closed in $\mathcal{O}_{x,y}$. (ii) can be checked directly from the above definition of the topology. □

Proposition 1.9. *There is an algebraic and topological isomorphism*

$$\mathbb{A}_y^{(r)} \cong \varprojlim_{j>0} \mathbb{A}_y^{(r)}/\mathbb{A}_y^{(r+j)}$$

Proof. Thanks to Proposition 1.8, we can apply directly [4, III §7.3, Corollary 1]. □

In particular $\mathbb{A}_y^{(r)} \cong t_y^r \mathbb{A}_y^{(0)}[[t_y]]$ and any Laurent power series in $t_y^r \mathbb{A}_y^{(0)}[[t_y]]$ is a truly convergent series. The open subgroups of $\mathbb{A}_y$ that form a local basis at 0 can be described in the following way: fix a sequence $\{U_i\}_{i\in\mathbb{Z}}$ of open sets in $\mathbb{A}_y^{(0)}$ with the property that there exists $k \in \mathbb{Z}$ such that $U_i = \mathbb{A}_y^{(0)}$ for $i \geq k$. Then we consider the open set

$$\sum{}' U_i t_y^i := \left\{ \text{Laurent series } \sum a_j t_y^j \text{ such that } a_j \in U_j \right\} .$$

Each open neighborhood $U \subset \mathbb{A}_y$ of 0 contains some $\mathbb{A}_y^{(r)}$.

2. The ring of completed adeles $\mathbf{A}_{\widehat{X}}$ and its subspaces

We want to define adeles for arithmetic surfaces in a way that preserves the most fundamental properties of the adelic theory and is compatible with Arakelov geometry. In particular, we have to consider points at infinity of the base and, corresponding to them, infinite fibers. When we add a fiber at infinity X_σ to the picture, we have to take in account *all* possible flags on the completed surface $\widehat{X}$: a point p on a fiber at infinity X_σ originates a flag $p \in X_\sigma$, but it can be seen also as an "intersection point" between a completed horizontal curve $\overline{y}$ and X_σ. (See Fig. 2.)

Let y be a curve on X, if y is vertical then we put $\overline{y} = y$, if y is horizontal, then by $\overline{y}$ we mean:

$$\overline{y} = y \cup \bigcup_{\sigma \in B_\infty} y_\sigma$$

where

$$y_\sigma = \varphi_\sigma^*(y) \in \mathrm{Div}(X_\sigma) .$$

By simplicity we also put $y_\infty := \cup_{\sigma\in B_\infty} y_\sigma$, so we have the decomposition $\overline{y} = y \cup y_\infty$. Any point $p \in X_\sigma$ lies on a completed horizontal curve $\overline{y}$ because we have the map $\varphi_\sigma : X_\sigma \to X_K \subset X$ and points of the generic fiber X_K are in bijective correspondence with horizontal curves. From now on, a curve on $\widehat{X}$ will be always a completed curve $\overline{y}$, and a point $x \in \overline{y}$ can be also a point lying on some "part at infinity" y_σ (when y is horizontal), if not explicitly said otherwise.

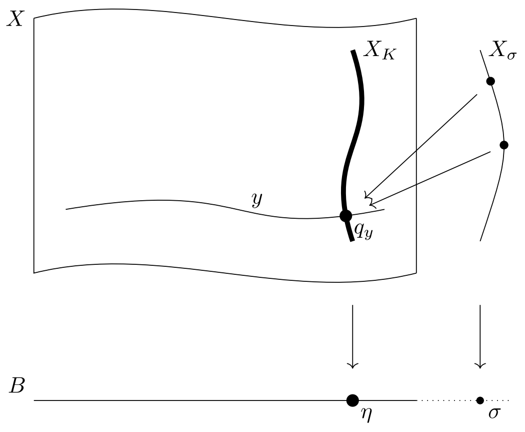


Fig. 2. A visual example where y_σ is made of two points (marked on the curve X_σ). The generic point of the curve y here is denoted by q_y.

The local data of the completed adelic ring will be the following ones:

- For any flag at infinity $p \in X_\sigma$ we put

$$K_{p,\sigma} := \mathrm{Frac}\left(\widehat{\mathcal{O}_{X_\sigma,p}}\right).$$

  In other words $K_{p,\sigma}$ is a local field isomorphic to $\mathbb{C}((t))$. The valuation ring of $K_{p,\sigma}$ is $\mathcal{O}_{p,\sigma} \cong \mathbb{C}[[t]]$ and $E_{p,\sigma} \cong \mathbb{C}$ is the residue field.
- If $p \in \overline{y}$ and $p \in y_\sigma$ for some $\sigma \in B_\infty$, we put

$$K_{p,\overline{y}} := K_{p,\sigma}, \quad \mathcal{O}_{p,\overline{y}} := \mathcal{O}_{p,\sigma}, \quad E_{p,\overline{y}} := E_{p,\sigma}.$$

- For any other point $x \in y$ we have:

$$K_{x,\overline{y}} := K_{x,y}, \quad \mathcal{O}_{x,\overline{y}} := \mathcal{O}_{x,y}, \quad E_{x,\overline{y}} := E_{x,y}, \quad k_{\overline{y}}(x) = k_y(x).$$

When p is a point at infinity we want to consider the fields $K_{p,\sigma}$ and $K_{p,\overline{y}}$ as 2-dimensional local fields, but if we use a completion/localization topology as described in equation (2), we obtain the usual one dimensional valuation topology. Therefore we fix some isomorphisms $K_{p,\sigma} \cong K_{p,\overline{y}} \cong \mathbb{C}((t))$ (parameterizations), we consider $\mathbb{C}$ with the standard topology given by its archimedean norm, and we endow $\mathbb{C}((t))$ with the ind/pro-topology (see appendix A.2). Then we carry such a topology on $K_{p,\sigma}$ and $K_{p,\overline{y}}$ through the parameterizations. The ind/pro-topology on $\mathbb{C}((t))$ is coarser than the 1-dimensional valuation topology. Let's emphasize the fact that in order to define a topology on the 2-dimension local fields at infinity we need to fix an isomorphism with $\mathbb{C}((t))$, so from now on we assume that such a choice has been made.

Remark 2.1. In [9] the construction of local fields at infinity is slightly different, indeed $K_{p,\sigma}$ is $\mathbb{R}((t))$ or $\mathbb{C}((t))$, depending whether σ is a real or complex embedding. This might seem a very natural choice, but in the framework of Arakelov geometry X_σ is always a Riemann surface, even if σ is real. We want to build deep link between Arakelov geometry and adelic geometry, therefore we prefer to put $K_{p,\sigma} \cong \mathbb{C}((t))$.

Remark 2.2. In the product $\prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} K_{x,\overline{y}}$ we find three different types of 2-dimensional local fields: $K_p((t))$, finite extensions of $K_p\{\{t\}\}$ and $\mathbb{C}((t))$.

We are going to define a new ring $\overline{\mathbf{A}}_X$ which will be a subspace of the big product $\prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} K_{x,\overline{y}}$. Let's first extend the spaces $\mathbb{A}_y^{(r)}$ for completed curves:

Definition 2.3. For any completed curve $\overline{y}$ let's put:

$$\mathbb{A}_{\overline{y}} := \mathbb{A}_y \oplus \prod_{p\in y_\infty} K_{p,\overline{y}}\,,$$

$$\mathbb{A}_{\overline{y}}^{(0)} := \mathbb{A}_y^{(0)} \oplus \prod_{p\in y_\infty} \mathcal{O}_{p,\overline{y}}\,,$$

$$\mathbb{A}_{\overline{y}}^{(r)} := \mathbb{A}_y^{(r)} \oplus \prod_{p\in y_\infty} \mathfrak{p}_{K_{p,\overline{y}}}^r \mathcal{O}_{p,\overline{y}}\,,$$

and endow them with the finite product topology.

Again each $\mathbb{A}_{\overline{y}}^{(r)}$ is closed in $\mathbb{A}_{\overline{y}}$ and the latter can be thought as a first restricted product performed on the completed curve $\overline{y}$. We can use the formal notation:

$$\mathbb{A}_{\overline{y}} = \prod_{x\in\overline{y}}{}' K_{x,\overline{y}}\,.$$

Let's assume by simplicity that y is a regular horizontal curve, then $K_{x,y} \cong E_{x,y}((t))$ where $E_{x,y}$ is a finite extension of $\mathbb{Q}_p$ and it is the completion of the field $k(y)$ with respect to the valuation induced by the inclusion $x \in y$. Moreover $y = \operatorname{Spec} O_L$ where L is a finite extension of K. In general if y is any horizontal curve admitting singular points, then $y = \operatorname{Spec} R$ where R is an order of L. For any curve y we put

$$\mathbf{A}_{\overline{y}} := \prod_{x\in y}{}' k(y)_x \oplus \prod_{q\in y_\infty} \mathbb{C}\,,$$

where the restricted product is with respect to the complete discrete valuation rings corresponding to the points $x \in y$. In other words $\mathbf{A}_{\overline{y}}$ is in general slightly bigger than the classical 1-dimensional adelic ring of $\overline{y}$. If the point $q \in y_\infty$ is present (recall that in the case of vertical curves there is no archimedean data) and corresponds to a real embedding σ, then the "q-component" of $\mathbf{A}_{\overline{y}}$ is $\mathbb{C}$ and not $\mathbb{R}$, i.e. we take $\mathbb{C}$ for all archimedean places.

The finite part of $\mathbf{A}_{\overline{y}}$, denoted by $\mathbf{A}_{\overline{y}}^{f}$, and the finite part of classical 1-dimensional adeles coincide. This of course descends from our choice of data at infinity (see Remark 2.1), but all adelic properties of $\mathbf{A}_{\overline{y}}$ are clearly the same of the one dimensional adeles. In particular all results of [23] hold for $\mathbf{A}_{\overline{y}}$.

Lemma 2.4. *Let y be a regular horizontal curve and let t be a uniformizing parameter of K_y. For any $r \in \mathbb{Z}$, $\mathbb{A}_y^{(r)}$ is equal to the following ring:*

$$\Xi_y^{(r)} := \left\{ (\alpha_{x,y})_{x\in y} \in \prod_{x\in y} K_{x,y} \colon \alpha_{x,y} \text{ satisfies the following conditions } (*) \text{ and } (**) \right\}$$

$(*)$ $\alpha_{x,y} \in t^r E_{x,y}[[t]]$.

$(**)$ *Assume that:*

$$\alpha_{x,y} = t^r \sum_{i\geq 0} \Gamma_{x,i} t^i \qquad \text{with } \Gamma_{x,i} \in E_{x,y} ,$$

then for any fixed index i the sequence $(\Gamma_{x,i})_{x\in y} \in \mathbf{A}_{\overline{y}}^{f}$. In other words for all but finitely many $x \in y$ we have that $\Gamma_{x,i} \in \mathcal{O}_{E_{x,y}}$.

Proof. *Inclusion* $\mathbb{A}_y^{(r)} \subseteq \Xi_y^{(r)}$. Let's start with $r = 0$, the general case will follow trivially. Consider an element $(\alpha_{x,y})_{x\in y}$, then clearly $(*)$ is true because $\mathcal{O}_{x,y} = E_{x,y}[[t]]$. Suppose that $\alpha_{x,y} = \sum_{i\geq 0} \Gamma_{x,i} t^i$, then there exists a decomposition:

$$\alpha_{x,y} = \sum_{i\geq 0} \Theta_{x,i} t^i + \sum_{i\geq 0} \Lambda_{x,i} t^i \in \mathcal{O}_x + \mathcal{O}_{x,y}$$

where $\Theta_{x,i} \in \mathcal{O}_{E_{x,y}}$, $\Lambda_{x,i} \in E_{x,y} \setminus \mathcal{O}_{E_{x,y}}$, and $\Gamma_{x,i} = \Theta_{x,i} + \Lambda_{x,i}$. Now fix an index $h \geq 0$, then the set

$$S_h := \{x \in y \colon \Lambda_{x,h} \neq 0\}$$

is finite, indeed note that $\mathcal{O}_x + \mathfrak{I}_{x,y}^{s} = \mathcal{O}_{E_{x,y}}[[t]] + t^s E_{x,y}[[t]]$, thus if $\Lambda_{x,h} \neq 0$, then $\alpha_{x,y} \notin \mathcal{O}_x + \mathfrak{I}_{x,y}^{h+1}$. In other words if for infinitely many $x \in y$ we had that $\Lambda_{x,h} \neq 0$, then for the same points $\alpha_{x,y} \notin \mathcal{O}_x + \mathfrak{I}_{x,y}^{h+1}$ against the definition of $\mathbb{A}_y^{(0)}$. We have shown that for all but finitely many $x \in y$, $\Gamma_{x,i} = \Theta_{x,i} \in \mathcal{O}_{E_{x,y}}$ which is equivalent to say that $(\Gamma_{x,i})_{x\in y} \in \mathbf{A}_{\overline{y}}^{f}$.
The case when $r \neq 0$ follows easily from the fact that $\widehat{\mathfrak{m}}_y^{r} \Xi_y^{(0)} = \Xi_y^{(r)}$.
Inclusion $\Xi_y^{(r)} \subseteq \mathbb{A}_y^{(r)}$. As above it is enough to write the proof for $r = 0$. Let $(\alpha_{x,y})_{x\in y} \in \Xi_y^{(0)}$, then for any index $i \geq 0$ define:

$$T_i := \left\{ x \in y \colon \Gamma_{x,i} \notin \mathcal{O}_{E_{x,y}}[[t]] \right\} ;$$

by the property $(**)$ T_i is a finite set. Now fix an index $h > 0$ then for all $x \in y \setminus \cup_{i=1}^{h-1} T_i$, (i.e. for all but finitely many $x \in y$) it holds that $\Gamma_{x,i} = \Theta_{x,i}$ when $i < h$, which means that

$$\alpha_{x,y} = \sum_{i \geq 0} \Theta_{x,i} t^i + \sum_{i \geq h} \Lambda_{x,i} t^i \in \mathcal{O}_x + \mathfrak{J}_{x,y}^h . \qquad \square$$

Proposition 2.5. *Let y be a regular horizontal curve and let t be a uniformizing parameter of K_y. For any $r \in \mathbb{Z}$, $\mathbb{A}_{\overline{y}}^{(r)} \cong t^r \mathbf{A}_{\overline{y}}[[t]]$. In particular $\mathbb{A}_{\overline{y}} \cong \mathbf{A}_{\overline{y}}((t))$ and $\mathbb{A}_{\overline{y}}^{(0)} \cong \mathbf{A}_{\overline{y}}[[t]]$.*

Proof. By Lemma 2.4 we have the equality $\mathbb{A}_y^{(r)} = \Xi_y^{(r)}$ and the map $\Xi_y^r \to t^r \mathbf{A}_{\overline{y}}^f[[t]]$ is given in the following way and it is well defined:

$$(\alpha_{x,y})_{x \in y} = \left(t_2^r \sum_{i \geq 0} \Gamma_{x,i} t_2^i \right)_{x \in y} \mapsto t^r \sum_{i \geq 0} (\Gamma_{x,i})_{x \in y} t^i .$$

It is routine check to show that is a ring isomorphism. $\square$

Remark 2.6. Proposition 2.5 is true also when y is a singular curve. The proof is based on a slightly modified version of Lemma 2.4; the only difference consists in the fact that if $x \in y$ is singular then $K_{x,y} = \prod_{\mathfrak{z} \in y(x)} K_{x,\mathfrak{z}}$ is a sum of 2-dimensional valuation fields and $\mathfrak{J}_{x,y}$ is the sum of the maximal ideals of $K_{x,\mathfrak{z}}$. Here we restricted the proof to the case of non-singular curves just by simplicity of notations.

Definition 2.7. The modified version of $\mathbf{A}_X$ which takes in account the completed curves is:

$$\overline{\mathbf{A}}_X := \left\{ (\beta_{\overline{y}})_{\overline{y} \subset \widehat{X}} \in \prod_{\overline{y} \subset \widehat{X}} \mathbb{A}_{\overline{y}} \colon \beta_{\overline{y}} \in \mathbb{A}_{\overline{y}}^{(0)} \text{ for all but finitely many } \overline{y} \right\} \subset \prod_{\substack{x \in \overline{y}, \\ \overline{y} \subset \widehat{X}}} K_{x,\overline{y}} .$$

We also introduce the formal notation

$$\overline{\mathbf{A}}_X = \prod_{\substack{x \in \overline{y} \\ \overline{y} \subset \widehat{X}}}'' K_{x,\overline{y}} .$$

The topology on $\overline{\mathbf{A}}_X$ is the restricted topology of the additive groups $\mathbb{A}_{\overline{y}}$ with respect to $\mathbb{A}_{\overline{y}}^{(0)}$.

Definition 2.8. The *completed adelic ring* attached to $\widehat{X}$ is

$$\mathbf{A}_{\widehat{X}} := \overline{\mathbf{A}}_X \oplus \prod_{\sigma \in B_\infty} \mathbf{A}_{X_\sigma}$$

where each $\mathbf{A}_{X_\sigma}$ is the adelic ring of the Riemann surface X_σ. The topology on $\mathbf{A}_{\widehat{X}}$ is the product topology.

Let Υ be the collection of all finite sets of completed curves of $\widehat{X}$, then for $S \in \Upsilon$ we define

$$\mathbf{A}_{\widehat{X}}(S) := \prod_{\overline{y} \in S} \mathbb{A}_{\overline{y}} \times \prod_{\overline{y} \notin S} \mathbb{A}_{\overline{y}}^{(0)} \times \prod_{\sigma \in B_\infty} \mathbf{A}_{X_\sigma}$$

then:

$$\bigcup_{S \in \Upsilon} \mathbf{A}_{\widehat{X}}(S) = \mathbf{A}_{\widehat{X}}, \quad \bigcap_{S \in \Upsilon} \mathbf{A}_{\widehat{X}}(S) = \prod_{\overline{y} \subset \widehat{X}} \mathbb{A}_{\overline{y}}^{(0)} \times \prod_{\sigma \in B_\infty} \mathbf{A}_{X_\sigma}.$$

The following proposition establishes a nice relationship between $\mathbf{A}_{\widehat{X}}$ and $\mathbf{A}_X$.

Proposition 2.9. *The following equality holds:*

$$\mathbf{A}_{\widehat{X}} = \mathbf{A}_X \oplus \prod_{\sigma \in B_\infty} (\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma}).$$

Proof. Let $\alpha \in \overline{\mathbf{A}}_X$, then it can be decomposed in the following way:

$$\alpha = (a_y)_{y \subset X} \times (a_{p,\sigma})_{\substack{p \in X_\sigma, \\ \sigma \in B_\infty}}$$

where:

- $a_y \in \mathbb{A}_y$ for all $y \subset X$ and $a_y \in \mathbb{A}_y^{(0)}$ for all but finitely many y.
- For any fixed σ we have $a_{p,\sigma} \in K_{p,\sigma}$ and $a_{p,\sigma} \in \mathcal{O}_{p,\sigma}$ for all but finitely many $p \in X_\sigma$.

