Intersection matrices for the minimal regular model of $X_0(N)$ and applications to the Arakelov canonical sheaf

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Abstract

Let $N > 1$ be an integer coprime to 6 and such that $N \notin \{5, 7, 13\}$. We compute the intersection matrices relative to special fibres of the minimal regular model of $X_0(N)$. Moreover we prove that self-intersection of the Arakelov canonical sheaf of $X_0(p^n)$, where $p > 3$ is a prime, is asymptotic to $3g \log p^n$ for $p^n \to +\infty$.

1 Introduction

1.1 Presentation of the results

The self-intersection of the Arakelov canonical sheaf of an arithmetic surface, here denoted by $\langle\overline{\omega}, \overline{\omega}\rangle$, is a crucial arithmetic invariant for several reasons (assuming that the genus of the generic fibre is at least 2):

- It appears in the arithmetic Noether formula proved in [MB89]. So it is closely related to the Faltings height of the Jacobian of the generic fibre of the arithmetic surface.

- The inequality $\langle \overline{\omega}, \overline{\omega} \rangle > 0$ is equivalent to the arithmetic Bogomolov conjecture for arithmetic surfaces proved by Ullmo and Zhang in [Ull98] and [Zha93].
- Suitable upper bounds for $\langle \overline{\omega}, \overline{\omega} \rangle$ imply an effective version of the Mordell conjecture (see for instance [MB90]).

The study of the quantity $\langle \overline{\omega}, \overline{\omega} \rangle$ is in general tremendously difficult since it requires a deep knowledge of the specific model of the curve under investigation together with some analytic information involving the curve seen as a point inside the compactified moduli space. Nevertheless, some results are known for Fermat curves, hyperelliptic curves, Belyi curves and modular curves (see for instance [dJ04],[CK09], [ECdJ+11], [Küh13], [Jav14]).

We restrict our attention to the case of $X_0(N)$ and its minimal regular model $\mathcal{X}$. It turns out that $\mathcal{X}$ has singular and reduced fibres only above the primes dividing the level N. In order to compute the "finite contribution" of $\langle \overline{\omega}, \overline{\omega} \rangle$, for every $p \mid N$ one needs the matrix containing all the possible intersection numbers between the irreducible components of the fibre $\mathcal{X}_p$. In Section 3.3 we compute these matrices for every $N > 1$ coprime to 6 such that $N \notin \{5, 7, 13\}$. The assumption $(N, 6) = 1$ on the level is substantial, since at the moment we don't have well behaved regular models of $X_0(N)$ when either $2 \mid N$ or $3 \mid N$. On the other, hand the restriction $N \notin \{5, 7, 13\}$ is just a matter of convenience, since these cases must be treated with slightly different methods. In the literature the intersection matrices of the minimal regular models of $X_0(N)$ have been computed only for N at the same time coprime to 6 and squarefree (see [DR73] and [Maz77, Appendix]) and for $N \in \{p^2, p^3, p^4\}$ (see [Edi90] and [BMC22]). At the moment we don't know how to calculate the exact value of the "contribution at infinity" of $\langle \overline{\omega}, \overline{\omega} \rangle$, but it has been *asimptotically estimated* for every $N > 1$ in [MvP22], improving the previous results of [AU97], [BBC20] and [BMC22], which had restrictions on N. Therefore it makes sense to ask what is the asymptotic behaviour of $\langle \overline{\omega}, \overline{\omega} \rangle$ for $N \to +\infty$. The problem has been thoroughly investigated for all the "classical" modular curves $X_0(N)$, $X_1(N)$, $X(N)$ and it turned out that

$$\langle \overline{\omega}, \overline{\omega} \rangle \sim 3g \log N, \qquad \text{for } N \to +\infty, \tag{1}$$

in the following cases:

- For $X_0(N)$ when N is square-free with $(N, 6) = 1$ (see [AU97] and [MU98]) and moreover when $N \in \{p^2, p^3, p^4\}$ for $p > 5$ prime (see [BBC20] and [BMC22]).
- For $X_1(N)$ when N is at the same time square-free, odd and divisible by at least two coprime integers bigger than 4. (see [May14]).
- For $X(N)$ when N is at the same time square-free, composite and odd (see [GvP22]).

