LIMITS OF FUNCTIONS

4.1 Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

(1) $$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

(2) $$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

(3) $$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

If X and/or Y are replaced by the real line, the complex plane, or by some euclidean space R^k, the distances d_X, d_Y are of course replaced by absolute values, or by norms of differences (see Sec. 2.16).

It should be noted that $p \in X$, but that p need not be a point of E in the above definition. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \to p} f(x)$.

We can recast this definition in terms of limits of sequences:

4.2 Theorem *Let X, Y, E, f, and p be as in Definition* 4.1. *Then*

(4) $$\lim_{x \to p} f(x) = q$$

if and only if

(5) $$\lim_{n \to \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that

(6) $$p_n \neq p, \qquad \lim_{n \to \infty} p_n = p.$$

Proof Suppose (4) holds. Choose $\{p_n\}$ in E satisfying (6). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$. Thus, for $n > N$, we have $d_Y(f(p_n), q) < \varepsilon$, which shows that (5) holds.

Conversely, suppose (4) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ), for which $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = 1/n$ $(n = 1, 2, 3, \ldots)$, we thus find a sequence in E satisfying (6) for which (5) is false.