This means that $\alpha \in \overline{\mathbf{A}}_X \subseteq \mathbf{A}_X \oplus \prod_{\sigma \in B\sigma} \mathbf{A}_{X_\sigma}$, so obviously

$$\mathbf{A}_{\widehat{X}} \subseteq \mathbf{A}_X \oplus \prod_{\sigma \in B_\sigma} (\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma}).$$

Vice versa, let $\alpha \in \mathbf{A}_X \oplus \prod_{\sigma \in B\sigma} \mathbf{A}_{X_\sigma}$ then:

$$\alpha = (a_y)_{y \subset X} \times (a_{p,\sigma})_{\substack{p \in X_\sigma, \\ \sigma \in B_\infty}}$$

where a_y and $a_{p,\sigma}$ satisfy the conditions listed above. Since each $\varphi_\sigma : X_\sigma \to X_K$ is surjective and points of X_K correspond to horizontal curves on X, we can write easily:

$$\alpha = (a_y)_{y \subset X} \times (a_{p,\sigma})_{\substack{p \in X_\sigma, \\ \sigma \in B_\infty}} = (a_y)_{y \subset X} \times ((a_{p,\sigma})_{p \in y_\infty})_{y_\infty \subset X_\infty} = (a_{\overline{y}})_{\overline{y} \subset \widehat{X}} \in \overline{\mathbf{A}}_X. \qquad \square$$

Remark 2.10. The above definition of $\mathbf{A}_{\widehat{X}}$ is new, but the object is very similar to the ring of completed adeles given in [9]. One difference was already emphasized in Remark 2.1; moreover in [9, 25.] the spaces $\mathbb{A}_y^{(r)}$ are obtained through some local lifting maps of $E_{x,y}$ to $\mathcal{O}_{x,y}$.

Remark 2.11. At first glance, one might think that a reasonable definition of the adelic ring $\mathbf{A}_{\widehat{X}}$ can be just $\mathbf{A}_X \oplus \prod_{\sigma\in B_\infty} \mathbf{A}_{X_\sigma}$. With such a definition of $\mathbf{A}_{\widehat{X}}$ we totally forget about the flags of the type $p \in \overline{y} \subset \widehat{X}$ where y is horizontal and $p \in X_\infty$. So, we only add the flags of the type $p \in X_\sigma \subset \widehat{X}$ to the usual geometric picture.

Now we give the definitions of the completed spaces $A_{\widehat{*}}$: denote by $K_{\overline{y}}$ the diagonal embedding of K_y inside $\prod_{x\in\overline{y}} K_{x,\overline{y}}$, then we put:

$$\overline{A}_{01} := \overline{\mathbf{A}}_X \cap \prod_{\overline{y}\subset\widehat{X}} K_{\overline{y}} \,.$$

Moreover for any σ let $A_0(\sigma)$ be the diagonal embedding $\mathbb{C}(X_\sigma) \hookrightarrow \prod_{p\in X_\sigma} K_{p,\sigma}$, then:

$$A_{\widehat{01}} := \overline{A}_{01} \oplus \prod_{\sigma\in B_\infty} A_0(\sigma) \,.$$

If $x \in X$ we have the natural embedding $K_x \hookrightarrow \prod_{\overline{y}\ni x} K_{x,\overline{y}}$; if $p \in X_\sigma$ then we consider the diagonal $\Delta_{p,\sigma} \subset K_{p,\overline{y}} \times K_{p,\sigma}$, where $\overline{y}$ is the unique horizontal curve containing p (remember that $K_{p,\overline{y}} = K_{p,\sigma}$). Thus we define:

$$A_{\widehat{02}} := \mathbf{A}_{\widehat{X}} \cap \left(\prod_{x\in X} K_x \times \prod_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}} \Delta_{p,\sigma} \right) .$$

$\overline{A}_0$ is the diagonal embedding of $K(X)$ in $\overline{\mathbf{A}}_X$ and:

$$A_{\widehat{0}} := \overline{A}_0 \oplus \prod_{\sigma\in B_\infty} A_0(\sigma) \,.$$

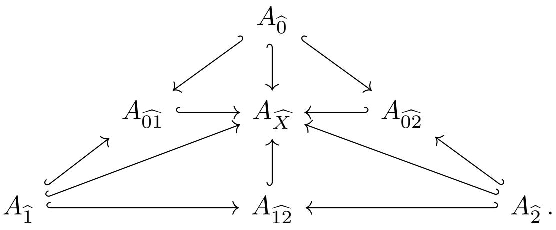
Note that: $A_{\widehat{01}} \cap A_{\widehat{02}} = A_{\widehat{0}}$.

Remark 2.12. On a completed arithmetic surface we have a "generalized version" of the function field, it is not just $K(X)$ because we have fibers at infinity. It should be intended as $K(X) \oplus \prod_\sigma \mathbb{C}(X_\sigma)$ and note that this coincides with $A_{\widehat{0}}$.

The other adelic subspaces are the followings:

$$A_{\widehat{12}} := \mathbf{A}_{\widehat{X}} \cap \left(\prod_{\substack{x\in\overline{y},\\ \overline{y}\in\widehat{X}}} \mathcal{O}_{x,\overline{y}} \times \prod_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}} \mathcal{O}_{p,\sigma} \right), \quad A_{\widehat{1}} := A_{\widehat{01}} \cap A_{\widehat{12}}, \quad A_{\widehat{2}} := A_{\widehat{02}} \cap A_{\widehat{12}},$$

and the containment relations are the same as the geometric case:



3. Residue theory

3.1. Local multiplicative residues

For any $b \in \widehat{B}$ we choose a (standard) character $\psi_b : K_b \to \mathbb{T}$ such that

$$\prod_{b\in\widehat{B}} \psi_b : \mathbf{A}_{\widehat{B}} \to \mathbb{T}$$

is a character which is trivial on the diagonal embedding of K inside the adeles (see [23, Lemma 4.1.5]).

Fix a completed curve $\overline{y} \subset \widehat{X}$, by considering all local branches in $y(x)$ we also define:

$$\Omega^1_{x,\overline{y}} := \bigoplus_{\mathfrak{z}\in y(x)} \Omega^1_{x,\mathfrak{z}}.$$

The structure of $\Omega^1_{x,\overline{y}}$ and the explicit expression of the $\operatorname{res}_{x,\mathfrak{z}}$ depend on the nature of $\overline{y}$:

$\overline{y}$ horizontal The local field $E_{x,\mathfrak{z}}$ is the constant field of $K_{x,\mathfrak{z}}$ i.e. $K_{x,\mathfrak{z}} \cong E_{x,\mathfrak{z}}((t))$ and $[E_{x,\mathfrak{z}} : K_b] < \infty$. In [16, 2.2] it is shown that there is an isomorphism

$$\Omega^1_{x,\mathfrak{z}} \cong E_{x,\mathfrak{z}}((t))dt \tag{3}$$

where t is a uniformizing parameter and moreover the local residue assumes the following form independently from the choice of the isomorphism (3)

$$\begin{aligned} \mathrm{res}_{x,\mathfrak{z}} : \Omega^1_{x,\mathfrak{z}} &\to K_b \\ adt &\mapsto \mathrm{Tr}_{E_{x,\mathfrak{z}}|K_b}(a_{-1}) \end{aligned}$$

where $a = \sum_{j\geq m} a_j t^j \in E_{x,\mathfrak{z}}((t))$. Moreover we put:

$$\mathrm{res}_{x,\overline{y}} := \sum_{\mathfrak{z}\in y(x)} \mathrm{res}_{x,\mathfrak{z}} : \Omega^1_{x,\overline{y}} \to K_b \,,$$

$$\mathrm{Cres}_{x,\overline{y}} := \psi_b \circ \mathrm{res}_{x,y} : \Omega^1_{x,\overline{y}} \to \mathbb{T}$$

where $\psi_b : K_b \to \mathbb{T}$ is the standard character.

$\overline{y} = y$ *vertical* $K_{x,\mathfrak{z}}$ is a finite extension of the standard field $L = K_p\{\{t\}\}$ where $[K_p : K_b] < \infty$ and t is a uniformizing parameter for the residue field $\overline{L} = \overline{K}_p((t))$. Thanks to [16, 2.3] we have an isomorphism

$$\Omega^{\mathrm{cts}}_{L|K_b} := \left(\Omega^1_{\mathcal{O}_L|\mathcal{O}_b}\right)^{\mathrm{sep}} \otimes_{\mathcal{O}_L} L \cong K_p\{\{t\}\}dt \tag{4}$$

and a local residue independent from isomorphism (4):

$$\begin{aligned} \mathrm{res}_L : \Omega^{\mathrm{cts}}_{L|K_b} &\to K_b \\ adt &\mapsto -\mathrm{Tr}_{K_p|K_b}(a_{-1}) \end{aligned}$$

where $a = \sum_{j\in\mathbb{Z}} a_j t^j \in K_p\{\{t\}\}$. By [16, Remark 2.6], we know that $\Omega^1_{x,\mathfrak{z}} = \Omega^{\mathrm{cts}}_{L|K_b} \otimes_L K_{x,\mathfrak{z}}$, so we obtain a well defined trace map

$$\mathrm{Tr}_{K_{x,\mathfrak{z}}|L} : \Omega^1_{x,\mathfrak{z}} \to \Omega^{\mathrm{cts}}_{L|K_b}$$

At this point we define:

$$\mathrm{res}_{x,\mathfrak{z}} := \mathrm{res}_L \circ \mathrm{Tr}_{K_{x,\mathfrak{z}}|L} : \Omega^1_{x,\mathfrak{z}} \to K_b \,,$$

$$\mathrm{res}_{x,\overline{y}} := \sum_{\mathfrak{z}\in y(x)} \mathrm{res}_{x,\mathfrak{z}} : \Omega^1_{x,\overline{y}} \to K_b \,,$$

$$\mathrm{Cres}_{x,\overline{y}} := \psi_b \circ \mathrm{res}_{x,\overline{y}} : \Omega^1_{x,\overline{y}} \to \mathbb{T} \,,$$

where $\psi_b : K_b \to \mathbb{T}$ is the standard character.

When $\overline{y}$ is a completed horizontal curve and $x = p \in y_\sigma \subset y_\infty$ is a point at infinity, then:

$$\Omega^1_{x,\overline{y}} := \Omega^1_{p,\sigma} = K_{p,\sigma}dt;$$

$$\mathrm{Cres}_{x,\overline{y}} := \psi_\sigma \circ \mathrm{res}_{p,\sigma} : \Omega^1_{p,\sigma} \to \mathbb{T} \,.$$

Where in the last line, $\mathrm{res}_{p,\sigma}$ is the one dimensional residue on the Riemann surface X_σ at the point p and $\psi_\sigma : \mathbb{C} \to \mathbb{T}$ is the standard character of $\mathbb{C}$.

Finally for a flag at infinity $p \in X_\sigma$:

$$\mathrm{Cres}_{p,\sigma} := \psi_\sigma \circ (-\mathrm{res}_{p,\sigma}) : \Omega^1_{p,\sigma} \to \mathbb{T}\ .$$

The detailed proofs of the independence of the various local residues maps from the parameterizations and standard fields can be found in [16].

Remark 3.1. The choice of the minus sign in the definition of $\mathrm{Cres}_{p,\sigma}$ is coherent with the main theory since X_σ is vertical curve on $\widehat{X}$ in our geometric construction.

The following proposition is the extension of [18, Lemma 3.3] to the adelic case. It says that it makes sense to take the product of residues along vertical curves; moreover by looking at its proof one immediately realizes that in the definition of two dimensional adeles, "the first restricted product" along a fixed curve is a crucial operation.

Proposition 3.2. *Let $\alpha \in \mathbf{A}_X$ and fix a vertical curve $y \subseteq X_b$. Then the series*

$$\sum_{x \in y} \mathrm{res}_{x,y}(\alpha_{x,y} dt)$$

converges in K_b. In particular $\mathrm{res}_{x,y}(\alpha_{x,y}dt) \in \mathcal{O}_b$ for all but finitely many $x \in y$.

Proof. For simplicity let's assume that y is nonsingular. We know that $(\alpha_{x,y})_{x\in y} \in \mathbb{A}_y^{(r)}$ for some $r \in \mathbb{Z}$, it means that $(\alpha_{x,y})_{x\in y} = (t_y^r \beta_{x,y})_{x \in y}$ where $(\beta_{x,y})_{x\in y} \in \mathbb{A}_y^{(0)}$. Now we use the definition of $\mathbb{A}_y^{(0)}$ to say that for any $s > 0$ we have $\mathrm{res}_{x,y}(\beta_{x,y}) \in \mathfrak{p}_{K_b}^{s+m}\mathcal{O}_b$ at almost all $x \in y$. It follows that for any $s > 0$, $\mathrm{res}_{x,y}(\alpha_{x,y}) \in \mathfrak{p}_{K_b}^{s+m+r}\mathcal{O}_b$ at almost all $x \in y$. This shows that $\sum_{x\in y} \mathrm{res}_{x,y}(\alpha_{x,y}dt)$ converges in K_b. $\square$

By the universal property of the module of differential forms we have a canonical map $\Omega^1_{K(X)|K} \to \Omega^1_{x,\overline{y}}$, therefore by abuse of notation, we can consider an element $\omega \in \Omega^1_{K(X)|K}$ as an element lying in $\Omega^1_{x,\overline{y}}$. Moreover, by base change we know that $\Omega^1_{\mathbb{C}(X_\sigma)|\mathbb{C}} \cong \Omega^1_{K(X)|K} \otimes_{K(X)} \mathbb{C}(X_\sigma)$, so we have a canonical composition map:

$$\Omega^1_{K(X)|K} \to \Omega^1_{\mathbb{C}(X_\sigma)|\mathbb{C}} \to \Omega^1_{p,\sigma}$$

and when clear from the context we can consider $\omega \in \Omega^1_{K(X)|K}$ as an element lying in $\Omega^1_{p,\sigma}$. In other words, it always makes sense to take a residue of a "rational" differential form $\omega \in \Omega^1_{K(X)|K}$ for flags in X and in $\widehat{X}$.

Theorem 3.3 *(2D arithmetic reciprocity laws). Let $\omega \in \Omega^1_{K(X)|K}$ and nonzero, then:*

(1) *Let $x \in X$, then $\sum_{\overline{y}\ni x} \mathrm{res}_{x,y}(\omega) = 0$ and $\mathrm{res}_{x,\overline{y}}(\omega) = 0$ for all but finitely many curves $\overline{y}$ containing x. In particular $\prod_{\overline{y}\ni x} \mathrm{Cres}_{x,\overline{y}}(\omega) = 1$ and $\mathrm{Cres}_{x,\overline{y}}(\omega) = 1$ for all but finitely many $x \in y$.*

(2) *Let $p \in X_\sigma$, and let $\overline{y_p}$ be the only completed horizontal curve containing p, then*

$$\operatorname{Cres}_{p,\sigma}(\omega) \cdot \prod_{\overline{y} \ni p} \operatorname{Cres}_{p,\overline{y}}(\omega) = \operatorname{Cres}_{p,X_\sigma}(\omega)\operatorname{Cres}_{p,\overline{y_p}}(\omega) = 1\,.$$

(3) *Let $\overline{y} \subset X$ be a vertical curve or $\overline{y} = X_\sigma$ for some $\sigma \in B_\infty$, then $\sum_{x\in\overline{y}} \operatorname{res}_{x,y}(\omega) = 0$. In particular $\prod_{x\in\overline{y}} \operatorname{Cres}_{x,\overline{y}}(\omega) = 1$ and $\operatorname{Cres}_{x,\overline{y}}(\omega) = 1$ for all but finitely many $x \in y$.*

(4) *Let $\overline{y} \in \widehat{X}$ be a horizontal curve, then $\prod_{x\in\overline{y}} \operatorname{Cres}_{x,\overline{y}}(\omega) = 1$.*

Proof. See [18, 2.4], [18, 5] and [18, 3] for (1), (4) and the non-archimedean part of (3) respectively. For the archimedean case of (3) see [24, Corollary of Theorem 3]. (2) Follows basically from the definitions of the local residues. □

Remark 3.4. Note that statements (1) and (2) of Theorem 3.3 describe reciprocity laws around a point, whereas statements (3) and (4) describe reciprocity laws for a fixed curve. Archimedean data are taken in account without any special treatment: points on X_σ are considered as points of $\widehat{X}$ and achimedean fibers are considered as vertical curves on $\widehat{X}$. We point out that statement (2) is new and it hasn't been published anywhere before.

3.2. Adelic residue

In this subsection we globalize the local residues in order to get a residue at the level of completed adeles. Fix a nonzero rational 1-form $\omega \in \Omega^1_{K(X)|K}$, then we define the adelic residue map:

$$\begin{array}{ccc} \xi^\omega\colon \mathbf{A}_{\widehat{X}} & \longrightarrow & \mathbb{T} \\ \cup & & \cup \\ (a_{x,\overline{y}})_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} \times (a_{p,\sigma})_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}} & \longmapsto & \prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} \operatorname{Cres}_{x,\overline{y}}(\omega a_{x,\overline{y}}) \prod_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}} \operatorname{Cres}_{p,\sigma}(\omega a_{p,\sigma}) \end{array} \tag{5}$$

Let's explain why ξ^ω is well defined (i.e. the product (5) is convergent): along all but finitely many curves $y \subset X$ the local residue is zero due to the restricted product with respect to the spaces $\mathbb{A}_y^{(0)}$. For the remaining curves we use the following arguments

- If y is horizontal it is enough to look at property $(**)$ of Lemma 2.4. It follows that the residue is 0 at all but finitely many points of y.
- If y is vertical we use Proposition 3.2.
- For curves at infinity it is enough to appeal to the 1-dimensional adelic restricted product.

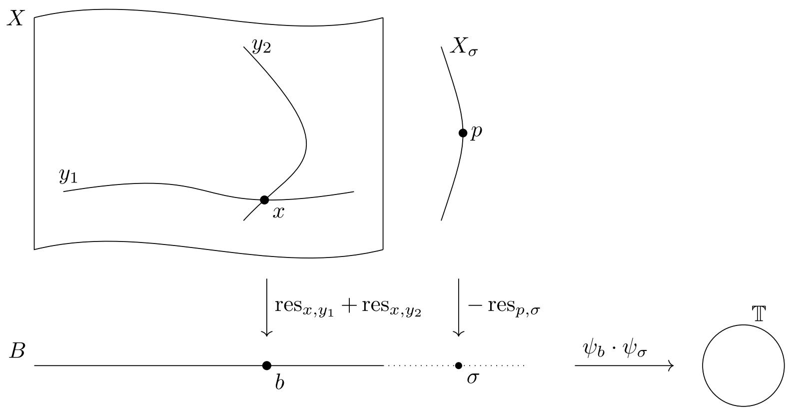


Fig. 3. A graphic representation of the action of the adelic residue on 3 different flags: $x \in y_1$, $x \in y_2$ and $p \in X_\sigma$.

In [18, Lemma 2.8, Remark 2.9] it was proved that the local residues $\mathrm{res}_{x,y} : \Omega_{x,y} \to K_b$ are continuous, moreover it is clear that the local residues at infinity $\mathrm{res}_{p,\sigma} : \Omega_{p,\sigma} \to \mathbb{C}$ are continuous (remember that here $\mathbb{C}$ has the archimedean topology and $K_{p,\sigma}$ the 2-dimensional topology). We are interested in the global theory of residues and we will show that the adelic residue ξ^ω is sequentially continuous. (See Fig. 3.)