Here we use the intersection matrices of $\mathcal{X}$ to study the asymptotics of $\langle \overline{\omega}, \overline{\omega} \rangle$ for the modular curves $X_0(p^n)$, where $p > 3$ is a prime. We prove the following theorem which extends (as expected) the aforementioned results:

Theorem 1.1. *Let $p > 3$ be a prime and let $\mathcal{X}$ be the minimal regular model of $X_0(p^n)$, then*

$$\langle \overline{\omega}, \overline{\omega} \rangle \sim 3g \log p^n, \qquad \text{for } p^n \to +\infty.$$

1.2 Overview of the paper

We follow the approach in [BBC20] and [BMC22] for the case $N \in \{p^2, p^3, p^4\}$, but with several refinements and improvements. Let $N > 1$ be an integer coprime to 6; first of all we make use of the Edixhoven's model for $X_0(N)$ (see [Edi90]): it is a regular model $\mathcal{X}' \to \operatorname{Spec} \mathbb{Z}$ with the property that for every prime $\ell \nmid N$ the fibre $\mathcal{X}'_\ell$ is smooth and irreducible. Above the primes dividing N we find the so called special fibres that have many irreducible components, completely described. Edixhoven's model is in general not minimal and in some cases it is necessary to perform several blow downs in order to obtain the minimal regular model $\mathcal{X}$. The intersection matrices of $\mathcal{X}$ are the coefficient matrices, whose dimensions depend on N, of some linear systems whose solutions allow us to write two special rational vertical divisors V_0 and V_∞ supported on the special fibres of $\mathcal{X}$. By using the Faltings-Hriljac's version of the Hodge's index theorem and the Manin-Drinfeld's theorem it is possible to write $\langle \overline{\omega}, \overline{\omega} \rangle$ just in terms of:

(*a*) the Arakelov intersection between H_0 and H_∞, which are the prolongation on $\mathcal{X}$ of the two cusps 0 and ∞ of $X_0(N)$;

(*b*) the intersection between V_0 and V_∞ and their respective self-intersections;

(*c*) a "height term" related to the Néron-Tate's height of some specific horizontal divisors on $\mathcal{X}$.

The pieces (*a*) and (*c*) encapsulate the data relative to the "self-intersection at infinity" of $\overline{\omega}$. Their asymptotics have already been estimated respectively in [MvP22] and [MU98]. In this paper we are able to exactly calculate the term (*b*) for $N = p^n$ where $p > 3$. The main difficulty was to find a closed formula for the divisors V_0 and V_∞. We observed that the exact solutions of the above mentioned linear systems, written in a suitable way, have a "simple" shape that we proved is the correct formula for every level p^n using the software Mathematica. However, at time of writing, we are not able to find a similar simple shape for a more general level, i.e., when N is not a prime power. Our method for finding V_0 and V_∞ is completely general and depends on the Zariski's lemma for vector spaces (see [Mor13, Lemma 2.2.1]). Whereas in [BBC20] and [BMC22] the authors use some *ad hoc* techniques.

The structure of the paper is the following: In Section 2 we fix the notation and we briefly recall the needed basic concepts. In Section 3.1 we give a detailed description of Edixhoven's models and in Section 3.2 we discuss the process of obtaining the minimal regular model, when necessary. The intersection matrices are given in Section 3.3. Finally Section 4 is devoted to the proof of Theorem 1.1.