Proposition 3.5. *The adelic residue ξ^ω is sequentially continuous.*

Proof. To prove the sequential continuity of ξ^ω it is not necessary to consider the residues along curves at infinity because we have only a finite number of them and the 1-dimensional adelic residue is continuous. So, it is enough to discuss the schematic part of ξ^ω which will be denoted as $\xi^\omega_S : \mathbf{A}_X \to \mathbb{T}$. Note that we can write $\xi^\omega_S = \psi_S \circ \theta^\omega$ where $\psi_S : \mathbf{A}_B \to \mathbb{T}$ is the schematic part of the 1-dimensional standard character and

$$\theta^\omega = (\theta^\omega_b)_{b \in B} : \mathbf{A}_X \to \mathbf{A}_B$$

with

$$\theta^\omega_b : \prod_{\substack{x \in y, \\ y \subset X, \\ x \mapsto b}} K_{x,y} \to K_b$$

$$\theta^\omega_b((\alpha_{x,y})) = \sum_{\substack{x \in X_b, \\ y \ni x}} \mathrm{res}_{x,y}(\omega\alpha_{x,y}) = \overbrace{\sum_{\substack{y \subset X_b, \\ x \in y}} \mathrm{res}_{x,y}(\omega\alpha_{x,y})}^{(i)} + \overbrace{\sum_{\substack{x \in X_b, \\ y \ni x, \\ y \text{ horiz.}}} \mathrm{res}_{x,y}(\omega\alpha_{x,y})}^{(ii)} \in K_b \,.$$

For any $n \in \mathbb{N}$, let $\alpha^{(n)} := (\alpha^{(n)}_{x,y})_{x,y} \in \mathbf{A}_X$ such that $\lim_{n\to\infty} \alpha^{(n)}_{x,y} = 0$. Moreover put $\beta^{(n)}_{x,y}dt := \omega\alpha^{(n)}_{x,y}$. Just for simplicity of notations we can assume that all curves involved in are nonsingular. We want to show that

$$\lim_{n\to\infty} \xi^{\omega}_{S}\left(\alpha^{(n)}\right) = 1\,.$$

Let's analyze carefully the summations (i) and (ii):

(i) We know that $K_{x,y}$ is a finite extension of $L = K_p\{\{t\}\}$ and we can write

$$\mathrm{Tr}_{K_{x,y}|L}\left(\beta^{(n)}_{x,y}\right) = \sum_{i=-\infty}^{\infty} \beta^{(n)}_{x,y}(i)t^i \quad \text{for } \beta^{(n)}_{x,y}(i) \in K_p\,.$$

Then

$$\mathrm{res}_{x,y}\left(\beta^{(n)}_{x,y}dt\right) = -\,\mathrm{Tr}_{K_p|K_b}\left(\beta^{(n)}_{x,y}(-1)\right)\,.$$

Since $\lim_{n\to\infty} \beta^{(n)}_{x,y} = 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $\beta^{(n)}_{x,y}(-1) \in \mathcal{O}_{K_p}$, i.e. $\mathrm{res}_{x,y}\left(\beta^{(n)}_{x,y}dt\right) \in \mathcal{O}_b$. This means that

$$\lim_{n\to\infty} \sum_{\substack{y\subset X_b,\\ x\in y}} \mathrm{res}_{x,y}\left(\beta^{(n)}_{x,y}dt\right) \in \mathcal{O}_b\,.$$

(ii) We know that $\beta^{(n)}_{x,y} = \sum_{i\geq m} \beta^{(n)}_{x,y}(i)t^i$, where $\beta^{(n)}_{x,y}(i) \in E_{x,y}$ and $E_{x,y}$ is a finite extension of K_b. Furthermore $\lim_{n\to\infty} \beta^{(n)}_{x,y} = 0$. We have:

$$\lim_{n\to\infty} \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} \mathrm{res}_{x,y}(\beta^{(n)}_{x,y}dt) = \lim_{n\to\infty} \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} \mathrm{Tr}_{E_{x,y}|K_b}(\beta^{(n)}_{x,y}(-1))\,. \tag{6}$$

Due to the adelic restricted product, for all $n \geq n_0$ we have that $\mathrm{res}_{x,y}\left(\beta^{(n)}_{x,y}\right) = 0$ along all but a fixed finite set of curves $y \subset X$, therefore we can exchange the summation and the limit in equation (6). So we get:

$$\lim_{n\to\infty} \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} \mathrm{res}_{x,y}(\beta^{(n)}_{x,y}dt) = \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} \lim_{n\to\infty} \mathrm{Tr}_{E_{x,y}|K_b}(\beta^{(n)}_{x,y}(-1)) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} \mathrm{Tr}_{E_{x,y}|K_b}\left(\lim_{n\to\infty} \beta^{(n)}_{x,y}(-1)\right) = 0\,.$$

We can write:

$$\lim_{n\to\infty} \xi^{\omega}_{S}(\alpha^{(n)}) = \lim_{n\to\infty} \psi_S(\theta^{\omega}(\alpha^{(n)}))\,.$$

For each $b \in B$ we have

$$\tau_b^{(n)} := \theta_b^\omega(\alpha^{(n)}) = \sum_{\substack{y \subset X_b, \\ x \in y}} \mathrm{res}_{x,y}(\beta_{x,y}^{(n)} dt) + \sum_{\substack{x \in X_b, \\ y \ni x, \\ y \text{ horiz.}}} \mathrm{res}_{x,y}(\beta_{x,y}^{(n)} dt)$$

and by (i) and (ii) we can conclude that:

$$\lim_{n\to\infty} \tau_b^{(n)} \in \mathcal{O}_b\,.$$

Finally:

$$\lim_{n\to\infty} \xi_S^\omega(\alpha^{(n)}) = \lim_{n\to\infty} \psi_S\left(\left(\tau_b^{(n)}\right)_{b\in B}\right) = \psi_S\left(\left(\lim_{n\to\infty} \tau_b^{(n)}\right)_{b\in B}\right) = 1\,. \qquad \square$$

From the sequential continuity of the adelic residue we can deduce a stronger version of reciprocity laws:

Proposition 3.6. *Fix a rational differential form $\omega \in \Omega^1_{K(X)|K}$. Then the following statements hold:*

(1) *Fix a point $x \in X$. For any $\alpha \in K_x$ we have $\prod_{\overline{y}\ni x} \mathrm{Cres}_{x,\overline{y}}(\alpha\omega) = 1$.*
(2) *Fix a curve $\overline{y} \subset \widehat{X}$. For any $\alpha \in K_{\overline{y}}$ we have $\prod_{x\in\overline{y}} \mathrm{Cres}_{x,\overline{y}}(\alpha\omega) = 1$.*

Proof. (1) $K_x = K(X)\mathcal{O}_x$, but $\mathscr{O}_{X,x}$ is sequentially dense into its completion $\mathcal{O}_x$. Then the claim follows from the sequential continuity of the adelic residue and Theorem 3.3(1).

(2) Again It follows from the fact that $K(X)$ is sequentially dense in its completion (with respect to y) K_y, the sequential continuity of the adelic residue, and Theorem 3.3(3)-(4). $\square$

4. Self-duality of completed adeles

This section is entirely dedicated to the proof that the additive group $\mathbf{A}_{\widehat{X}}$ is self-dual. We will reduce the problem to show the self-duality of $\mathbb{A}_y$ and $\mathbf{A}_{X_\sigma}$.

The following two lemmas characterize the characters of $\mathbf{A}_{\widehat{X}}$:

Lemma 4.1. *Let $\chi \in \widehat{\mathbf{A}_{\widehat{X}}}$, then $\chi\left(\mathbb{A}_{\overline{y}}^{(0)}\right) = 1$ for all but finitely many curves $\overline{y} \subset \widehat{X}$. In particular if $\beta := (\beta_{\overline{y}})_{\overline{y}} \times (\beta_\sigma)_\sigma \in \mathbf{A}_{\widehat{X}}$ we have that*

$$\chi(\beta) = \prod_{\overline{y}\subset\widehat{X}} \chi(\beta_{\overline{y}}) \prod_{\sigma\in B_\infty} \chi(\beta_\sigma)\,.$$

(In the above formula we clearly embedded each $\beta_{\overline{y}}$ and β_σ naturally in $\mathbf{A}_{\widehat{X}}$.)

Proof. Let $U \subset \mathbb{T}$ be an open neighborhood of 1 which contains no subgroups of $\mathbb{T}$ other than $\{1\}$ and let $V \subset \mathbf{A}_{\widehat{X}}$ be an open subset such that $\chi(V) \subset U$. By the definition of restricted product as direct limit with the final topology, we know that for any finite set S of completed curves in $\widehat{X}$ the subset $V \cap \mathbf{A}_{\widehat{X}}(S)$ is open in $\mathbf{A}_{\widehat{X}}(S)$. In particular by the definition of product topology, it contains an open subset of the following form:

$$W = \prod_{\overline{y} \notin S'} \mathbb{A}_{\overline{y}}^{(0)} \times \prod_{\overline{y} \in S'} W'_{\overline{y}} \times \prod_{\sigma \in B_\infty} W'_\sigma$$

where S' is another finite set of completed curves in $\widehat{X}$ and $W'_{\overline{y}} \subset \mathbb{A}_{\overline{y}}$, $W'_\sigma \subset \mathbf{A}_{X_\sigma}$ are open. It follows that $H := \chi\left(\prod_{\overline{y} \notin S'} \mathbb{A}_{\overline{y}}^{(0)}\right) \subset U$, but H is a subgroup of $\mathbb{T}$, thus $H = \{1\}$ by the choice of U. In particular $\chi\left(\mathbb{A}_{\overline{y}}^{(0)}\right) = 1$ for any $\overline{y} \notin S'$. The last assertion of the lemma is straightforward. $\square$

Lemma 4.2. *Let $\chi_{\overline{y}} \in \widehat{\mathbb{A}_{\overline{y}}}$ and let $\chi_\sigma \in \widehat{\mathbf{A}_{X_\sigma}}$. If $\chi_{\overline{y}}\left(\mathbb{A}_{\overline{y}}^{(0)}\right) = 1$ for all but finitely many curves $\overline{y} \subset \widehat{X}$, then*

$$\chi := \prod_{\overline{y} \subset \widehat{X}} \chi_{\overline{y}} \prod_{\sigma \in B_\infty} \chi_\sigma \in \widehat{\mathbf{A}_{\widehat{X}}}.$$

Proof. The only thing that is not straightforward is the continuity of χ, and there is no need to consider the fibers at infinity since they are finitely many. Let $U \subset \mathbb{T}$ be an open neighborhood of 1 and choose $V \subset U$ such that[2] $\coprod_m V \subset U$. Now pick a finite set of completed curves $S \subset \Upsilon$ of cardinality m, and for any $\overline{y} \in S$ take $W_{\overline{y}} \subset \chi_{\overline{y}}^{-1}(V)$. Then $\prod_{\overline{y} \in S} W_{\overline{y}} \times \prod_{\overline{y} \notin S} \mathbb{A}_{\overline{y}}^{(0)}$ is contained in the preimage of $\prod_{\overline{y}} \chi_{\overline{y}}$. $\square$

The following proposition is basically the "reduction argument" that allows us to restrict our attention to $\mathbb{A}_y$ and $\mathbf{A}_{X_\sigma}$.

Proposition 4.3. *The following isomorphism of topological groups holds:*

$$\widehat{\mathbf{A}_{\widehat{X}}} \cong \prod_{\overline{y} \subset \widehat{X}}{}' \widehat{\mathbb{A}_{\overline{y}}} \times \prod_{\sigma \in B_\infty} \widehat{\mathbf{A}_{X_\sigma}}$$

where on the right hand side the restricted product is taken with respect to the subgroups $\left(\mathbb{A}_{\overline{y}}^{(0)}\right)^\perp \subset \widehat{\mathbb{A}_{\overline{y}}}$.

[2] By $\coprod$ we denote the actual complex multiplication of all elements in the open sets. In this particular case we are taking the "m-th power of V".

Proof. Consider the map:

$$\Psi : {\prod_{\overline{y}\subset\widehat{X}}}' \widehat{\mathbb{A}_{\overline{y}}} \times \prod_{\sigma\in B_\infty} \widehat{\mathbf{A}_{X_\sigma}} \to \widehat{\mathbf{A}_{\widehat{X}}}$$
$$(\chi_{\overline{y}})_{\overline{y}\subset\widehat{X}} \times (\chi_\sigma)_{\sigma\in B_\infty} \mapsto \prod_{\overline{y}\subset\widehat{X}} \chi_{\overline{y}} \prod_{\sigma\in B_\infty} \chi_\sigma$$

where clearly we naturally considered $\chi_{\overline{y}}, \chi_\sigma \in \widehat{\mathbf{A}_{\widehat{X}}}$. From Lemmas 4.1 and 4.2 it follows immediately that Ψ is an isomorphism of groups, so we have to prove that it is continuous and open. Let U be an open neighborhood of 1 and consider the compact of $\mathbf{A}_{\widehat{X}}$:

$$C = \prod_{\overline{y}\in S} C_{\overline{y}} \times \prod_{\overline{y}\notin S} \mathbb{A}_{\overline{y}}^{(0)} \times \prod_{\sigma\in B_\infty} C_\sigma$$

where $C_\sigma, C_{\overline{y}}$ are compacts, and we assume that S has cardinality m. Then $\mathcal{W}(C,U)$ is a basic open neighborhood of $\mathbf{A}_{\widehat{X}}$ around the identity character. Take now $V \subset U$ such that $\prod_{m+\#B_\infty} V \subset U$ and consider:

$$W = \prod_{\overline{y}\in S} \mathcal{W}(C_{\overline{y}}, V) \times \prod_{\overline{y}\notin S} \left(\mathbb{A}_{\overline{y}}^{(0)}\right)^\perp \times \prod_{\sigma\in B_\infty} \mathcal{W}(C_\sigma, V)\,.$$

Then clearly $\Psi(W) \subseteq \mathcal{W}(C,U)$. The proof of openness is similar. □

Remark 4.4. So in order to show the self-duality of $\widehat{\mathbf{A}_{\widehat{X}}}$ we are reduced to show two things:

- The self-duality of $\mathbf{A}_{X_\sigma}$.
- There are topological and algebraic isomorphisms $\theta_{\overline{y}} : \mathbb{A}_{\overline{y}} \to \widehat{\mathbb{A}_{\overline{y}}}$ mapping homeomorphically $\mathbb{A}_{\overline{y}}^{(0)}$ onto $\left(\mathbb{A}_{\overline{y}}^{(0)}\right)^\perp$ for all but finitely many completed curves.

For the self-duality of $\mathbf{A}_{X_\sigma}$ we will use the following general results about Laurent power series over a self-dual group.

Lemma 4.5. *Let G be a ST ring and suppose that $(G,+)$ is endowed with a standard character. Then $G((t))$ has a standard character with conductor equal to 0 (see appendix A.2 to see how $G((t))$ is topologised and for the definition of conductor).*

Proof. Let ξ be a standard character of G. First of all let's find explicitly a nontrivial character of $G((t))$ which has conductor equal to 0. Consider:

$$\psi^0 : G((t)) \to \mathbb{T}$$
$$\sum_{i\geq m} a_i t^i \mapsto \xi(a_{-1})$$

Let $\psi \in \widehat{G((t))}$, we want to show that there exists a uniquely determined $\alpha \in G((t))$ such that $\psi = \psi^0_\alpha$. Assume that $c_\psi = i$, for any $b \in G$ the map $b \mapsto \psi(bt^{i-1})$ defines a character on G that by hypothesis is equal to ξ_{a_0} for a uniquely determined $a_0 \in G$. So consider the character:

$$\psi^1(x) := \frac{\psi(x)}{\psi^0(xa_0t^{-i})} \quad \text{for } x \in G((t))\,,$$

it is easy to verify that $\psi^1\left(t^{i-1}G[[t]]\right) = 1$. By iterating the above argument, for any $j \geq 1$ one finds a uniquely determined $a_j \in G$ such that

$$\psi^{j+1}(x) := \frac{\psi^j(x)}{\psi^0(xa_jt^{-i+j})} = \frac{\psi(x)}{\psi^0\left(x\sum_{h=0}^{j} a_h t^{-i+h}\right)}$$

is a character trivial on $t^{i-1-j}G[[t]]$. By taking the limit for $j \to \infty$ we obtain:

$$1 = \lim_{j\to\infty} \psi^j(x) = \frac{\psi(x)}{\psi^0\left(x\sum_{h\geq 0} a_h t^{-i+h}\right)}\,.$$

So we put $\alpha := \sum_{h\geq 0} a_h t^{-i+h}$ and it follows that $\psi(x) = \psi^0(x\alpha)$.

Now we show the continuity and the openness of the map $G((t)) \to \widehat{G((t))}$ defined by $\alpha \mapsto \psi^0_\alpha$. It is enough to prove the following simple things:

(*a*) Given a compact $C \subset G((t))$ and an open $U \ni 1$ in $\mathbb{T}$, there exist an open set $V \ni 0$ in $G((t))$ such that: $a \in V \Rightarrow aC \subseteq \psi^{-1}(U)$.
(*b*) Given an open $U \ni 0$ in A there exist a compact $C \subset G((t))$ and an open $V \ni 1$ in $\mathbb{T}$, such that: $aC \subseteq \psi^{-1}(V) \Rightarrow a \in U$.

The explicit proofs of (*a*) and (*b*) are a respectively a very special case of the proofs of continuity and openness assertions of Theorem 4.7, so they are omitted here. □

Proposition 4.6. *The additive group $\mathbf{A}_{X_\sigma}$ is self-dual for every $\sigma \in B_\infty$.*

Proof. For any point $p \in X_\sigma$, we have $K_{p,\sigma} \cong \mathbb{C}((t))$, therefore we can apply Lemma 4.5 to conclude that $K_{p,\sigma}$ is self-dual and that a standard character with conductor equal to 0 is given by $a \mapsto \mathrm{Cres}_{p,\sigma}(adt)$. At this point it is enough to follow line by line the argument in Tate's thesis that shows that adeles over a number field are self-dual (see for example [22, 5.1]) to prove that $\mathbf{A}_{X_\sigma}$ is self-dual. Actually one needs the 1-dimensional version of Lemmas 4.1, 4.2 and 4.3, but recall that we have a 2-dimensional topological structure on $K_{p,\sigma}$ and $\mathbf{A}_{X_\sigma}$. □

When $\overline{y}$ is horizontal one can apply Lemma 4.5 and the explicit expression of $\mathbb{A}_{\overline{y}}$ to show that $\mathbb{A}_{\overline{y}}$ is self-dual, but when $\overline{y}$ is vertical, the proof is more problematic because we don't have any nice expression of $\mathbb{A}_{\overline{y}}$ in terms of one dimensional adeles. A deeper analysis of the proof of Lemma 4.5 unravels that the only real advantage of having the expression $A = G((t))$, is the ind-pro structure of A over a self-dual group G with a standard character. In general also $\mathbb{A}_y$ has such property, and the following theorem is a generalization of Lemma 4.5, where $\mathbb{A}_y^{(-1)}/\mathbb{A}_y^{(0)}$ plays the role of G and $\mathbb{A}_y^{(0)}$ plays the role of $G[[t]]$. We will heavily employ the topological properties described in subsection 1.3.

Theorem 4.7. *The additive group $\mathbb{A}_{\overline{y}}$ is self-dual with a standard character ψ^0. Moreover $\psi^0 \in \left(\mathbb{A}_{\overline{y}}^{(0)}\right)^{\perp}$ and $\Theta_{\psi^0}\left(\mathbb{A}_{\overline{y}}^{(0)}\right) = \left(\mathbb{A}_{\overline{y}}^{(0)}\right)^{\perp}$.*

Proof. It is enough to work with $\mathbb{A}_y$. For simplicity of notations let's put $A_r := \mathbb{A}_y^{(r)}$, $A := \mathbb{A}_y$ and $t := t_y$. Let's summarize some properties (all categorical limits are in the category of linearly topologised groups):

(1) A_r is complete and $A_r = \varprojlim_{j\geq 1} A_r/A_{r+j}$.
(2) A_r/A_{r+1} is locally compact and self-dual with a standard character.
(3) A_r/A_{r+j} is locally compact for every $j > 0$.
(4) $A = \varinjlim_r A_r = \bigcup_r A_r$ and $\bigcap_r A_r = \{0\}$.
(5) Any open neighborhood of 0 in A contains some A_r.

Fix a standard character $\overline{\xi} \in \widehat{A_{-1}/A_0}$. Then consider the following commutative diagram of topological groups with exact short sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_{-1}/A_0 & \longrightarrow & A_{-2}/A_0 & \longrightarrow & A_{-2}/A_{-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & A_{-1} & & A_{-2} & & A_{-2} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & A_0 & & A_0 & & A_{-1} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the dual functor is exact on the category of LCA groups, we get the following diagram with exact short sequences:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \widehat{A_{-1}/A_0} & \longleftarrow & \widehat{A_{-2}/A_0} & \longleftarrow & \widehat{A_{-2}/A_{-1}} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \widehat{A_{-1}} & & \widehat{A_{-2}} & & \widehat{A_{-2}} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \widehat{A_0} & & \widehat{A_0} & & \widehat{A_{-1}} & &
\end{array}
$$

In other words $\overline{\xi}$ lifts to a character $\xi^1 \in \widehat{A_{-1}}$ which is trivial on A_0, then we can lift ξ^1 to a character $\xi^2 \in \widehat{A_{-2}}$ which extends ξ^1. By iterating this process we clearly construct a character $\xi^n \in \widehat{A_{-n}}$ extending ξ^1. Now we can define a character $\psi : A \to \mathbb{T}$ in the following way:

$$\psi^0(a) := \xi^n(a) \quad \text{ if } \quad a \in A_n \setminus A_{n+1}\,.$$

By construction ψ^0 is trivial on A_0. A more explicit expression of ψ^0 can be given by using the identification $A_r = t^r A_0[[t]]$: if $a = \sum_{i\geq r} a_i t^i \in A_r$, then $\psi^0(a) = \overline{\xi}(\overline{a_{-1}t^{-1}})$ where $\overline{a_{-1}t^{-1}}$ is the natural projection of $a_{-1}t^{-1}$ onto A_{-1}/A_0. We want to prove that ψ^0 is a standard character for A, so that the map:

$$
\begin{aligned}
\Theta_{\psi^0} : A &\to \widehat{A} \\
a &\mapsto \psi^0_a
\end{aligned}
$$

is an algebraic and topological isomorphism.

Surjectivity. Since $\overline{\xi}$ is a standard character of A_{-1}/A_0, any other character in $\widehat{A^{-1}}$ which is trivial on A_0 is of the form $\xi^1(g\,\cdot)$ for $g \in A_{-1}$. Consider any $\psi \in \widehat{A}$ and let $i = c_\psi$ the minimum integer $i \in \mathbb{Z}$ such that $\psi(A_i) = 1$, note that this integer always exists thanks to (5) and the fact that $\mathbb{T}$ has no small subgroups. Then:

$$\psi_{|A_{i-1}}(\cdot\, t^{-i}) = \xi^1(\cdot\, a_0 t^{-i}) \quad \text{for } a_0 \in A_{-1}.$$

Let's define the following character

$$\psi^1(\cdot) = \frac{\psi(\cdot)}{\psi^0(\cdot\, a_0 t^{-i})}\,,$$

then for any $t^{i-1}b \in A_{i-1}$ ($b \in A_0$):

$$\psi^1_{|A_{i-1}}(t^{i-1}b) = \frac{\psi(t^{i-1}b)}{\psi^0(a_0 t^{-i} t^{i-1} b)} = \frac{\xi^1(a_0 t^{-1} b)}{\psi^0(a_0 t^{-1} b)} = 1\,.$$

In other words ψ^1 is trivial on A_{i-1}. By iterating the above process for $j > 1$, we find elements $a_h \in A_{-1}$ and characters:

$$\psi^j(\cdot) = \frac{\psi(\cdot)}{\psi^0(\cdot \sum_{h=0}^{j} a_h t^{-i+h})}$$

which are trivial on A_{i-1-j}. Now for $g \in A$ take the limit:

$$1 = \lim_{j\to\infty} \psi^j(g) = \frac{\psi(g)}{\psi^0(a \sum_{g\geq 0} a_h t^{-i+h})} \,.$$

We conclude that $\psi(\cdot) = \psi^0(\cdot\, \alpha)$ for $\alpha := \sum_{h\geq 0} a_h t^{-i+h}$. The partial sums defining α form a Cauchy sequence in A_{-1}, which is complete, so α is actually an element of A_{-1}.

Injectivity. For every $a \in A \setminus 0$, there exists $r \in \mathbb{Z}$ such that $\ker \psi_a$ is trivial on A_r but not on A_{r-1}.

Continuity. We have to show that given a compact $K \subset A$ and an open $U \ni 1$ in $\mathbb{T}$, there exist an open set $V \ni 0$ in A such that: $\psi(VK) \subseteq U$. Since K is contained in some A_m, by simplicity we can "shift" K thanks to the multiplication by t^{m-1} and assume $K \subset A_{-1}$. Then $K = \varprojlim_j K_j$ with $K_j \in A_{-1}/A_j$. Now, since $\overline{\xi}$ is a standard character for A_{-1}/A_0, it is not difficult to show by induction that the multiplication in A and the character ψ^0 induce an algebraic and topological isomorphism $A_{-1}/A_j \cong \widehat{A_{-j}/A_1}$ for any $j \geq 0$. Thus we induce perfect pairing of LCA groups:

$$e_j : A_{-1}/A_j \times A_{-j}/A_1 \to \mathbb{T} \,.$$

Consider the orthogonal complement $W_j = K_j^{\perp} := \{a \in A_{-j}/A_1 \colon e_j(K_j, a) = 1\}$, then W_j is open in A_{-j}/A_1. Let $V_j \subset A_{-j}$ be the lift of W_j, it follows that the open set $V = \bigcup_j V_j$ is the open set we were looking for.

Openness. We have to show that given an open $U \ni 0$ in A there exist a compact $K \subset A$ and an open $V \ni 1$ in $\mathbb{T}$, such that $aK \subseteq \psi^{-1}(V) \Rightarrow a \in U$. The open set U is contained a basic open subgroup $\sum' U_i t^i$ where we assume that $U_i = A_0$ for $i \geq m$. Since A_{-1}/A_0 has a standard character, for any $i < m$ there exists a compact $C_i \subset A_{-1}/A_0$ and an open pen $V_i \ni 1$ in $\mathbb{T}$ such that:

$$\overline{\xi}(\overline{a}\overline{C_i}) \subset V_i \Rightarrow \overline{a} \in \overline{U_i t^{-1}} \subset A_{-1}/A_0 \,.$$

Since $\mathbb{T}$ has no small subgroups, we can actually choose $\overline{V_i}$ in a way that

$$\overline{\xi}(\overline{a}\overline{C_i}) = 1 \Rightarrow \overline{a} \in \overline{U_i t^{-1}} \,.$$

Now, since for any $r \geq 1$ we have surjective homomorphisms of LCA groups $A_{-1}/A_r \to A_{-1}/A_0$, we can lift $\overline{C_i}$ to $C_i^r \in A_{-1}/A_r$ which in turn gives $C_i = \varprojlim_r C_i^r$ compact in A_{-1}. We put $K_i = C_i t \in A_0$. For $i \geq m$ we choose $K_i = 0$, so we construct the compact set $K = \sum_i K_i t^{-i}$ in A. It is easy to show that K and a small enough $V \subset T$ containing 1 satisfy the requirements needed to show openness.

Clearly $\Theta_{\psi^0}(A_0) \subseteq (A_0)^\perp$. Let $\psi_a^0 \notin (A_0)^\perp$, then there exists $b \in A_0$ such that $\psi^0(ab) \neq 1$, but this means that $a \notin A_0$ otherwise we would have $\psi^0(ab) = 1$. □

Corollary 4.8. $\mathbf{A}_{\widehat{X}}$ *is self-dual.*

Proof. The proof follows directly from Propositions 4.3, 4.6 and Theorem 4.7. For more clarity, see also Remark 4.4. □

5. Properties of the adelic differential pairing

Fix a nonzero rational differential form $\omega \in \Omega^1_{K(X)|X}$, then the *adelic differential pairing (associated to ω)* is defined as:

$$\begin{aligned} d_\omega \colon \mathbf{A}_{\widehat{X}} \times \mathbf{A}_{\widehat{X}} &\to \mathbb{T} \\ (\alpha, \beta) &\mapsto \xi^\omega(\alpha\beta)\,. \end{aligned}$$

For any subset $S \subseteq \mathbf{A}_{\widehat{X}}$ we define the *orthogonal complement of S with respect to d_ω*:

$$S^\perp := \{\beta \in \mathbf{A}_{\widehat{X}} \colon d_\omega(S, \beta) = 1\}\,. \tag{7}$$

The operator $\perp$ in this section shouldn't be confused with the one for topological groups and their duals.

Proposition 5.1. *The map d_ω has the following properties:*

(1) *It is symmetric and sequentially continuous.*
(2) *For any couple of subgroups $H_1, H_2 \subseteq \mathbf{A}_{\widehat{X}}$ we have $H_1^\perp \cap H_2^\perp = (H_1 + H_2)^\perp$.*

Proof. (1) Symmetry is obvious, sequential continuity follows easily from the fact that ξ^ω and the product are sequentially continuous.

(2) If $h \in H_1^\perp \cap H_2^\perp$, then $d_\omega(h, H_1 + H_2) = d(h, H_1) + d(h, H_2) = 1$, so one inclusion is proved. Vice versa assume that $h \in (H_1 + H_2)^\perp$, then $d(h, H_i) = 0$ for $i = 1, 2$, so we have also the other inclusion. □

Now we show that the spaces $A_{\widehat{01}}$ and $A_{\widehat{02}}$ are equal to their orthogonal complements. Compare these results with the “geometric counterpart” in [10].

Theorem 5.2. $A_{\widehat{01}}^\perp = A_{\widehat{01}}$.

Proof. We show the equality by showing two inclusions. First we prove that $A_{\widehat{01}} \subseteq A_{\widehat{01}}^{\perp}$. It is essentially a consequence of our reciprocity laws for completed arithmetic surfaces. We have to show that for any $\alpha, \beta \in A_{\widehat{01}}$, $d_\omega(\alpha, \beta) = \xi^\omega(\alpha\beta) = 1$. Let $a = \alpha\beta$, then

$$\xi^\omega(a) = \prod_{\substack{x \in \overline{y},\\ \overline{y} \subset \widehat{X}}} \mathrm{Cres}_{x,\overline{y}}(\omega a_{x,\overline{y}}) \prod_{\substack{p \in X_\sigma,\\ \sigma \in B_\infty}} \mathrm{Cres}_{p,\sigma}(\omega a_{p,\sigma}) .$$

The first product is equal to 1 thanks to Proposition 3.6(2); the second product is 1 thanks to the one dimensional reciprocity law.

Next we show the inclusion $A_{\widehat{01}}^{\perp} \subseteq A_{\widehat{01}}$. We take an element $a = (a_{\overline{y}}) \times (a_\sigma) \in A_{\widehat{01}}^{\perp}$. We need to show that $a_{\overline{y}} \in K_{\overline{y}}$ and $a_\sigma \in \mathbb{A}_0(\sigma)$. We consider 3 cases.

Curves at infinity. Pick any $g \in \mathbb{A}_0(\sigma) \subset A_{\widehat{01}}$, then since $a \in A_{\widehat{01}}^{\perp}$

$$d_\omega(a, g) = \prod_{p \in X_\sigma} \mathrm{Cres}_{p,\sigma}(a_{p,\sigma} g\omega) = 1 .$$

If ψ is the standard character of $\mathbb{C}$, it follows that

$$\sum_{p \in X_\sigma} \mathrm{res}_{p,\sigma}(a_{p,\sigma} g\omega) \in \ker \psi = \frac{1}{2}\mathbb{Z} + \mathbb{R}i, \quad \forall g \in A_0(\sigma) . \tag{8}$$

By equation (8) for any $\lambda \in \mathbb{R}$ we have

$$\sum_{p \in X_\sigma} \mathrm{res}_{p,\sigma}(a_{p,\sigma} \lambda\omega) = \lambda \sum_{p \in X_\sigma} \mathrm{res}_{p,\sigma}(a_{p,\sigma}\omega) \in \frac{1}{2}\mathbb{Z} + \mathbb{R}i .$$

It follows that $\sum_{p \in X_\sigma} \mathrm{res}_{p,\sigma}(a_{p,\sigma}\omega) = 0$. We can replace a_σ with $a_\sigma h$ for any $h \in A_0(\sigma)$ to get $\sum_{p \in X_\sigma} \mathrm{res}_{p,\sigma}(a_{p,\sigma} h\omega) = 0$. In other words a_σ lies in the orthogonal complement of $A_0(\sigma)$ with respect to the pairing:

$$\begin{aligned} T_\omega : \mathbf{A}_{X_\sigma} \times \mathbf{A}_{X_\sigma} &\to \mathbb{C} \\ ((\alpha_{p,\sigma}), (\beta_{p,\sigma})) &\mapsto \sum_{p \in X_\sigma} \mathrm{res}_{p,\sigma}(\alpha_{p,\sigma}\beta_{p,\sigma}\omega) \end{aligned}$$

But we know that $\mathbb{A}_0(\sigma)$ is equal to its orthogonal complement (with respect to T_ω). Such a result was proved for number fields in [23, Theorem 4.1.4], but see for example [7, Theorem 2.21] for the function field case. Therefore we conclude that $a_\sigma \in \mathbb{A}_0(\sigma)$.

$\overline{y}$ horizontal. We know that $\mathbb{A}_{\overline{y}} = \mathbf{A}_{\overline{y}}((t_y))$, where t_y is a local parameter for $K_{\overline{y}} \subset A_{\widehat{01}}$ and therefore any $a_{\overline{y}}$ has the following expression:

$$\mathbb{A}_{\overline{y}} = \mathbf{A}_{\overline{y}}((t)) \ni a_{\overline{y}} = \sum_{i \geq m} a_i t_y^i \quad \text{with } a_i \in \mathbf{A}_{\overline{y}} .$$

We can also take $\omega = f dt_y$. Then for any $r \in \mathbb{Z}$ and any $c \in k(y)$:

$$d_\omega(a_{\overline{y}}, cf^{-1}t_y^r) = \prod_{x\in\overline{y}} \mathrm{Cres}_{x,\overline{y}}(a_{x,\overline{y}} t_y^r c dt_y) = \xi^{dt_y}(a_{\overline{y}} t_y^r c) = 1\,. \tag{9}$$

Then $\xi^{dt_y}(a_{\overline{y}} t_y^r c)$ is a standard character of the one dimensional adeles $\mathbf{A}_{\overline{y}}$ calculated at ca_{-r-1}. Since equation (9) holds for every $r \in \mathbb{Z}$, and $k(y)$ is equal to $k(y)^\perp$ in $\mathbf{A}_{\overline{y}}$ (again [23, Theorem 4.1.4]), we can conclude that $a_i \in k(y)$ for every i. This means that $a_{\overline{y}} \in K_{\overline{y}}$.

$\overline{y} = y$ *vertical.* Let $\overline{t} \in k(y)$ be a uniformizing parameter and consider a lift $t \in \mathscr{O}_{X,y}$. Put $\omega = f dt$ and let $L = K_p\{\{t\}\}$ a standard subfield of $K_{x,y}$. We know that there exists an integer s such that $\mathrm{Tr}_{K_p|K_b}(\mathfrak{p}_{K_p}^s) \subseteq \mathfrak{p}_{K_b}$. Fix $r \in \mathbb{Z}$ such that $\mathrm{Tr}_{K_{x,y}|L}(p^r a_{x,y}) \in \mathfrak{p}_L^s$. For any $m \in \mathbb{Z}$ we have $f^{-1}p^r t^m \in K_y \subset A_{\widehat{01}}$, so since $a \in A_{\widehat{01}}^\perp$ we obtain that

$$d(a, f^{-1}p^r t^m) = \sum_{x\in y} \mathrm{res}_{x,y}(p^r a_{x,y} \cdot t^m dt) \in \mathcal{O}_b\,.$$

Since $\mathrm{Tr}_{K_{x,y}|L}(p^r a_{x,y}) \in \mathfrak{p}_L^s$ and $\mathrm{Tr}_{K_p|K_b}(\mathfrak{p}_{K_p}^s) \subseteq \mathfrak{p}_{K_b}$:

$$\sum_{x\in y} \overline{\mathrm{res}_{x,y}(p^r a_{x,y} \cdot t^m dt)} = 0.$$

Now we apply [16, Corollary 2.23] to write

$$0 = \overline{\sum_{x\in y} \mathrm{res}_{x,y}(p^r a_{x,y} \cdot t^m dt)} = \sum_{x\in y} \overline{\mathrm{res}_{x,y}(p^r a_{x,y} \cdot t^m dt)} =$$

$$= \sum_{x\in y} e_{x,y}\, \mathrm{res}_{x,y}^{(1)}(\overline{p^r a_{x,y}} \cdot \overline{t^m} d\overline{t}) = \sum_{x\in y} \mathrm{res}_{x,y}^{(1)}(e_{x,y}\overline{p^r a_{x,y}} \cdot \overline{t}^m d\overline{t})$$

where:

- $\mathrm{res}_{x,y}^{(1)} : E_{x,y} \to k(b)$ is the one dimensional residue on $E_{x,y}$.
- $e_{x,y} := e(K_{x,y}|K_b)$ is the ramification degree.

The above relation holds for any $m \in \mathbb{Z}$, and moreover we apply the same one-dimensional argument used in the case of the curves at infinity to conclude that $k(y)$ is equal to $k(y)^\perp$ in $\mathbf{A}_y$. It follows that $(\overline{a_{x,y}})_{x\in y} \in k(y)$, therefore $a_y \in K_y$. □

Before proving that $A_{\widehat{02}}$ is self-orthogonal we need to study with more detail the structure of a neighborhood of a point $x \in X$ such that $\varphi(x) = b$. Let's denote with $\mathrm{Spec}^1 \mathcal{O}_x$ the set of prime ideals of height 1 in $\mathcal{O}_x$, then a curve y passing by x corresponds to the set of local branches $y(x) \subset \mathrm{Spec}^1 \mathcal{O}_x$. But there might be some elements $\mathfrak{q} \in$

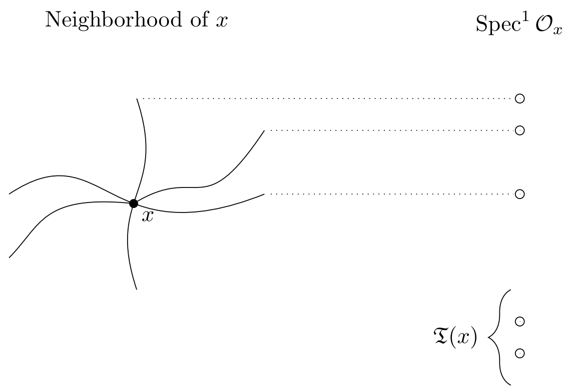


Fig. 4. A visual representation of the correspondence between prime ideals of $\mathcal{O}_x$ and curves passing by x. For simplicity we assumed that the curves are nonsingular at x, hence $y(x)$ is exactly a point in $\operatorname{Spec} \mathcal{O}_x$ for any y.