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2 Preliminaries

2.1 Arakelov theory

In this section we briefly recall the basic notions of Arakelov geometry on arithmetic surfaces, for more details the reader can consult the original Arakelov paper [Ara74] together with Faltings seminal paper [Fal84], or the more recent books [Lan88] and [Mor14]. For our purposes, it is enough to deal with the theory for curves defined over $\mathbb{Q}$, but everything can be easily written down when the ground field is any number field, in fact it is enough to add more places at infinity.

Let X be a smooth, geometrically integral, projective curve over $\mathbb{Q}$ of genus $g > 0$, and let $f\colon \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be any regular model of X. On $\mathcal{X}$ it is not possible to define a meaningful intersection pairing between divisors that descends to the Picard group. Nevertheless, there is a "partial" $\mathbb{Z}$-valued intersection pairing which is well defined when at least one of the divisors is supported over a prime p (i.e., contained in one fiber of f). For every $D \in \operatorname{Div}(\mathcal{X})$ and every $D' \in \operatorname{Div}(\mathcal{X})$ such that $\operatorname{supp}(D) \subseteq \mathcal{X}_p$ we denote this pairing by $D \cdot D'$; for further details and properties the reader can check [Liu06, Chapter 9]. In order to intersect horizontal divisors we need to take in account the base change curve $X_\mathbb{C}$ as a fiber lying over the archimedean place of $\mathbb{Z}$, i.e., as a *fiber at infinity*. At this point one needs to fix a Kähler form Ω on $X_\mathbb{C}$, and in general the intersection numbers will depend on such a choice. Arakelov proposed a canonical choice for this Kähler form which here we denote by Ω^{can} (see for instance [Mor14, page 112]). One can extend the notion of line bundle on $\mathcal{X}$ and introduce the concept of *admissible hermitian line bundle* which is a couple $\overline{\mathcal{L}} = (\mathcal{L}, h)$ where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and h is a C^∞-hermitian metric on the base change $\mathcal{L}_\mathbb{C}$ (on $X_\mathbb{C}$) satisfying:

$$c_1(\overline{\mathcal{L}}) = a\Omega^{\mathrm{can}}, \quad \text{for } a \in \mathbb{R},$$

where c_1 is the first Chern form of $\overline{\mathcal{L}}$. By using the Deligne pairing and the notion of Arakelov degree for hermitian line bundles on $\operatorname{Spec}\mathbb{Z}$ one can define a $\mathbb{R}$-valued intersection pairing between admissible hermitian line bundles which is denoted by $\langle \overline{\mathcal{L}}, \overline{\mathcal{L}}' \rangle$.

There is a map that relates admissible hermitian line bundle to *Arakelov divisors* which are the couples of the type $\overline{D} = (D, \alpha)$ where $D \in \operatorname{Div}(\mathcal{X})$ and $\alpha \in \mathbb{R}$. Here it is important to mention that the real constant α appearing in $\overline{D}$ can be recovered as the integral of a Green function on $X_{\mathbb{C}}$. The Arakelov divisors form a group under the obvious notion of addition and one can form the Arakelov-Chow group $\overline{\mathrm{CH}}^1(X)$ by performing the quotient by an adequate notion of principal Arakelov divisors. In [Ara74] is defined a $\mathbb{R}$-valued intersection pairing between Arakelov divisors denoted as $\langle \overline{D}, \overline{D}' \rangle$. We point out that an ordinary divisor $D \in \mathrm{Div}(\mathcal{X})$ can be identified with the Arakelov divisor $(D, 0)$ and, therefore the notation $\langle D, D' \rangle$ makes sense. In particular, if D is any divisor and D' is a divisor supported over the prime p and we have the following fundamental relationship:

$$\langle D, D' \rangle = (D \cdot D') \log p.$$

A crucial result of the theory (see for instance [Mor14, Sections 4.3 and 4.4]) shows that the aforementioned map between admissible line bundles and Arakelov divisors descends to a group isomorphism between the isometry classes of admissible line bundles and $\overline{\mathrm{CH}}^1(X)$ and moreover preserves the intersection pairings.

The relative canonical sheaf ω of $f\colon \mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ can be endowed with a canonical hermitian metric thanks to the adjunction isomorphism so that we obtain the *Arakelov canonical sheaf* $\overline{\omega}$ that satisfies an arithmetic version of the adjunction formula (see for instance [Mor14, Section 4.5]). Let $\mathcal{K}$ be a canonical divisor of $\mathcal{X}$ i.e., every divisor whose corresponding line bundle is isomorphic to ω, then one can easily prove that as an Arakelov divisor corresponds to $\overline{\omega}$ through the above mentioned isomorphism. Therefore we conclude that $\langle \mathcal{K}, \mathcal{K} \rangle = \langle \overline{\omega}, \overline{\omega} \rangle$. Let Δ be diagonal of $X_{\mathbb{C}} \times X_{\mathbb{C}}$, then there exists a unique (symmetric) C^∞-function $\mathcal{G}\colon (X_{\mathbb{C}} \times X_{\mathbb{C}}) - \Delta \to \mathbb{R}$ such that:

(i) around any point $P \in X_{\mathbb{C}}$ we can write $\mathcal{G}(P, \cdot) = -\log |z|^2 + u$, where z is a chart centered in P and u is a C^∞-function;

(ii) $\frac{1}{2\pi i} \partial \bar{\partial} \mathcal{G}(P, \cdot) = \Omega^{\mathrm{can}}$;

(iii) $\int_{X_{\mathbb{C}}} \mathcal{G}(P, \cdot) \Omega^{\mathrm{can}} = 0$.

This function is called the *(canonical) Green function* on $X_{\mathbb{C}}$ and it is crucial to compute the contribution at infinity of the Arakelov intersection pairing. For instance assume that $\overline{P}$ and $\overline{Q}$ are the closures on $\mathcal{X}$ of two rational points $P, Q \in X(\mathbb{Q})$, then one can show that

$$2\langle \overline{P}, \overline{Q} \rangle = -\mathcal{G}(P, Q).$$

2.2 Modular curves

In this subsection we explain our notation and recall some basic facts about modular curves. Our main reference for this subsection is [DS05]. Let $\mathcal{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ be the complex upper half-plane and denote by $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ the extended complex upper half-plane. There is an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}^*$ given, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}^*$, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

Let N be a positive integer, the subgroup

$$\Gamma(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \}$$

is called *principal congruence subgroup of level* N. Each subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$ is called *congruence subgroup of level* N. We define the modular curve over $\mathbb{C}$ associated to a congruence subgroup Γ:

$$X_\Gamma := \Gamma \backslash \mathcal{H}^*.$$

We also define, for every positive integer N, the congruence subgroup

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N \},$$

(where any $*$ in the matrices means "no restriction") and we define the modular curve:

$$X_0(N) := \Gamma_0(N) \backslash \mathcal{H}^*.$$

The curve $X_0(N)$ is defined over $\mathbb{C}$, but it can be defined over $\mathbb{Q}$ as well (see [DS05, Chapter 7]).

Remark 2.1. The genus g of $X_0(N)$ is given, for $N > 1$, by

$$g = 1 + \frac{d(N)}{12} - \frac{\varepsilon_2(N)}{4} - \frac{\varepsilon_3(N)}{3} - \frac{\varepsilon_\infty(N)}{2},$$

where

$$d(N) = N \prod_{p|N} \frac{p+1}{p},$$

$$\varepsilon_2(N) = \begin{cases} \prod_{p|N}(1 + \left(\frac{-1}{p}\right)), & \text{if } 4 \nmid N \text{ and } N \neq 2, \\ 0, & \text{if } 4 \mid N \text{ and } N \neq 2, \\ 1, & \text{if } N = 2, \end{cases}$$