$\operatorname{Spec}^1 \mathcal{O}_x$ which don't correspond to any curve passing by x, those are exactly those ideals:

$$\mathfrak{T}(x) := \{\mathfrak{q} \in \operatorname{Spec}^1 \mathcal{O}_x \colon \mathfrak{q} \cap \mathscr{O}_{X,x} = (0)\}\,.$$

The elements of $\mathfrak{T}(x)$ are called *transcendental curves (passing by x)*. (See Fig. 4.)

Also for any $\mathfrak{q} \in \mathfrak{T}(x)$ it is possible to construct a 2-dimensional local field $K_{x,\mathfrak{q}}$ and the residues $\operatorname{res}_{x,\mathfrak{q}} : \Omega_{x,\mathfrak{q}} \to K_b$, $\operatorname{Cres}_{x,\mathfrak{q}} : \Omega_{x,\mathfrak{q}} \to \mathbb{T}$ in the usual way. But transcendental curves have the following pathological behavior:

Lemma 5.3. *Fix $\omega \in \Omega^1_{K(X)|K}$ and let $\mathfrak{q} \in \mathfrak{T}(x)$, then $\operatorname{Cres}_{x,\mathfrak{q}}(\omega) = 1$. Moreover if $g \in K'_x$, then $\operatorname{Cres}_{x,\mathfrak{q}}(g\omega) = 1$.*

Proof. The first claim follows immediately from the fact that $K(X) \subseteq (\mathcal{O}_x)_\mathfrak{q}$. For the second one it is enough to notice that $K_x = K(X)\mathcal{O}_x$ is sequentially dense in K'_x and use the first part of the lemma. □

The presence of transcendental curves is a subtlety in the adelic theory. In fact, in general K_x is obviously a proper subring of K'_x, but the following result from commutative algebra ensures that K_x and K'_x coincide if and only if there are no transcendental curves passing by x.

Proposition 5.4. *Let A be a Noetherian, regular, local domain and let $\widehat{A}$ be the completion with respect to its maximal ideal. Then the product $\widehat{A} \operatorname{Frac} A$ is a field if and only if for any nonzero prime $\mathfrak{q} \subset \widehat{A}$, $\mathfrak{q} \cap A \neq (0)$.*

Proof. Since A is regular and local, also $\widehat{A}$ is regular and local, which implies that $\widehat{A}$ and $\widehat{A} \operatorname{Frac} A$ are integral domains as well. It follows that $\widehat{A} \otimes_A \operatorname{Frac} A \cong \widehat{A} \operatorname{Frac} A$. Then it is well known (e.g. [14, p. 47]) that we have an homeomorphism:

$$\operatorname{Spec}(\widehat{A} \otimes_A \operatorname{Frac} A) \cong S := \left\{ \mathfrak{q} \in \operatorname{Spec} \widehat{A} \colon \mathfrak{q} \cap A = (0) \right\} . \tag{10}$$

($\Rightarrow$) $\widehat{A} \otimes_A \operatorname{Frac} A$ contains only a prime ideal, the trivial one, so by the homeomorphism (10), S contains only one element, which is (0). ($\Leftarrow$) If S contains only (0), then by the homeomorphism (10) the only prime ideal of $\widehat{A} \otimes_A \operatorname{Frac} A$ is (0), which means that $\widehat{A} \otimes_A \operatorname{Frac} A$ is a field. □

Corollary 5.5. *Fix a closed point $x \in X$, then $K_x = K'_x$ if and only if there are no transcendental curves passing by x.*

Proof. By definition $K_x = K(X)\mathcal{O}_x$ is the smallest ring containing $K(X)$ and $\mathcal{O}_x$, so the claim follows from Proposition 5.4. □

Now let's put

$$\mathbf{A}_{X,x} := \mathbf{A}_X \cap \prod_{y \ni x} K_{x,y} ,$$

$$\mathbf{A}'_{X,x} := \prod_{\mathfrak{q} \in \operatorname{Spec}^1 \mathcal{O}_x}{}' K_{x,\mathfrak{q}} \quad \text{with resp. to } \mathcal{O}_{x,\mathfrak{q}} ,$$

and note that $\mathbf{A}'_{X,x} \supseteq \mathbf{A}_{X,x}$. Lemmas 5.8 and 5.9 below will be used to show the inclusion $A_{\widehat{02}}^{\perp} \subseteq A_{\widehat{02}}$. The first one will be a modified version of [12, Lemma 3.3], so we present a proof. The second one will be just [12, Lemma 3.4] rewritten with our notation, so for its proof we remand the reader to the appropriate reference.

Remark 5.6. The paper [12] shows only some local calculations regarding residues on the space $\mathbf{A}'_{X,x}$. Moreover the space denoted as K_x in [12] is exactly our K'_x.

Lemma 5.7. *Let R be a ring, then*

$$\operatorname{Frac}(R[[t]]) = F := \left\{ \sum_{i \geq m} a_i t^i \in \operatorname{Frac}(R)((t)) \colon \exists r \in R \text{ such that } a_i \in R[1/r] , \forall i \right\} .$$

In particular, we deduce that in general $\operatorname{Frac}(R[[t]])$ *is strictly contained in* $\operatorname{Frac}(R)((t))$.

Proof. Since F is a field containing $R[[t]]$, we have to show only the inclusion $\operatorname{Frac}(R[[t]]) \subseteq F$. Let $\phi(t) = \frac{f(t)}{g(t)} \in \operatorname{Frac}(R[[t]])$ with $f(t), 0 \neq g(t) \in R[[t]]$. Write $g(t) = t^k(r - t\gamma(t)) = t^k r(1 - \frac{t}{r}\gamma(t))$ with $k \geq 0$, $0 \neq r \in R$ and $\gamma(t) \in R[[t]]$. Then $\frac{1}{g(t)} = t^{-k} \sum \frac{t^n}{r^n}(\gamma(t))^n$ and $\phi(t) = \sum_{i=m}^{\infty} c_i t^i$ where $m \in \mathbb{Z}$ depends on ϕ and each c_i is of the form $c_i = \frac{\rho_i}{r^{\nu_i}}$ with $\rho_i \in R$ and $\nu_i \in \mathbb{N}$. □

The morphism $\varphi : X \to B$ sending x to b induces a finite ring extension $\mathcal{O}_b[[t]] \hookrightarrow \mathcal{O}_x$, therefore from now on we can always identify $\mathcal{O}_b[[t]]$ with its image in $\mathcal{O}_x$ when $\varphi(x) = b$.

Lemma 5.8. *Assume that $\mathcal{O}_x = \mathcal{O}_b[[t]]$. Fix a rational differential form $\omega \in \Omega^1_{K(X)|K}$ and let $a = (a_{x,\mathfrak{q}}) \in \mathbf{A}'_{X,x}$ such that:*

$$\prod_{\mathfrak{q}\in \operatorname{Spec}^1 \mathcal{O}_x} \operatorname{Cres}_{x,\mathfrak{q}}(g a_{x,\mathfrak{q}} \omega) = 1 \quad \textit{for any } g \in K'_x\,, \tag{11}$$

then $a \in K'_x$.

Proof. There is a well known classification result for the elements $\mathfrak{q} \in \operatorname{Spec}^1 \mathcal{O}_b[[t]] = \operatorname{Spec}^1 \mathcal{O}_x$ (see for example [19, Lemma 5.3.7]):

- $\mathfrak{q} = \mathfrak{q}_v := \pi_b \mathcal{O}_x$, where π_b is the uniformizing parameter of $\mathcal{O}_x$. This is the only prime ideal such that $K_{x,\mathfrak{q}_v}$ is of mixed characteristic.
- $\mathfrak{q} = h_\mathfrak{q} \mathcal{O}_x$ where $h_\mathfrak{q} \in \mathcal{O}_b[t]$ is an irreducible Weierstrass polynomial, i.e. $h_\mathfrak{q} = t^d + a_1 t^{d-1} + \ldots + a_d$ with $a_i \in \mathfrak{p}_{K_b}$.

Without loss of generality we may assume that $a_{x,\mathfrak{q}} \in \mathcal{O}_{x,\mathfrak{q}}$ for any $\mathfrak{q} \neq \mathfrak{q}_v$ since multiplying $(a_{x,\mathfrak{q}})$ by any nonzero element in K'_x amounts to an equivalent problem. Moreover, for the same reason we can also assume for simplicity that $\omega = 1dt$.

For any $\mathfrak{q} \neq \mathfrak{q}_v$ and any uniformizing parameter $\pi_\mathfrak{q}$ for the 2-dimensional local fields $K_{x,\mathfrak{q}}$, we can choose the following isomorphism:

$$\begin{aligned} K_{x,\mathfrak{q}} &\xrightarrow{\cong} E_{x,\mathfrak{q}}((h_\mathfrak{q})) \\ \pi_\mathfrak{q} &\mapsto h_\mathfrak{q}(t)\,. \end{aligned}$$

In other words t can be identified with a root of the polynomial equation $h_\mathfrak{q}(t) = \pi_\mathfrak{q}$. By Hensel's lemma we deduce that such a root exists and it is integral, thus we can write:

$$t = \sum_{i\geq 0} c_i \pi_\mathfrak{q}^i \quad \text{with } c_i \in E_{x,\mathfrak{q}}\,.$$

The following two easy results are fundamental:

(*i*) $h_\mathfrak{q} \in \mathcal{O}^\times_{x,\mathfrak{q}'}$ for any $\mathfrak{q}' \neq \mathfrak{q}, \mathfrak{q}_v$. This is obvious from the definition of $\mathcal{O}^\times_{x,\mathfrak{q}'}$.

(*ii*) $t \in \mathcal{O}_{x,\mathfrak{q}'}$ for any $\mathfrak{q}' \neq \mathfrak{q}$. Assume by contradiction that $t \notin \mathcal{O}_{x,\mathfrak{q}'}$ and let $h_\mathfrak{q} = t^d + a_1 t^{d-1} + \ldots + a_d$, then by (*i*):

$$\begin{aligned} 0 = v_{x,\mathfrak{q}'}(t^d + a_1 t^{d-1} + \ldots + a_d) &= \min\left\{v_{x,\mathfrak{q}'}(t^d), v_{x,\mathfrak{q}'}(a_1 t^{d-1}), \ldots, v_{x,\mathfrak{q}'}(a_d)\right\} = \\ &= \min\left\{d v_{x,\mathfrak{q}'}(t), (d-1) v_{x,\mathfrak{q}'}(t), \ldots, 0\right\} = d v_{x,\mathfrak{q}'}(t) < 0 \end{aligned}$$

which cannot be true.

If for any $\mathfrak{q}' \neq \mathfrak{q}_v$ we write

$$a_{x,\mathfrak{q}'} = \sum_{i \geq 0} a_{i,\mathfrak{q}'} \pi_{\mathfrak{q}'}^i \,, \quad a_{i,\mathfrak{q}'} \in E_{x,\mathfrak{q}'} \,,$$

by (i)–(ii) and equation (11), for any $n \geq 0$ we have

$$\prod_{\mathfrak{q}' \in \mathrm{Spec}^1 \mathcal{O}_x} \mathrm{Cres}_{x,\mathfrak{q}'}(h_{\mathfrak{q}}^{-1} t^n a_{x,\mathfrak{q}'} \omega) = \mathrm{Cres}_{x,\mathfrak{q}}(h_{\mathfrak{q}}^{-1} t^n a_{x,\mathfrak{q}} \omega) \cdot \mathrm{Cres}_{x,\mathfrak{q}_v}(h_{\mathfrak{q}}^{-1} t^n a_{x,\mathfrak{q}_v} \omega) = 1 \,.$$

Therefore, we have the equality

$$\mathrm{Cres}_{x,\mathfrak{q}}(h_{\mathfrak{q}}^{-1} t^n a_{x,\mathfrak{q}} \omega) = \mathrm{Cres}_{x,\mathfrak{q}_v}(h_{\mathfrak{q}}^{-1} t^n a_{x,\mathfrak{q}_v} \omega)^{-1} \,, \tag{12}$$

but by definition

$$\mathrm{Cres}_{x,\mathfrak{q}}(h_{\mathfrak{q}}^{-1} t^n a_{x,\mathfrak{q}} \omega) = \psi_b \left(\mathrm{Tr}_{E_{x,\mathfrak{q}}|K_b}(c_0^n a_{0,\mathfrak{q}}) \right) \,. \tag{13}$$

Since we can take $1, c_0, ..., c_0^{\deg h_{\mathfrak{q}} - 1}$ as a basis of $E_{x,\mathfrak{q}}$ over K_b, equations (12) and (13) imply that $\mathrm{Tr}_{E_{x,\mathfrak{q}}|K_b}(\lambda a_{0,\mathfrak{q}})$ is determined by $a_{x,\mathfrak{q}_v}$ for any $\lambda \in E_{x,\mathfrak{q}}$. By using non-degeneracy of the trace pairing

$$\begin{aligned} E_{x,\mathfrak{q}} \times E_{x,\mathfrak{q}} &\to K_b \\ (u, s) &\mapsto \mathrm{Tr}_{E_{x,\mathfrak{q}}|K_b}(us) \end{aligned}$$

we conclude that the element $a_{0,\mathfrak{q}}$ is uniquely determined by $a_{x,\mathfrak{q}_v}$. We can conduct the same calculations for $h_{\mathfrak{q}}^{-i-1} t^n a_{x,\mathfrak{q}}$, to see that $a_{i,\mathfrak{q}}$ is determined by $a_{x,\mathfrak{q}_v}$ for any positive integer i. It leads us to a conclusion that $a_{x,\mathfrak{q}}$ is uniquely determined by $a_{x,\mathfrak{q}_v}$ for any $\mathfrak{q} \neq \mathfrak{q}_v$.

So, we are reduced to show that $a_{x,\mathfrak{q}_v}$ is in K_x'. Recall that $K_{x,\mathfrak{q}_v} \cong K_b\{\{t\}\}$, so we can write

$$a_{x,\mathfrak{q}_v} = \sum_{i \in \mathbb{Z}} a_{i,\mathfrak{q}_v} t^i \,, \quad a_{i,\mathfrak{q}_v} \in K_b \,.$$

Now, by putting $\mathfrak{p}_0 = t\mathcal{O}_x$ and reasoning similarly as above we obtain

$$\mathrm{Cres}_{x,\mathfrak{q}_v}(t^{i-1} a_{x,\mathfrak{q}_v} \omega)^{-1} = \mathrm{Cres}_{x,\mathfrak{p}_0}(t^{i-1} a_{x,\mathfrak{p}_0} \omega) = 1 \,, \quad \text{for all } i \geq 1 \,.$$

It means that $a_{-i,\mathfrak{q}_v}^{-1} \in \mathcal{O}_b$ for any $i \geq 1$. By definition of $K_b\{\{t\}\}$, we know that there exists $N > 0$ such that $a_{-i,\mathfrak{q}_v} \in \mathcal{O}_b$ for $i \geq N$. In other words if $i \geq N$ and $a_{-i,\mathfrak{q}_v} \neq 0$, then $a_{-i,\mathfrak{q}_v} \in \mathcal{O}_b^{\times}$. Since $\lim_{j \to -\infty} a_{j,\mathfrak{q}_v} = 0$, we conclude that it has to exist $M > 0$ such that $a_{-i,\mathfrak{q}_v} = 0$ for $i \geq M$. This proves that $a_{x,\mathfrak{q}_v} \in K_b((t))$. Again thanks to the definition of $K_b\{\{t\}\}$, we know that there exists $m \in \mathbb{Z}$ such that $v_b(a_{i,\mathfrak{q}_v}) \geq m$, which means that for any choice of uniformizing parameter $s \in \mathcal{O}_b$, then $a_{i,\mathfrak{q}_v} = s^{m+j} g$ with

$j \geq 0$ and $g \in \mathcal{O}_b^\times$. We distinguish two cases:

- If $m < 0$, then $a_{i,\mathfrak{q}_v} = (1/s)^{-m} \cdot s^j g \in \mathcal{O}_b[1/s]$
- If $m \geq 0$, then $a_{i,\mathfrak{q}_v} = (1/s) \cdot s^{m+j+1} g \in \mathcal{O}_b[1/s]$

Thanks to Lemma 5.7 we conclude that $a_{x,\mathfrak{q}_v} \in K'_x$. □

Let $x \in X$ such that $\varphi(x) = b$, then we put $\mathcal{O}_x^\# := \mathcal{O}_b[[t]] \subseteq \mathcal{O}_x$ (recall that $\mathcal{O}_b[[t]]$ is canonically embedded in $\mathcal{O}_x$). For any prime $u \in \operatorname{Spec}^1 \mathcal{O}_x^\#$ we have the 2-dimensional local field $K_{x,u}^\#$ obtained by the usual process of completion/localization. In general we can construct all local adelic objects relative to the flags $x \in u \in \operatorname{Spec}^1 \mathcal{O}_x^\#$. Such objects arising from the special ring $\mathcal{O}_x^\#$ will be marked with the symbol # to distinguish them from the usual ones. Let $\mathfrak{q} \in \operatorname{Spec}^1 \mathcal{O}_x$ be a prime sitting over u, then we have a finite field extension $K_{x,\mathfrak{q}}|K_{x,u}^\#$ and a trace map $\operatorname{Tr}_{K_{x,\mathfrak{q}}|K_{x,u}^\#}$ which extends directly at the level of differential forms:

$$\begin{aligned}\operatorname{Tr}_{K_{x,\mathfrak{q}}|K_{x,u}^\#} : \Omega^1_{x,\mathfrak{q}} &\to \Omega^{1,\#}_{x,u} \\ f dt &\mapsto \operatorname{Tr}_{K_{x,\mathfrak{q}}|K_{x,u}^\#}(f) dt\end{aligned}$$

Such a map is exactly the abstract trace map for differential forms defined in [16] and mentioned in section 3. We recall that in [16] it is also proved that the residue is functorial with respect to the trace, which in our case means that $\operatorname{res}_{x,\mathfrak{q}} = \operatorname{res}^\#_{x,u} \circ \operatorname{Tr}_{K_{x,\mathfrak{q}}|K_{x,u}^\#}$. The local trace map defined above can be further generalized to an adelic trace:

$$\begin{aligned}\operatorname{Tr}_x \colon\; \mathbf{A}'_{X,x} &\to \left(\mathbf{A}'_{X,x}\right)^\# \\ (a_{x,\mathfrak{q}})_\mathfrak{q} &\mapsto \left(\sum_{\mathfrak{q}|u} \operatorname{Tr}_{K_{x,\mathfrak{q}}|K_{x,u}^\#}(a_{x,\mathfrak{q}})\right)_u\end{aligned}$$

where with the notation $\mathfrak{q}|u$ we denote all ideals $\mathfrak{q} \in \operatorname{Spec}^1 \mathcal{O}_x$ sitting over u.