$$\varepsilon_3(N) = \begin{cases} \prod_{p|N}(1 + \left(\frac{-3}{p}\right)), & \text{if } 9 \nmid N \text{ and } N \neq 2, \\ 0, & \text{if } 9 \mid N \text{ or } N = 2, \end{cases}$$

$$\varepsilon_\infty(N) = \sum_{d|N} \phi\left(\gcd\left(d, \frac{N}{d}\right)\right),$$

with ϕ the Euler's totient function (see for example [DS05, Section 3.9, pag. 107]). In our case, i.e., $N > 1$ coprime with 6, we have

$$g = \frac{N}{12} + o(N), \quad \text{for } N \to +\infty. \tag{2}$$

3 Regular models of $X_0(p^n M)$

In [Edi90], Edixhoven found a regular model of the curve $X_0(N)$, for N coprime to 6. We denote Edixhoven's model by $\mathcal{X}'$. In most of the cases this regular model is minimal. But in some cases some blow downs are necessary to get the minimal regular model (see Section 3.2 below for more details about this). In both cases we denote by $\mathcal{X}$ the minimal regular model of $X_0(N)$.

3.1 Edixhoven's models

In this is subsection we fix $N > 1$ and $(N, 6) = 1$.

Here we describe in a very detailed way the Edixhoven's model $\mathcal{X}'$. For a prime $\ell \nmid N$, the fiber $\mathcal{X}'_\ell$ of $\mathcal{X}'$ at ℓ is smooth (see [DS05, Theorem 8.6.1 and the discussion below]) and it is isomorphic to $X_0(N)/\mathbb{F}_\ell$. The fiber $\mathcal{X}'_p$ at p a prime dividing N is more complicate. Let $N = p^n M$, with $p \nmid M$. The fiber $\mathcal{X}'_p$ has $n+1$ *Igusa components* denoted by C'_a and indexed by an integer $a \in \{0, 1, \ldots, n\}$. Each C'_a is a curve isomorphic to $X_0(M)/\mathbb{F}_p$ with multiplicity $\phi(p^{\min(a, n-a)})$, where ϕ is the Euler's totient function. Let

$$\xi(m) := \frac{1 - \left(\frac{m}{p}\right)}{2}, \tag{3}$$

where $\left(\frac{\cdot}{p}\right)$ is the Kronecker symbol. There are

$$k := \frac{p-1}{12} d(M) - \frac{\xi(-1)\varepsilon_2(M)}{2} - \frac{\xi(-3)\varepsilon_3(M)}{3} \tag{4}$$

points where each Igusa component intersects all the others (the notation is the same as Remark 2.1). Among these k points, $\left\lfloor \frac{p}{12} \right\rfloor$ correspond to the supersingular j-invariants modulo p for $j \notin \{0, 1728\}$. The remaining (if any) comes from the ramification points over the possibly supersingular j-invariant $j \in \{0, 1728\}$. Moreover, there are some *extra components* whose number and properties depend on p mod 12 and are related to the elliptic points of $X_0(M)$.

Case $p \equiv 1 \pmod{12}$**.** In this case for each C'_a with $a \notin \{0,n\}$, there are $\varepsilon_2(M)$ extra components denoted by $E'_{a,i}$, with $i = 1,\ldots,\varepsilon_2(M)$, such that each $E'_{a,i}$ has multiplicity $\frac{1}{2}\phi(p^{\min(a,n-a)})$ and intersects only C'_a at the i-th elliptic point of $X_0(M)$ corresponding to the j-invariant $j = 1728$. Moreover, for each C'_a with $a \notin \{0,n\}$, there are $\varepsilon_3(M)$ extra components denoted by $F'_{a,i}$, with $i = 1,\ldots,\varepsilon_3(M)$, such that $F'_{a,i}$ has multiplicity $\frac{1}{3}\phi(p^{\min(a,n-a)})$ and intersects only C'_a at the i-th elliptic point of $X_0(M)$ corresponding to the j-invariant $j = 0$.

Case $p \equiv 5 \pmod{12}$**.** In this case for each C'_a with $a \notin \{0,n\}$, there are $\varepsilon_2(M)$ extra components denoted by $E'_{a,i}$, with $i = 1,\ldots,\varepsilon_2(M)$, such that each $E'_{a,i}$ has multiplicity $\frac{1}{2}\phi(p^{\min(a,n-a)})$ and intersects only C'_a at the i-th elliptic point of $X_0(M)$ corresponding to the j-invariant $j = 1728$. Moreover, if n is even there are $\varepsilon_3(M)$ extra components denoted by $F'_{\infty,i}$, with $i = 1,\ldots,\varepsilon_3(M)$, each with multiplicity $\frac{1}{3}(p+1)p^{\frac{n}{2}-1}$ that intersects in a single point every C'_a for $0 \le a < \frac{n}{2}$, intersects in a separate point $C'_{\frac{n}{2}}$ and intersects in a separate single point every C'_a for $\frac{n}{2} < a \le n$; these intersection points on the Igusa components correspond to the i-th elliptic point of $X_0(M)$ associated to the j-invariant $j = 0$. If n is odd there are $2\varepsilon_3(M)$ extra components denoted by $F'_{0,i}$ and $F'_{n,i}$, with $i = 1,\ldots,\varepsilon_3(M)$, each with multiplicity $p^{\frac{n-1}{2}}$ and such that $F'_{0,i}$ intersects $F'_{n,i}$ (same i) in one point, $F'_{0,i}$ intersects in a single point every C'_a for $0 \le a < \frac{n}{2}$ and $F'_{n,i}$ intersects in a single point every C'_a for $\frac{n}{2} < a \le n$; these intersection points on the Igusa components correspond to the i-th elliptic point of $X_0(M)$ associated to the j-invariant $j = 0$.

Case $p \equiv 7 \pmod{12}$**.** In this case there are $\varepsilon_2(M)$ extra components denoted by $E'_{\infty,i}$, with $i = 1,\ldots,\varepsilon_2(M)$, each with multiplicity $\frac{1}{2}(p+1)p^{\frac{n}{2}-1}$ if n is even and $p^{\frac{n-1}{2}}$ if n is odd and that intersects in a single point every C'_a for $0 \le a < \frac{n}{2}$, intersects in a separate single point every C'_a for $\frac{n}{2} < a \le n$ and intersects (only when n is even) in a separate point $C'_{\frac{n}{2}}$; these intersection points on the Igusa components correspond to the i-th elliptic point of $X_0(M)$ associated to the j-invariant $j = 1728$. Moreover, for each C'_a with $a \notin \{0,n\}$, there are $\varepsilon_3(M)$ extra components denoted by $F'_{a,i}$, with $i = 1,\ldots,\varepsilon_3(M)$, such that $F'_{a,i}$ has multiplicity $\frac{1}{3}\phi(p^{\min(a,n-a)})$ and intersects only C'_a at the i-th elliptic point of $X_0(M)$ corresponding to the j-invariant $j = 0$.