Lemma 5.9. *Let $f \in \mathbf{A}'_{X,x}$ such that $\operatorname{Tr}_x(fg) \in (K'_x)^\#$ for any $g \in K'_x$, then $f \in K'_x$.*

Proof. See [12, Lemma 3.4]. □

Theorem 5.10. $A_{\widehat{02}}^\perp = A_{\widehat{02}}$.

Proof. First of all let's prove that $A_{\widehat{02}} \subseteq A_{\widehat{02}}^\perp$. We have to show that for any $\alpha, \beta \in A_{\widehat{02}}$, $d_\omega(\alpha, \beta) = \xi^\omega(\alpha\beta) = 1$. Let $a = \alpha\beta$, then

$$\begin{aligned}\xi^\omega(a) &= \prod_{\substack{x \in \overline{y}, \\ \overline{y} \subset \widehat{X}}} \operatorname{Cres}_{x,\overline{y}}(\omega a_{x,\overline{y}}) \prod_{\substack{p \in X_\sigma, \\ \sigma \in B_\infty}} \operatorname{Cres}_{p,\sigma}(\omega a_{p,\sigma}) = \\ &= \prod_{\substack{x \in X, \\ \overline{y} \ni x}} \operatorname{Cres}_{x,\overline{y}}(\omega a_{x,\overline{y}}) \prod_{\substack{p \in X_\sigma, \\ \overline{y} \ni p, \\ \sigma \in B_\infty}} \operatorname{Cres}_{p,\sigma}(\omega a_{p,\sigma}) \operatorname{Cres}_{p,\overline{y}}(\omega a_{p,\overline{y}}) .\end{aligned}$$

We can conclude $\xi^\omega(a) = 1$ thanks to Proposition 3.6(1) and from the explicit definition of $A_{\widehat{02}}$ at infinity.

Now we show the inclusion $A_{\widehat{02}}^\perp \subseteq A_{\widehat{02}}$. Fix $a = (a_{x,\overline{y}}) \times (a_{p,\sigma}) \in A_{\widehat{02}}^\perp$ and assume $\omega = f dt$, we consider two cases:

$x = p$ is a point on X_σ. For any $g \in \mathbb{C}((t))$ we consider the element $(f^{-1}g, f^{-1}g) \in \Delta_{p,\sigma}$, then if $\overline{y}$ is the unique horizontal curve containing p we obtain

$$\mathrm{Cres}_{p,\overline{y}}(a_{p,\overline{y}} f^{-1} g\omega) \cdot \mathrm{Cres}_{p,\sigma}(a_{p,\sigma} f^{-1} g\omega) = 1\,.$$

This means

$$\mathrm{res}_{p,\overline{y}}(a_{p,\overline{y}} g dt) - \mathrm{res}_{p,\sigma}(a_{p,\sigma} g dt) \in \ker \psi_\sigma = \frac{1}{2}\mathbb{Z} + \mathbb{R} i\,. \tag{14}$$

Since equation (14) holds for any $g \in \mathbb{C}((t))$, it is clear that it must be $(a_{p,\overline{y}}, a_{p,\sigma}) \in \Delta_{p,\sigma}$.

x is a point on X. Recall that $\mathcal{O}_x$ is a finite ring extension of $\mathcal{O}_b[[t]]$.

We first treat the case where there are transcendental curves passing by x; let's extend the element $(a_{x,\overline{y}})_{\overline{y}\ni x}$ to an element $(a'_{x,\mathfrak{q}})_\mathfrak{q} \in \mathbf{A}'_{X,x}$ in the following way: for a transcendental curve $\mathfrak{q} \in \mathfrak{T}(x)$ let's insert $a'_{x,\mathfrak{q}} \in K_x$; at all other primes nothing changes. Now let $g \in K_x$, then:

$$\prod_{\mathfrak{q}\in \mathrm{Spec}^1 \mathcal{O}_x} \mathrm{Cres}_{x,\mathfrak{q}}(a'_{x,\mathfrak{q}} g\omega) = \underbrace{\prod_{\overline{y}\ni x} \mathrm{Cres}_{x,\overline{y}}(a_{x,\overline{y}} g\omega)}_{(i)} \underbrace{\prod_{\mathfrak{q}\in\mathfrak{T}(x)} \mathrm{Cres}_{x,\mathfrak{q}}(a'_{x,\mathfrak{q}} g\omega)}_{(ii)} = 1 \tag{15}$$

where $(i) = 1$ because $(a_{x,\overline{y}}) \times (a_{p,\sigma}) \in A_{\widehat{02}}^\perp$ and $(ii) = 1$ thanks to Lemma 5.3. Since K_x is sequentially dense in K'_x, equation (15) implies that for any $h \in K'_x$

$$\prod_{\mathfrak{q}\in \mathrm{Spec}^1 \mathcal{O}_x} \mathrm{Cres}_{x,\mathfrak{q}}(a'_{x,\mathfrak{q}} h\omega) = 1\,. \tag{16}$$

Now we use equation (16) and the functoriality of the residue with respect to the trace map:

$$\mathcal{O}_b \ni \sum_{\mathfrak{q}} \mathrm{res}_{x,\mathfrak{q}}(a'_{x,\mathfrak{q}} h\omega) = \sum_u \sum_{\mathfrak{q}|u} \mathrm{res}^{\#}_{x,u}\left(\mathrm{Tr}_{K_{x,\mathfrak{q}}|K^{\#}_{x,u}}(a'_{x,\mathfrak{q}} h\omega)\right) =$$

$$= \sum_u \mathrm{res}^{\#}_{x,u}\left(\sum_{\mathfrak{q}|u} \mathrm{Tr}_{K_{x,\mathfrak{q}}|K^{\#}_{x,u}}(a'_{x,\mathfrak{q}} h\omega)\right) = \sum_u \mathrm{res}^{\#}_{x,u}\left(\sum_{\mathfrak{q}|u} \mathrm{Tr}_{K_{x,\mathfrak{q}}|K^{\#}_{x,u}}(a'_{x,\mathfrak{q}}) h\omega\right)\,.$$

By Lemma 5.8 we can conclude that $\mathrm{Tr}_x\left(a'_{x,\mathfrak{q}}\right)_\mathfrak{q} \in (K')^{\#}_x$ diagonally. By replacing $a_{x,\overline{y}}$ with $ca'_{x,\overline{y}}$ for any $c \in K'_x$ we can again conclude that $\mathrm{Tr}_x\left(ca'_{x,\mathfrak{q}}\right)_\mathfrak{q} \in (K')^{\#}_x$ diagonally.

At this point we can use Lemma 5.9 to conclude that $(a'_{x,\mathfrak{q}})_\mathfrak{q} \in K'_x$. It means that $(a_{x,\overline{y}})_{\overline{y}\ni x} \in K_x$ by the choice of $(a'_{x,\mathfrak{q}})_\mathfrak{q}$.

If there are no transcendental curves passing by x, then $\mathbf{A}'_{X,x} = \mathbf{A}_{X,x}$ and $K_x = K'_x$ by Corollary 5.5. Then we can apply a simplified version of the argument used above to conclude the proof. □

Remark 5.11. We were informed by I. Fesenko that there is an alternative proof of Theorem 5.10 which uses an arithmetic version of his argument in [10].

6. Idelic interpretation of Arakelov intersection theory

A prerequisite for this section is the whole appendix B. In [8], it is described how to get a lift of the Deligne pairing (i.e. the schematic part of the Arakelov intersection number) at the level of ideles. Let's summarize the result: first of all we consider the idelic complex attached to the (uncompleted) surface X

$$\begin{array}{llll} \mathcal{A}_X^\times : & A_0^\times \oplus A_1^\times \oplus A_2^\times \xrightarrow{d^0_\times} & A_{01}^\times \oplus A_{02}^\times \oplus A_{12}^\times \xrightarrow{d^1_\times} & A_{012}^\times \\ & (a_0, a_1, a_2) \longmapsto & (a_0 a_1^{-1}, a_2 a_0^{-1}, a_1 a_2^{-1}) & \\ & & (a_{01}, a_{02}, a_{12}) \longmapsto & a_{01}a_{02}a_{12} \end{array} \tag{17}$$

and we note that we have a surjective map:

$$\begin{aligned} p: \quad & \ker(d^1_\times) \to \operatorname{Div}(X) \\ (\alpha, \beta, \alpha^{-1}\beta^{-1}) & \mapsto \sum_{y\subset X} v_y(\alpha_{x,y})[y] \, . \end{aligned}$$

Then by globalizing the Kato's symbol, we define ad idelic Deligne pairing $\langle\, , \rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \operatorname{Pic}(B)$ which descends to the Deligne pairing $\langle\, , \rangle : \operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(B)$. In turn, the Deligne pairing is strictly related to intersection theory because for any two divisors $D, E \in \operatorname{Div}(X)$, the class in $\operatorname{Pic}(B)$ of the divisor

$$\langle D, E\rangle = \varphi_* i(D,E) = \sum_{x\in X} [k(x) : k(\varphi(x))] \, i_x(D,E) \, [\varphi(x)]$$

is equal to $\langle \mathscr{O}_X(D), \mathscr{O}_X(D)\rangle$. Note that we have used the brackets $\langle\, , \rangle$ to denote two different (but strictly related) objects, but the clash of notations shouldn't confuse the reader.

The contribution at infinity to the Arakelov intersection pairing is given by the $*$-product between Green functions, so the next step in our theory is to find an idelic description of it. The part at infinity of the full adelic ring $\mathbf{A}_X \oplus \prod_{\sigma \in B_\sigma}(\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma})$ is given by $\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma}$ (for each σ), so we want to find a surjective map:

$$(\mathbf{A}_{X_\sigma}^\times \oplus \mathbf{A}_{X_\sigma}^\times) \supseteq S \to \mathbb{Z}G(X_\sigma)$$

where S is an adequate subset of $\mathbf{A}_{X_\sigma}^\times \oplus \mathbf{A}_{X_\sigma}^\times$ still to be determined and $\mathbb{Z}G(X_\sigma)$ is the vector space of Green functions on X_σ with integer orders.

Remark 6.1. First of all let's introduce a notation. For any $a = (a_x) \in \mathbf{A}_{X_\sigma}$, with $a(x)$ we denote the projection of a_x onto the residue field $\mathbb{C}$ (when it is well defined).

Let $\mathcal{F}(X_\sigma, \mathbb{R})'$ be the set of functions $f : U \subseteq X_\sigma \to \mathbb{R}$ whose domain U is the whole X_σ minus a finite set of points, then we have the following map:

$$\begin{aligned}\Theta : \mathbf{A}_{X_\sigma}^\times \times \mathbf{A}_{X_\sigma}^\times &\to \mathcal{F}(X_\sigma, \mathbb{R})' \\ (a, b) &\mapsto -\log(ba\overline{a}) := [x \mapsto -\log\left(b(x)a(x)\overline{a(x)}\right)]\end{aligned}$$

where $\overline{a(x)}$ denotes the complex conjugate. Note that $\mathbb{Z}G(X_\sigma) \subset \mathcal{F}(X_\sigma, \mathbb{R})'$, then put

$$G(\mathbf{A}_{X_\sigma}^\times) := \{(a, b) \in \Theta^{-1}(\mathbb{Z}G(X_\sigma))\colon v_x(a_x) = \mathrm{ord}_x^G(\Theta(a, b)),\ \forall x \in X_\sigma\}\,.$$

We get the map:

$$\pi_\sigma := \Theta|_{G(\mathbf{A}_{X\sigma}^\times)} : G(\mathbf{A}_{X\sigma}^\times) \to \mathbb{Z}G(X_\sigma)\,.$$

Proposition 6.2. *The map π_σ is surjective.*

Proof. Let $g \in \mathbb{Z}G(X_\sigma)$, by Proposition B.6, there exist a C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$ on X and a meromorphic section $s = \{(s_j, U_j)\}$ of $\mathscr{L}$ such that we can write:

$$g = -\log(h(s, s))\,.$$

We can choose $a \in \mathbf{A}_{X_\sigma}^\times$ such that $a(x) = s(x)$ (when $s(x)$ is well defined) and $v_x(a_x) = \mathrm{ord}_x(s)$ for any $x \in X_\sigma$. Now we can write

$$g(x) = -\log(h_x(a(x), a(x)))\,.$$

Since $z \mapsto h_x(z\overline{z})$ is a complex absolute value, we have $h_x(z\overline{z}) = w_x z\overline{z}$ with $w_x \in \mathbb{C}$. Let's choose $b = (b_x) \in \mathbf{A}_{X_\sigma}^\times$ such that $b(x) = w_x$, then

$$g(x) = -\log\left(b(x)a(x)\overline{a(x)}\right)\,.$$

The fact that $v_x(a_x) = \mathrm{ord}_x^G(g)$ follows directly from the fact that for any hermitian metric h and meromorphic section s we have the equality:

$$\mathrm{div}^G(-\log(h(s,s))) = \mathrm{div}(s)\,.$$

(See Proposition B.3.) □

So far, we have the idelic description of Green functions with integer orders thanks to the projection π_σ. Now let's fix a (normalized) Kähler fundamental form Ω_σ on X_σ and consider $G_0^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times) := \pi_\sigma^{-1}(\mathbb{Z}G_0^{\Omega_\sigma}(X_\sigma))$, $G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times) := \pi_\sigma^{-1}(\mathbb{Z}G^{\Omega_\sigma}(X_\sigma))$. For pairs $(\alpha,\beta) \in G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times) \times G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times)$ such that $\mathrm{div}^G(\pi_\sigma(\alpha))$ and $\mathrm{div}^G(\pi_\sigma(\beta))$ have no common components we want to find a product $\alpha *_i \beta$ such that the following equality holds:

$$\begin{array}{ccc} (\alpha,\beta) & & \\ \downarrow & \searrow & \\ (\pi_\sigma(\alpha),\pi_\sigma(\beta)) & \longmapsto & \alpha *_i \beta = \pi_\sigma(\alpha) * \pi_\sigma(\beta) \end{array}$$

As a consequence of the symmetry of the $*$-product we will get also the symmetry of $*_i$. For any $\alpha = (a,b) \in G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times)$ let's put:

$$\xi(\alpha) := e^{\int_{X_\sigma} \log(ba\overline{a})\Omega_\sigma}\,.$$

Definition 6.3. Let $\alpha = (a,b), \beta = (c,d) \in G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times)$, then the idelic $*$-product is defined as:

$$\alpha *_i \beta := -\sum_{x\in X_\sigma} v_x(c_x)\log\Big(b(x)a(x)\overline{a(x)}\xi(\alpha)\Big) + \log(\xi(\alpha))\,\mathrm{ideg}(c) + \log(\xi(\beta))\,\mathrm{ideg}(a)\,,$$

where ideg is the idelic degree map defined as:

$$\begin{aligned} \mathrm{ideg} : \mathbf{A}_{X_\sigma}^\times &\to \mathbb{Z} \\ (\alpha_x)_x &\mapsto \sum_{x\in X_\sigma} v_x(\alpha_x)\,. \end{aligned}$$

Proposition 6.4. $(\alpha,\beta) \in G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times) \times G^{\Omega_\sigma}(\mathbf{A}_{X_\sigma}^\times)$ *such that* $\mathrm{div}^G(\pi_\sigma(\alpha))$ *and* $\mathrm{div}^G(\pi_\sigma(\beta))$ *have no common component; then* $\alpha *_i \beta = \pi_\sigma(\alpha) * \pi_\sigma(\beta)$.

Proof. Put $g_1 = \pi_\sigma(\alpha)$ and $g_2 = \pi_\sigma(\beta)$, then by Proposition B.8 we can write $g_1 = g_{1,0} + c_1$ and $g_2 = g_{2,0} + c_2$ for, $g_{1,0}, g_{2,0} \in G_0^{\Omega_\sigma}(X_\sigma)$, $c_1 = \log(\xi(\alpha))$ and $c_2 = \log(\xi(\beta))$. An easy calculation shows that:

$$g_1 * g_2 = \sum_{x\in X_\sigma} \mathrm{ord}_x^G(g_{2,0})g_{1,0}(x) + c_1 \sum_{x\in X_\sigma} \mathrm{ord}_x^G(g_{2,0}) + c_2 \sum_{x\in X_\sigma} \mathrm{ord}_x^G(g_{1,0})\,.$$

Then it is enough to note the following equalities:

$$\operatorname{ord}_x^G(g_{1,0}) = \operatorname{ord}_x^G(g_1) = v_x(a_x)\,,$$
$$\operatorname{ord}_x^G(g_{2,0}) = \operatorname{ord}_x^G(g_2) = v_x(c_x)\,,$$
$$g_{1,0}(x) = g_1(x) - \log(\xi(\alpha)) = -\log(b(x)a(x)\overline{a(x)}) - \log(\xi(\alpha))\,. \qquad \square$$

Let's write an element $\alpha \in \mathbf{A}_{\widehat{X}}^\times = \mathbf{A}_X^\times \oplus \prod_{\sigma \in B_\infty}(\mathbf{A}_{X_\sigma}^\times \oplus \mathbf{A}_{X_\sigma}^\times)$ in the following way:

$$\alpha = \alpha_X \times (\alpha_\sigma)_\sigma$$

with $\alpha_X \in \mathbf{A}_X^\times$ and $\alpha_\sigma \in \mathbf{A}_{X_\sigma}^\times \oplus \mathbf{A}_{X_\sigma}^\times$, then we have a surjective map:

$$\widehat{p} : \ker(d_\times^1) \oplus \prod_\sigma G(\mathbf{A}_{X_\sigma}^\times) \to \operatorname{Div}(X) \oplus \bigoplus_\sigma G(X_\sigma)$$
$$\alpha = \alpha_X \times (\alpha_\sigma)_\sigma \mapsto \left(p(\alpha_X), \sum_\sigma \pi_\sigma(\alpha_\sigma) X_\sigma\right)$$

where $p : \ker(d_\times^1) \to \operatorname{Div}(X)$ is the usual projection on usual divisors and $\pi_\sigma : G(\mathbf{A}_{X_\sigma}^\times) \to G(X_\sigma)$ is the projection on Green functions.

Definition 6.5. Let's put

$$\operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^\times\right) := \widehat{p}^{\,-1}\left(\operatorname{Div}_{\mathrm{Ar}}(X, \Omega)\right)\,,$$

and let $\alpha, \beta \in \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^\times\right)$ such that $(\widehat{p}(\alpha), \widehat{p}(\beta)) \in \Upsilon_{\mathrm{Ar}}$ then the *idelic Arakelov intersection pairing* is given by

$$\alpha.\beta := \deg\left(\langle \alpha_X, \beta_X \rangle_i\right) + \frac{1}{2}\sum_\sigma \varepsilon_\sigma\, \alpha_\sigma *_i \beta_\sigma$$

where deg is the usual degree of line bundles, $\langle\, ,\, \rangle_i$ is the idelic Deligne pairing and $\alpha_\sigma *_i \beta_\sigma$ is the idelic $*$-product.

We have to check that Definition 6.5 gives the correct extension of the Arakelov pairing.

Theorem 6.6. *Let $\alpha, \beta \in \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^\times\right)$ such that $\widehat{p}(\alpha) = \widehat{D}$ and $\widehat{p}(\beta) = \widehat{E}$, with $(\widehat{D}, \widehat{E}) \in \Upsilon_{\mathrm{Ar}}$, then $\alpha.\beta = \widehat{D}.\widehat{E}$. In other words the idelic Arakelov intersection pairing extends to a pairing:*

$$\operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^\times\right) \times \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^\times\right) \to \mathbb{R}$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Div}\left(\mathbf{A}^{\times}_{\widehat{X}}\right) \times \mathrm{Div}\left(\mathbf{A}^{\times}_{\widehat{X}}\right) & & \\
\Big\downarrow{\scriptstyle \hat{p}\times\hat{p}} & \searrow & \\
\mathrm{Div}_{\mathrm{Ar}}(X) \times \mathrm{Div}_{\mathrm{Ar}}(X) & \longrightarrow & \mathbb{R}
\end{array}$$

Proof. It follows easily from the definitions. □

Appendix A. Semi-topological algebraic structures

A.1. Basic notions

Definition A.1. A topological abelian group (G, τ) is *linearly topologised* (or has a *linear topology*) if there is a local basis at 0 made of subgroups. A morphism between linearly topologised groups is a continuous homomorphism. The category of linearly topologised group is denoted by **LTAb**.

Proposition A.2. *Let G be an abelian group and fix a non-empty collection of subgroups $\mathcal{F} = \{U_i\}_{i\in I}$. If G is endowed with the topology τ generated by $\{x + U_i\}_{i\in I, x\in G}$, then it becomes a linearly topologised group.*

Proof. First we show that G is a topological group: we want the inversion $\iota : G \to G$ and the sum $\sigma : G \times G \to G$ to be continuous. We check this for the subbase $\{x+U_i\}_{i\in I,x\in G}$. Obviously $\iota^{-1}(U_i + x) = U_i - x \in \tau$. Then we prove that the following equality holds:

$$\sigma^{-1}(U_i + x) = \bigcup_{y\in G} (U_i + y) \times (U_i + x - y)\,.$$

The inclusion $\supseteq$ is evident, so let $(z, z') \in \sigma^{-1}(U_i + x)$, then $z = u + (x - z')$ for $u \in U_i$. If we write $z' = 0 + x - (x - z')$ and we put $y = x - z'$ we finally get $(z, z') = (u + y, 0 + x - y) \in (U_i + y) \times (U_i + x - y)$.
For the last statement consider the family

$$\mathcal{B} := \{U \in \tau \colon \text{ U is finite intersection of elements of } \mathcal{F}\}\,.$$

Then $\mathcal{B}$ is a local basis at 0 made of subgroups. □

Definition A.3. The linear topology on an abelian group G obtained from a family of subgroups $\{U_i\}_{i\in I}$, as it is described in Proposition A.2, is called *the linear topology generated by $\{U_i\}_{i\in I}$*.

In this setting, concepts like initial and final topologies are well defined. Let G be an abelian group and consider some homomorphisms of groups $\{\varphi_\alpha : G \to H_\alpha\}_\alpha$ and

$\{\psi_\beta : H_\beta \to G\}_\beta$, where the H_α and H_β are all linearly topologised. The *initial linear topology* on G with respect to $\{\varphi_\alpha\}_\alpha$ is the linear topology generated by

$$\left\{\varphi_\alpha^{-1}(V_\alpha)\colon V_\alpha \subseteq H_\alpha \text{ is an open subgroup}\right\}_\alpha .$$

This is the coarsest linear topology which makes all the φ_α continuous. The *final linear topology* on G with respect to $\{\psi_\beta\}_\beta$ is the linear topology generated by

$$\Big\{U \subseteq G\colon U \text{ is a subgroup and } \psi_\beta^{-1}(U) \text{ is open for any } \beta\Big\} .$$

This is the finest linear topology which makes all the ψ_β continuous.

Proposition A.4. LTAb *is an additive category and moreover it admits inverse and direct limits.*

The nontrivial statements are those involving the categorical limits. In particular $\varprojlim_i G_i$ and $\varinjlim_j G_j$ are the usual limits in the category of groups, endowed respectively with the initial and final linear topology.

Remark A.5. By commodity, in the category of linearly topologised groups, we call the limits $\varprojlim_i G_i$ and $\varinjlim_j G_j$ respectively *linear inverse limit* and *linear direct limit*.

Definition A.6. A *ST ring* (ST stands for semi-topological) is a ring A endowed with a topology satisfying the following two properties:

- $(A, +)$ is a linearly topologised abelian group.
- For any $a \in A$ the map $\lambda_a : A \to A$, such that $\lambda_a(x) = ax$, is continuous.

A morphism of ST rings is continuous homomorphisms of rings. The category of ST rings is denoted as **STRing**. Moreover B is a ST A-algebra if there is a morphism of ST rings $\varphi : A \to B$. The category of ST A-algebras is A-**STAlg**.

Proposition A.7. STRing *and A-***STAlg** *admit inverse and direct limits.*

Proof. We show it only for rings. Let $A = \varprojlim_i A_i$ be the usual inverse limit in the category of rings and topologise its additive structure by taking the linear inverse limit topology. Thus we have the coarsest linear topology on $(A, +)$ such that the projections $\pi_j : A \to A_j$ are continuous. Assume that $\Lambda_{(a_i)}$ is the multiplication by $(\dots, a_i, a_{i+1}, \dots)$ in A and consider the composition: $A \xrightarrow{\Lambda_{(a_i)}} A \xrightarrow{\pi_j} A_j$, given by

$$x = (\dots x_i, x_{i+1}, \dots) \mapsto (\dots a_i x_i, a_{i+1} x_{i+1}, \dots) \mapsto a_j x_j .$$

Since $\pi_j \circ \Lambda_{(a_i)}(x) = \lambda_{a_j} \circ \pi_j(x)$, we can conclude that $\pi_j \circ \Lambda_{(a_j)}$ is continuous. Finally if $\pi_j^{-1}(V_j) \subset A$ is an element in the subbase of A, then $\Lambda_{(a_i)}^{-1}(\pi_j^{-1}(V_j))$ is open in A.

Let $A = \varinjlim_i A_i$ be the usual direct limit in the category of rings and topologise its additive structure by taking the linear direct limit topology. Thus we have the finest linear topology on $(A, +)$ such that the maps $\phi_i : A_i \to A$ are continuous. Let's denote with $\mu_{ij} : A_i \to A_j$ the continuous homomorphisms in the directed set $\{A_i\}_i$; moreover $\Lambda_{[(j,a)]}$ is the multiplication in $A = (\sqcup_i A_i)/_\sim$ for the fixed element $[(j, a)]$ where $a \in A_j$. Note that the composition: $A_i \xrightarrow{\phi_i} A \xrightarrow{\Lambda_{[j,a]}} A$, given by

$$x \mapsto [(i, x)] \mapsto [k, \mu_{jk}(a)\mu_{ik}(x)]$$

is continuous. Thus if $U \subset A$ is open, then $\phi_1^{-1}\left(\Lambda_{[j,a]}^{-1}(U)\right)$ is open and by definition of final linear topology we can conclude that $\Lambda_{[j,a]}^{-1}(U)$ is open in A. □

Definition A.8. Let A be a ST ring. A *ST A-module* is an A-module satisfying the following properties:

- M is a linearly topologised abelian group.
- For any $a \in A$ and any $m \in M$ the maps $\lambda_a^M : M \to M$ and $\rho_m : A \to M$ such that $\lambda_a(x) = ax$ and $\rho_m(x) = xm$ are continuous.

A morphism of ST modules is a continuous homomorphism of A-modules. If A is a ST field then M is called a *ST vector space*.

Given a ST A-module M, the subset $\overline{\{0\}}$ is a submodule because of the continuity of λ_a, therefore we define

$$M^{\text{sep}} := M/\overline{\{0\}}$$

which is again a ST A-module if endowed with the quotient topology.

Proposition A.9. *Let A be a ST ring, and M an A-module. If M is endowed with the final linear topology with respect to the group homomorphisms $\rho_m : A \to M$, then M is a ST A-module.*

Proof. See [25, p. 17]. □

Definition A.10. The topology on M described in Proposition A.9 is called the *fine A-module topology*.

A.2. Ind/pro topologies

Now we present the crucial part of this very general theory. Given a ST ring A, we describe two procedures called (C) and (L) that give canonical topologies of ST rings respectively on $\varprojlim_r A/\mathfrak{p}^r$ and $A_\mathfrak{p}$ for any prime ideal $\mathfrak{p} \subset A$. We need the following lemma:

Lemma A.11. *Let $\varphi : A \to B$ be a ring homomorphism where A is a ST ring. Consider B as an A-module endowed with the fine A-module topology, then B is a ST ring.*

Proof. [25, Proposition 1.2.9.(b)]. □

(C) For any $r > 0$ we put on $A/\mathfrak{p}^r$ the fine A-module topology, so by Lemma A.11 $A/\mathfrak{p}^r$ is a ST ring. By Proposition A.7 we can endow $\varprojlim_r A/\mathfrak{p}^r$ with a structure of ST ring.

(L) $A_\mathfrak{p}$ is naturally an A-module, so we endow it with the fine A-module topology. Again by Lemma A.11 we conclude that $A_\mathfrak{p}$ is a ST ring.

Let R be a ST ring and put on $A = R[t]$ the fine R-module topology. Consider the ring of formal Laurent power series $R((t))$, then as linear projective limit we have:

$$R[[t]] = \varprojlim_r \frac{R[t]}{t^r R[t]} .$$

Therefore we consider on $R((t))$ the topology induced in the following way:

$$A = R[t] \overset{(C)}{\rightsquigarrow} R[[t]] \overset{(L)}{\rightsquigarrow} R((t)) . \tag{A.1}$$

This is called the ind/pro-topology. We have an isomorphism of ST R-modules

$$R((t)) \cong \left(\bigoplus_{n\in\mathbb{N}} R \right) \oplus \prod_{n\in\mathbb{N}} R$$

and each subgroup of the form $t^r R[[t]]$, for $r \in \mathbb{Z}$, is closed in $R((t))$.

Remark A.12. If we start with a discrete field $K = R$, then the ind/pro-topology on $R((t))$ is the discrete valuation topology.

Let $\xi \in \widehat{R((t))}$ be a nontrivial character. The *conductor of* ξ is

$$c_\xi := \min \left\{ i \in \mathbb{Z} : \xi \in \left(t^i R[[t]] \right)^\perp \right\} .$$

Appendix B. Arakelov geometry

This section is just a collection of basic results about Arakelov geometry for arithmetic surfaces, a more detailed exposition of Arakelov geometry can be found for example in [15]. We will maintain the same notations used so far for the arithmetic surface $\varphi : X \to B$. Moreover we assume the reader to be familiar with complex analytic geometry for Riemann surfaces.

B.1. Green functions and $*$-product

Let's fix a connected Riemann surface C.

Definition B.1. A *Green function* on C is a map $g : U \subseteq C \to \mathbb{R}$ satisfying the following properties:

(1) $U = C \setminus \{x_1, \ldots, x_r\}$ for $r \in \mathbb{N}$.
(2) g is a C^∞ function on U.
(3) For any point $x \in \{x_1, \ldots, x_r\}$ there exist a real number $a \in \mathbb{R}$ and a C^∞ function u on an open neighborhood of x such that the equality:

$$g = a \log |z|^2 + u$$

holds in an open neighborhood of x contained in a holomorphic chart (V, z) centered in x.

The number $a \in \mathbb{R}$ arising in condition (3) of Definition B.1 depends only on the point x and it is uniquely defined.

Definition B.2. Let g be a Green function on C such that around a point $x \in C$ it can be written as $g = a \log |z|^2 + u$. Then we put $\operatorname{ord}_x^G(g) := -a$ and we call it *the Green order of g at x*. If x is a point in the domain of g, then $\operatorname{ord}_x^G(g) := 0$.

Clearly $\operatorname{ord}_x^G(g) \neq 0$ only at a finite number of points. The Green functions on C form a real vector space $G(C)$, and for any $g, g' \in G(C)$

$$\operatorname{ord}_x^G(\lambda g) = \lambda \operatorname{ord}_x^G(g) \quad \text{for any } \lambda \in \mathbb{R}\,,$$

$$\operatorname{ord}_x^G(g + g') = \operatorname{ord}_x^G(g) + \operatorname{ord}_x^G(g')\,.$$

Let's denote with $\operatorname{Div}(C)_{\mathbb{R}} := \operatorname{Div}(C) \otimes_{\mathbb{Z}} \mathbb{R}$ the vector space of $\mathbb{R}$-divisors on C, then we have a $\mathbb{R}$-linear map:

$$\begin{aligned}\mathrm{div}^G : G(C) &\to \mathrm{Div}(C)_{\mathbb{R}} \\ g &\mapsto \sum_{x\in C} \mathrm{ord}_x^G(g)[x].\end{aligned}$$

For any Green function $g \in G(C)$ and any $\mathbb{R}$-divisor $D = \sum_x \lambda_x[x]$, we put $\widetilde{g}(D) := \sum_x \lambda_x g(x)$ when the values $g(x)$ are well defined.

Proposition B.3. *Let $(\mathscr{L}, h)$ be a C^∞ hermitian invertible sheaf on C, and let s be a nonzero meromorphic section of $\mathscr{L}$, then the map $-\log(h(s,s))$ is a Green function on C such that $\mathrm{div}^G(-\log(h(s,s))) = \mathrm{div}(s)$.*

Proof. See [15, lemma 4.8]. □

The following result is an immediate consequence of Proposition B.3:

Proposition B.4. *The map $\mathrm{div}^G : G(C) \to \mathrm{Div}(C)_{\mathbb{R}}$ is surjective.*

Let's define a very important subspace of $\mathbb{Z}G(C)$:

Definition B.5. The vector space of Green functions with integer orders on C is:

$$\mathbb{Z}G(C) := \left\{ g \in G(C) \colon\ \mathrm{ord}_x^G(g) \in \mathbb{Z}\ \ \forall x \in C \right\}.$$

The next result shows that any Green function which induces a divisor on C is actually of the form $-\log(h(s,s))$ for some meromorphic section s of a C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$.

Proposition B.6. *Let $g \in \mathbb{Z}G(C)$, then there exist a C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$ on C and a meromorphic section s of $\mathscr{L}$ such that $g = -\log(h(s,s))$.*

Proof. See again [15, lemma 4.8]. □

From now on, in this subsection we fix a Kähler fundamental form Ω on C such that $\int_C \Omega = 1$. Let's define some subsets of $G(C)$:

$$\begin{aligned}G^\Omega(C) &:= \{g \in G(C) \colon \Delta_{\bar\partial}(g) \text{ is constant}\}, \\ G_0^\Omega(C) &:= \{g \in G^\Omega(C) \colon \int_C g\Omega = 0\}, \\ \mathbb{Z}G^\Omega(C) &:= \mathbb{Z}G(C) \cap G^\Omega(C), \\ \mathbb{Z}G_0^\Omega(C) &:= \mathbb{Z}G(C) \cap G_0^\Omega(C).\end{aligned}$$

Theorem B.7. *The map $\mathrm{div}^G|_{G_0^\Omega(C)} : G_0^\Omega(C) \to \mathrm{Div}(C)_{\mathbb{R}}$ is an isomorphism.*

Proof. See [15, Theorem 4.10]. □

Proposition B.8. *For any $g \in G^{\Omega}(C)$ there exists a unique decomposition $g = g_0 + c$ for $g_0 \in G_0^{\Omega}(C)$ and $c \in \mathbb{R}$.*

Proof. See again [15, Theorem 4.10]. □

Definition B.9. The inverse map of $\operatorname{div}^G|_{G_0^{\Omega}(X)}$ is denoted as:

$$\begin{aligned} \mathcal{G}^{\Omega} : \operatorname{Div}(X)_{\mathbb{R}} &\to G_0^{\Omega}(X) \\ D &\mapsto \mathcal{G}^{\Omega}(D) \end{aligned}$$

and we can define the following function:

$$\begin{aligned} g^{\Omega} : (X \times X) \setminus \Delta_{X \times X} &\to \mathbb{R} \\ (p, q) &\mapsto g^{\Omega}(p, q) := \mathcal{G}^{\Omega}([p])(q) \end{aligned}$$

where $\Delta_{X \times X}$ denotes the diagonal subset of $X \times X$.

By construction g^{Ω} is C^{∞} in the variable q, but, as we will see soon (Corollary B.14), g^{Ω} turns out to be symmetric, therefore it is C^{∞}. Since $g^{\Omega}(p, \cdot) \in G_0^{\Omega}(X) \subset G^{\Omega}(X)$, then $dd^c(g^{\Omega}(p, \cdot)) = \alpha\Omega$ for a constant $\alpha \in \mathbb{C}$, but

$$1 = \deg^G(g^{\Omega}(p, \cdot)) = \int_X dd^c(g^{\Omega}(p, \cdot)) = \int_X \alpha\Omega = \alpha\,.$$

Hence $\alpha = 1$ and

$$dd^c(g^{\Omega}(p, \cdot)) = \Omega. \tag{B.1}$$

Thus, amongst all Green functions, those of the form $g^{\Omega}(p, \cdot)$ satisfy the Poisson differential equation (B.1). this feature will be very useful for intersection theory.

Another important property is that for any fixed $p \in X$:

$$\int_X g^{\Omega}(p, \cdot)\Omega = \int_X \mathcal{G}^{\Omega}([p])\Omega = 0 \tag{B.2}$$

because $\mathcal{G}^{\Omega}([p]) \in G_0^{\Omega}(X)$.

Remark B.10. g^{Ω} can be defined as the *unique* function on $(X \times X) \setminus \Delta_{X \times X}$ with values in $\mathbb{R}$ satisfying the following properties:

(1) Around any point $p \in X$ we can write $g^{\Omega}(p, \cdot) = -\log |z|^2 + u$, where z is a chart centered in p and u is C^{∞}.
(2) $dd^c(g^{\Omega}(p, \cdot)) = \Omega$.
(3) $\int_X g^{\Omega}(p, \cdot)\Omega = 0$.

This is how Arakelov defined g^{Ω} in [2] and [1]. In the literature g^{Ω} is usually called *the Green function of X (with respect to Ω).*[3] Here we used a different approach (and notations), indeed g^{Ω} was constructed directly by using the isomorphism $\mathrm{Div}(X)_{\mathbb{R}} \cong G_0^{\Omega}(X)$.

Definition B.11. Let $g_1, g_2 \in G(C)$ such that $\mathrm{div}^G(g_1)$ and $\mathrm{div}^G(g_2)$ have no common components then the *$*$-product* between g_1 and g_2 is the real number:

$$g_1 * g_2 := \widetilde{g_1}(\mathrm{div}^G(g_2)) + \int_C dd^c(g_1)g_2\,,$$

where $dd^c = \frac{1}{2\pi}\partial\overline{\partial}$.

Remark B.12. It is necessary to assume that $\mathrm{div}^G(g_1)$ and $\mathrm{div}^G(g_2)$ have no common components otherwise $\widetilde{g_1}(\mathrm{div}(g_2))$ wouldn't be well defined.