Case $p \equiv 11 \pmod{12}$**.** In this case there are $\varepsilon_2(M)$ extra components denoted by $E'_{\infty,i}$, with $i = 1,\ldots,\varepsilon_2(M)$, each with multiplicity $\frac{1}{2}(p+1)p^{\frac{n}{2}-1}$ if n is even and $p^{\frac{n-1}{2}}$ if n is odd and that intersects in a single point every C'_a for $0 \le a < \frac{n}{2}$, intersects in a separate single point every C'_a for $\frac{n}{2} < a \le n$ and intersects (only when n is even) in a separate point $C'_{\frac{n}{2}}$; these intersection points on the Igusa components correspond to the i-th elliptic point of $X_0(M)$ associated to the j-invariant $j = 1728$. Moreover, if n is even there are $\varepsilon_3(M)$ extra components denoted by $F'_{\infty,i}$, with $i = 1,\ldots,\varepsilon_3(M)$, each with multiplicity $\frac{1}{3}(p+1)p^{\frac{n}{2}-1}$ that intersects in a single point every C'_a for $0 \le a < \frac{n}{2}$, intersects in a separate point $C'_{\frac{n}{2}}$ and intersects in a separate single point every C'_a for $\frac{n}{2} < a \le n$; these intersection points on the Igusa components correspond to the i-th elliptic point of $X_0(M)$ associated to the j-invariant $j = 0$. If n is odd there are $2\varepsilon_3(M)$ extra components denoted by $F'_{0,i}$ and $F'_{n,i}$, with $i = 1,\ldots,\varepsilon_3(M)$, each with multiplicity $p^{\frac{n-1}{2}}$ and such that $F'_{0,i}$ intersects $F'_{n,i}$ (same i) in one point, $F'_{0,i}$ intersects in a single point every C'_a for $0 \le a < \frac{n}{2}$ and $F'_{n,i}$ intersects in a single point every C'_a for $\frac{n}{2} < a \le n$; these intersection points on the Igusa components correspond to the i-th elliptic point of $X_0(M)$ associated to the j-invariant $j = 0$.

Hence, we have that:

- if $p \equiv 1 \pmod{12}$,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) \left(C'_a + \frac{1}{2} \sum_{i=1}^{\varepsilon_2(M)} E'_{a,i} + \frac{1}{3} \sum_{i=1}^{\varepsilon_3(M)} F'_{a,i} \right);$$

- if $p \equiv 5 \pmod{12}$ and n is even,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) \left(C'_a + \frac{1}{2} \sum_{i=1}^{\varepsilon_2(M)} E'_{a,i} \right) + \frac{1}{3}(p+1)p^{\frac{n}{2}-1} \sum_{i=1}^{\varepsilon_3(M)} F'_{\infty,i};$$

- if $p \equiv 5 \pmod{12}$ and n is odd,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) \left(C'_a + \frac{1}{2} \sum_{i=1}^{\varepsilon_2(M)} E'_{a,i} \right) + p^{\frac{n-1}{2}} \sum_{i=1}^{\varepsilon_3(M)} \left(F'_{0,i} + F'_{n,i} \right);$$

- if $p \equiv 7 \pmod{12}$ and n is even,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) \left(C'_a + \frac{1}{3} \sum_{i=1}^{\varepsilon_3(M)} F'_{a,i} \right) + \frac{1}{2}(p+1)p^{\frac{n}{2}-1} \sum_{i=1}^{\varepsilon_2(M)} E'_{\infty,i};$$

- if $p \equiv 7 \pmod{12}$ and n is odd,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) \left(C'_a + \frac{1}{3} \sum_{i=1}^{\varepsilon_3(M)} F'_{a,i} \right) + p^{\frac{n-1}{2}} \sum_{i=1}^{\varepsilon_2(M)} E'_{\infty,i};$$

- if $p \equiv 11 \pmod{12}$ and n is even,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) C'_a + (p+1)p^{\frac{n}{2}-1} \left(\frac{1}{2} \sum_{i=1}^{\varepsilon_2(M)} E'_{\infty,i} + \frac{1}{3} \sum_{i=1}^{\varepsilon_3(M)} F'_{\infty,i} \right);$$

- if $p \equiv 11 \pmod{12}$ and n is odd,

$$\mathcal{X}'_p = C'_0 + C'_n + \sum_{a=1}^{n-1} \phi(p^{\min(a,n-a)}) C'_a + p^{\frac{n-1}{2}} \left(\sum_{i=1}^{\varepsilon_2(M)} E'_{\infty,i} + \sum_{i=1}^{\varepsilon_3(M)} \left(F'_{0,i} + F'_{n,i} \right) \right).$$