Theorem B.13. *Let $g_1, g_2 \in G(C)$ such that $\mathrm{div}^G(g_1)$ and $\mathrm{div}^G(g_2)$ have no common components, then $g_1 * g_2 = g_2 * g_1$.*

Proof. See [15, Proposition 4.12]. □

Corollary B.14. *$g^{\Omega}(p, q) = g^{\Omega}(q, p)$ for any $p \neq q$.*

Proof. By using the properties of the elements in G_0^{Ω}, it is easy to verify that

$$\mathcal{G}^{\Omega}([p]) * \mathcal{G}^{\Omega}([q]) = g^{\Omega}(p, q); \quad \mathcal{G}^{\Omega}([q]) * \mathcal{G}^{\Omega}([p]) = g^{\Omega}(q, p)\,.$$

Hence the conclusion follows immediately from Theorem B.13. □

Note that for any three different points $p, q, t \in X$ and coefficients $a, b \in \mathbb{R}$ we have that:

$$\mathcal{G}^{\Omega}(a[p] + b[q]) * \mathcal{G}^{\Omega}([t]) = a\mathcal{G}^{\Omega}([p]) * \mathcal{G}^{\Omega}([t]) + b\mathcal{G}^{\Omega}([q]) * \mathcal{G}^{\Omega}([t])\,.$$

[3] Actually the conditions which uniquely define g^{Ω} in [2] and [1] are slightly different from the ones listed here, and moreover they may vary in other references. For instance it is common to find different constants for the differential Poisson equation, or the Green function might be defined as $G = \exp(g^{\Omega})$. Of course these discrepancies are fixed when the Green function is applied for intersection theory.

Therefore if $D = \sum_{p\in X} a_p[p]$ and $E = \sum_{q\in X} b_q[q]$ are two real divisors of X with no common components, then it is customary to define:

$$g^{\Omega}(D,E) := \sum_{p\neq q} a_p b_q g^{\Omega}(p,q)\,. \tag{B.3}$$

Remark B.15. The important point to emphasize here is that for Green functions $g_1, g_2 \in G_0^{\Omega}$, i.e. coming from some real divisors on X, the integral appearing in $g_1 * g_2$ vanishes. This means that for such kind of Green functions, the nature of the $*$-product is "less analytic", indeed it depends only on the value of g_1 or g_2 at a finite set of points.

B.2. Arakelov intersection pairing

On each Riemann surface X_σ we fix a Kähler form Ω_σ such that $\int_{X_\sigma} \Omega_\sigma = 1$, and we put $\Omega := \{\Omega_\sigma\}_{\sigma\in B_\infty}$. For any divisor $D \in \mathrm{Div}(X)$, $D_\sigma := \varphi_\sigma^* D \in \mathrm{Div}(X_\sigma)$ denotes its pullback through φ_σ. Consider the additive group $\mathbf{G}(X) := \oplus_{\sigma\in B_\infty} G(X_\sigma)$ and its subgroup, depending on Ω, $\mathbf{G}(X,\Omega) := \bigoplus_{\sigma\in B_\infty} G^{\Omega_\sigma}(X_\sigma)$. By commodity we write any element of $\mathbf{G}(X)$ (or of $\mathbf{G}(X,\Omega)$) as a finite formal linear combination $\sum_\sigma g_\sigma X_\sigma$ for $g_\sigma \in \mathbf{G}(X)$ (or $g_\sigma \in \mathbf{G}(X,\Omega)$).

Definition B.16. The group of *Arakelov divisors* on $\widehat{X}$ is:

$$\mathrm{Div}_{\mathrm{Ar}}(X,\Omega) := \left\{ \left(D, \sum_\sigma g_\sigma X_\sigma \right) \in \mathrm{Div}(X) \times \mathbf{G}(X,\Omega) \colon\ \mathrm{div}^G(g_\sigma) = D_\sigma \right\}.$$

We often denote the element $(0, X_\sigma) \in \mathrm{Div}_{\mathrm{Ar}}(X,\Omega)$ simply with the symbol X_σ.

It is important to understand the geometry lying behind the above apparently mysterious definition. Fix an Arakelov divisor $(D, \sum_\sigma g_\sigma X_\sigma)$, by Theorem B.7 and Proposition B.8 we can write

$$g_\sigma = \mathcal{G}^{\Omega_\sigma}(D_\sigma) + \alpha_\sigma \tag{B.4}$$

where $\alpha_\sigma \in \mathbb{R}$ is uniquely determined. Fig. 5 highlights the fact that D_σ, which is a finite set of points on X_σ, can be interpreted as the "prolongation" of D on the curve X_σ; thus, it makes sense to define the Arakelov divisor

$$\overline{D} := \left(D, \sum_\sigma \mathcal{G}^{\Omega_\sigma}(D_\sigma) X_\sigma \right) \in \mathrm{Div}_{\mathrm{Ar}}(X,\Omega)$$

which will be called *completion* of D in $\widehat{X}$ (this is consistent with the notion of completed horizontal curve given before). By equation (B.4) we have the following unique

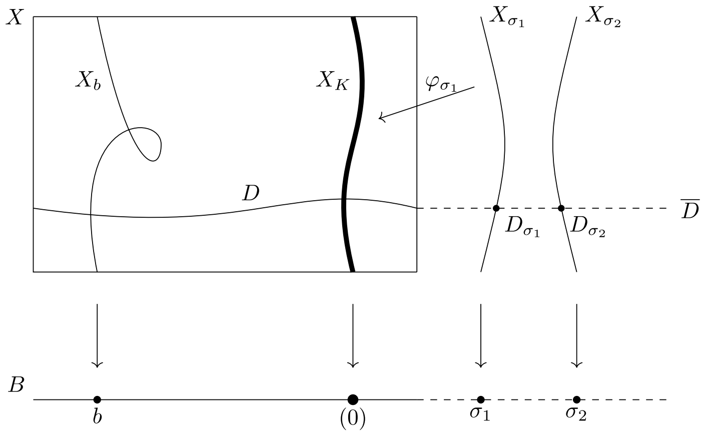


Fig. 5. A schematization of an arithmetic surfaces $\varphi : X \to B$ such that $B_\infty = \{\sigma_1, \sigma_2\}$ (for instance $B = \operatorname{Spec} \mathbb{Z}[i]$). X_b is a vertical divisor over the closed point b, D is a horizontal divisor such that D_{σ_1} and D_{σ_2} are prime divisors respectively on X_{σ_1} and X_{σ_2}.

decomposition of $(D, \sum_\sigma g_\sigma X_\sigma)$ in $\mathrm{Div}_{\mathrm{Ar}}(X)$:

$$\left(D, \sum_\sigma g_\sigma X_\sigma\right) = \overline{D} + \sum_\sigma \alpha_\sigma X_\sigma \tag{B.5}$$

where the linear combination $\sum_\sigma \alpha_\sigma X_\sigma$ can be evidently read as a "real divisor" on $\widehat{X}$ with support made of curves at infinity. In perfect analogy with the usual notion of divisor, equation (B.5) tells us that an Arakelov divisor can be interpreted as a formal linear combination of "curves" in $\widehat{X}$, such that the coefficients of the curves at infinity are in $\mathbb{R}$. The presence of this real coefficients underlines once again the fact that the curves at infinity have an analytic nature. From the above discussion we recover the original definition of the group of Arakelov divisors given in [2] and [1]:

Proposition B.17. *There is an isomorphism of groups:*

$$\mathrm{Div}_{\mathrm{Ar}}(X, \Omega) \cong \mathrm{Div}(X) \oplus \mathbb{R}^{(B_\infty)}.$$

Proof. Thanks to equation (B.5) we can define the isomorphism:

$$\left(D, \sum_\sigma g_\sigma X_\sigma\right) \mapsto D + \sum_\sigma \alpha_\sigma [\sigma]. \qquad \square$$

Now we want to introduce the concept of principal Arakelov divisor, in other words we want to define an Arakelov divisor associated to an element of $K(X)$. Recall that

$K(X)$ is also the function field of X_K, so the morphism $\varphi_\sigma : X_\sigma \to X_K$ induces a field embedding

$$\varphi_\sigma^\# : K(X) \hookrightarrow \mathbb{C}(X_\sigma)\,.$$

For any rational function $f \in K(X)$ we put by simplicity $f_\sigma := \varphi_\sigma^\#(f)$. Moreover let $\mathscr{O}_\sigma$ be the sheaf of regular functions on X_σ, then as usual f_σ can be identified with a holomorphic map $X_\sigma \to \mathbb{C}$ at all but finitely many points:

$$p \mapsto f_{\sigma,p} \mapsto \overline{f}_{\sigma,p} \in k(p) \cong \mathbb{C}.$$

Then it is easy to see that $-\log|f_\sigma|^2$ is a Green function on X_σ such that $\partial\bar{\partial}(-\log|f_\sigma|^2) = 0$, therefore $-\log|f_\sigma|^2 \in G^{\Omega_\sigma}(X_\sigma)$.

Proposition B.18. *Let $f \in K(X)^\times$, then $\mathrm{div}^G(-\log|f_\sigma|^2) = (f)_\sigma$, where $(f)_\sigma$ is the pullback of the principal divisor (f).*

Proof. Fix a point $p \in X_\sigma$, let $x = \varphi_\sigma(p)$ and consider f as a rational function on X_K. If ϖ_σ is a local parameter in $\mathscr{O}_{\sigma,p}$ and ϖ is a local parameter in $\mathscr{O}_{X_K,x}$, then

$$f_\sigma = \varpi_\sigma^{v_p(\varphi_\sigma^\#(\varpi))v_x(f)} u \quad \text{for } u \in \mathscr{O}_{\sigma,p}\,.$$

This implies that $\mathrm{ord}_p^G(-\log|f_\sigma|^2) = v_p(\varphi_\sigma^\#(\varpi))v_x(f)$, but $v_p(\varphi_\sigma^\#(\varpi))$ is precisely the ramification index $e_{\varphi_\sigma,p}$, hence $\mathrm{ord}_p^G(-\log|f_\sigma|^2) = e_{\varphi_\sigma,p}v_x(f)$. So, we finally have:

$$\mathrm{div}^G(-\log|f_\sigma|^2) = \sum_{p\in X_\sigma} e_{\varphi_\sigma,p} v_{\varphi_\sigma(p)}(f)[p] = (f)_\sigma\,. \qquad \square$$

Now the following definition makes sense:

Definition B.19. Let $f \in K(X)^\times$ be a rational function. It induces an Arakelov divisor in the following way:

$$\widehat{(f)} := \left((f), \sum_\sigma -\log|f_\sigma|^2 X_\sigma \right) \in \mathrm{Div}_{\mathrm{Ar}}(X,\Omega)\,.$$

The group

$$\mathrm{Princ}_{\mathrm{Ar}}(X,\Omega) := \left\{ \widehat{(f)} \;:\; f \in K(X) \right\}$$

is called the group of *principal Arakelov divisor* and $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega) := \frac{\mathrm{Div}_{\mathrm{Ar}}(X,\Omega)}{\mathrm{Princ}_{\mathrm{Ar}}(X,\Omega)}$ is the *Arakelov Chow group*. Two Arakelov divisor are said *linearly equivalent* if they are contained in the same class in $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)$.

Moreover for any principal Arakelov divisor $\widehat{(f)}$ we get the following decomposition:

$$\widehat{(f)} = \overline{(f)} + \sum_{\sigma} \left(\int_{X_\sigma} -\log |f_\sigma|^2 \Omega_\sigma \right) X_\sigma \, .$$

Proposition B.20. *Let D, E be two finite divisors on X with no common components, then for any $\sigma \in B_\infty$ the divisors D_σ and E_σ on X_σ have no common components.*

Proof. Omitted. □

Let's denote as $\Upsilon_{\mathrm{Ar}} \subset \mathrm{Div}_{\mathrm{Ar}}(X, \Omega) \times \mathrm{Div}_{\mathrm{Ar}}(X, \Omega)$ the set of couples of Arakelov divisors with no common components on X, then we can define the Arakelov intersection pairing on Υ_{Ar}:

Definition B.21. Let $\widehat{D} := (D, \sum_\sigma g_\sigma X_\sigma)$, $\widehat{E} := (E, \sum_\sigma l_\sigma X_\sigma)$ be two Arakelov divisors such that $(\widehat{D}, \widehat{E}) \in \Upsilon_{\mathrm{Ar}}$. Thanks to Proposition B.20 we can define an Arakelov divisor on B:[4]

$$\left\langle \widehat{D}, \widehat{E} \right\rangle_{\mathrm{Ar}} := \langle D, E \rangle + \sum_{\sigma} g_\sigma * l_\sigma \, [\sigma] \quad \in \mathrm{Div}_{\mathrm{Ar}}(B)$$

where

$$\langle D, E \rangle := \varphi_* i(D, E) = \sum_{x \in X} [k(x) : k(\varphi(x))] \, i_x(D, E) \, [\varphi(x)]$$

and $*$ is the product between Green functions. If $\widehat{d} = \sum_{\mathfrak{p} \in B} n_{\mathfrak{p}} [\mathfrak{p}] + \sum_{\sigma \in B_\infty} \alpha_\sigma [\sigma]$ is an Arakelov divisor on the base B, its degree is defined as:

$$\deg_{\mathrm{Ar}}(\widehat{d}) := \sum_{\mathfrak{p} \in B} n_{\mathfrak{p}} \log \mathfrak{N}(\mathfrak{p}) + \frac{1}{2} \sum_{\sigma \in B_\infty} \epsilon_\sigma \alpha_\sigma \, .$$

In particular we use the notation $D.E := \deg_{\mathrm{Ar}}(\langle D, E \rangle)$, and the *Arakelov intersection number* of $\widehat{D}$ and $\widehat{E}$ is:

$$\widehat{D}.\widehat{E} := \deg_{\mathrm{Ar}} \left(\left\langle \widehat{D}, \widehat{E} \right\rangle_{\mathrm{Ar}} \right) = D.E + \frac{1}{2} \sum_{\sigma} \epsilon_\sigma \, g_\sigma * l_\sigma \quad \in \mathbb{R} \, .$$

The following proposition summarizes some properties of the Arakelov intersection pairing:

[4] Note that we assume D and E to have no common components in order to ensure that the $*$-product between green functions is well defined for any $\sigma \in B_\infty$.

Proposition B.22. *Let* $(\widehat{D}, \widehat{E}), (\widehat{D_j}, \widehat{E_j}) \in \Upsilon_{\mathrm{Ar}}$ *with* $j = 1, 2$, *then*

(1) $\widehat{D}.\widehat{E} = \widehat{E}.\widehat{D}$ *(symmetry).*
(2) $(\widehat{D_1} + \widehat{D_2}).(\widehat{E_1} + \widehat{E_2}) = \sum_{j,k=1}^{2} \widehat{D_j}.\widehat{E_k}$ *($\mathbb{Z}$-bilinearity).*
(3) *If* $\widehat{D} = (D, \sum_\sigma g_\sigma X_\sigma)$ *and* $f \in K(X)^\times$ *such that* $(D, (f)) \in \Upsilon$, *then*

$$\left\langle \widehat{D}, \widehat{(f)} \right\rangle_{\mathrm{Ar}} = (\widehat{N_D(f)}) \ \in \mathrm{Princ}_{\mathrm{Ar}}(B)\,.$$

In particular $\widehat{D}.\widehat{(f)} = 0$.

Proof. See [15, section 4.4]. □

The Arakelov intersection number can be extended to an intersection pairing on the whole $\mathrm{Div}_{\mathrm{Ar}}(X, \Omega)$ and induces a natural intersection pairing on $\mathrm{CH}^1_{\mathrm{Ar}}(X, \Omega)$.

Proposition B.23. *The Arakelov intersection number extends to any two Arakelov divisors in* $\mathrm{Div}_{\mathrm{Ar}}(X, \Omega) \times \mathrm{Div}_{\mathrm{Ar}}(X, \Omega)$ *and moreover descends naturally to pairing on* $\mathrm{CH}^1_{\mathrm{Ar}}(X, \Omega) \times \mathrm{CH}^1_{\mathrm{Ar}}(X, \Omega)$.

Proof. See [15, section 4.4]. □

Now we interpret the Arakelov intersection pairing in a more geometric way by using the decomposition given in equation (B.5). Fix two Arakelov divisors $\widehat{D}, \widehat{E} \in \Upsilon_{\mathrm{Ar}}$, then we can write

$$\widehat{D} = \overline{D} + \sum_\sigma \alpha_\sigma X_\sigma = \left(D, \sum_\sigma \mathcal{G}^{\Omega_\sigma}(D_\sigma) X_\sigma \right) + \left(0, \sum_\sigma \alpha_\sigma X_\sigma \right)\,,$$

$$\widehat{E} = \overline{E} + \sum_\sigma \beta_\sigma X_\sigma = \left(E, \sum_\sigma \mathcal{G}^{\Omega_\sigma}(E_\sigma) X_\sigma \right) + \left(0, \sum_\sigma \beta_\sigma X_\sigma \right)\,.$$

In order to find explicitly $\widehat{D}.\widehat{E}$, by bilinearity and symmetry of the intersection pairing it is enough to understand how calculate the following three terms:

(i) $\overline{D}.\overline{E}$; namely the intersection of two completed divisors.
(ii) $\overline{D}.(0, \sum_\sigma \beta_\sigma X_\sigma)$; namely the intersection between a completed divisor and a divisor at infinity. Clearly $(0, \sum_\sigma \alpha_\sigma X_\sigma).\overline{E}$ is obtained in the same way.
(iii) $(0, \sum_\sigma \alpha_\sigma X_\sigma).(0, \sum_\sigma \beta_\sigma X_\sigma)$; that is the intersection of divisors composed only by curves at infinity.

For (i) let's evaluate $\mathcal{G}^{\Omega_\sigma}(D_\sigma)*\mathcal{G}^{\Omega_\sigma}(E_\sigma)$. By the bare definition of the $*$-product and g^{Ω_σ}:

$$\mathcal{G}^{\Omega_\sigma}(D_\sigma) * \mathcal{G}^{\Omega_\sigma}(E_\sigma) = g^{\Omega_\sigma}(D_\sigma, E_\sigma) + \int_{X_\sigma} dd^c \left(\mathcal{G}^{\Omega_\sigma}(D_\sigma)\right) \mathcal{G}^{\Omega_\sigma}(E_\sigma) \,,$$

but since $\mathcal{G}^{\Omega_\sigma}(D_\sigma), \mathcal{G}^{\Omega_\sigma}(E_\sigma) \in G_0^{\Omega_\sigma}(X_\sigma)$, it is straightforward to verify that the integral on the right hand side is 0. Therefore we get:

$$\overline{D}.\overline{E} = D.E + \frac{1}{2}\sum_\sigma \epsilon_\sigma g^{\Omega_\sigma}(D_\sigma, E_\sigma)\,. \tag{B.6}$$

In order to calculate (ii) we need $\mathcal{G}^{\Omega_\sigma}(D_\sigma) * \beta_\sigma$:

$$\mathcal{G}^{\Omega_\sigma}(D_\sigma) * \beta_\sigma = \beta_\sigma * \mathcal{G}^{\Omega_\sigma}(D_\sigma) = \beta_\sigma \deg(D_\sigma) + \int_{X_\sigma} dd^c(\beta_\sigma)\mathcal{G}^{\Omega_\sigma}(D_\sigma) = \beta_\sigma \deg(D_\sigma)\,,$$

thus we obtain

$$\overline{D}.(0, \sum_\sigma \beta_\sigma X_\sigma) = \frac{1}{2}\sum_\sigma \epsilon_\sigma \beta_\sigma \deg(D_\sigma)\,. \tag{B.7}$$

Finally (iii) is trivial since $\alpha_\sigma * \beta_\sigma = 0$ and we have:

$$(0, \sum_\sigma \alpha_\sigma X_\sigma).(0, \sum_\sigma \beta_\sigma X_\sigma) = 0\,. \tag{B.8}$$

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