By [Edi90], we know the local equation of each component. Let P be a point not corresponding to an elliptic point of $X_0(M)$, then the local equation of C'_a near P is

$$\begin{cases} x - y^{p^n} = 0, & \text{if } a = 0, \\ x^{p^n} - y = 0, & \text{if } a = n, \\ (x^{p^{a-1}} - y^{p^{n-a-1}})^{p-1} = 0, & \text{if } a \neq 0, n. \end{cases}$$

Let P be a point corresponding to an elliptic point over $j = 1728$. If $a < \frac{n}{2}$, then the local equation near P of E'_a is $t = 0$ and the local equation near P of C'_a is

$$u - t^{\frac{p^{n-2a}-1}{2}} = 0,$$

where $t = y^2$ and $u = \frac{x}{y}$. If $a > \frac{n}{2}$, then the local equation near P of E'_a is $s = 0$ and the local equation near P of C'_a is

$$s^{\frac{p^{2a-n}-1}{2}} - v = 0,$$

where $s = x^2$ and $v = \frac{y}{x}$. Let P be a point corresponding to an elliptic point over $j = 0$. If $a < \frac{n}{2}$, then the local equation near P of F'_a is $t = 0$ and the local equation near P of C'_a is

$$u - t^{\frac{p^{n-2a}-1}{3}} = 0,$$

where $t = y^3$ and $u = \frac{x}{y}$. If $a > \frac{n}{2}$, then the local equation near P of F'_a is $s = 0$ and the local equation near P of C'_a is

$$s^{\frac{p^{2a-n}-1}{3}} - v = 0,$$

where $s = x^3$ and $v = \frac{y}{x}$.

Recalling that if A is a local algebra over a field k with maximal ideal $\mathfrak{m}$ and M is a finitely generated A-module, we have that

$$\text{length}_A(M)\dim_k(A/\mathfrak{m}) = \dim_k(M).$$

See for example [Liu06, Exercise 1.6 (c), Chapter 7]. Hence, in order to find the local intersection numbers we compute

$$\dim_{\mathbb{F}_p} A/(f,g),$$

where P is the point of intersection, $A = \mathcal{O}_{\mathcal{X}',P}$ and $f = 0$ and $g = 0$ are the local equations at P of the two prime components considered. Then we sum these numbers for each intersection point of the two prime components. For instance, if $a < a' < n/2$ and P is a supersingular point that is not an elliptic point, we have that

$$A = \mathcal{O}_{\mathcal{X}',P} = \mathbb{F}_p[x,y]_{(x,y)},$$

C'_a has multiplicity $(p-1)p^{a-1}$ and local equation at P

$$x - y^{p^{n-2a}} = 0,$$

$C'_{a'}$ has multiplicity $(p-1)p^{a'-1}$ and local equation at P

$$x - y^{p^{n-2a'}} = 0.$$

Hence

$$A/(f,g) = \mathbb{F}_p[x,y]_{(x,y)}/(x - y^{p^{n-2a}}, x - y^{p^{n-2a'}}).$$

Since in A we have

$$\begin{aligned}
&\left(x - y^{p^{n-2a}}, x - y^{p^{n-2a'}}\right) = \left(y^{p^{n-2a'}} - y^{p^{n-2a}}, x - y^{p^{n-2a'}}\right) = \\
&= \left(y^{p^{n-2a'}}(1 - y^{p^{2a'-2a}}), x - y^{p^{n-2a'}}\right) = \left(y^{p^{n-2a'}}, x - y^{p^{n-2a'}}\right) = \\
&= \left(y^{p^{n-2a'}}, x\right),
\end{aligned}$$

then

$$\dim_{\mathbb{F}_p} \mathbb{F}_p[x,y]_{(x,y)}/(y^{p^{n-2a'}}, x) = p^{n-2a'}.$$

Finally, since we have k supersingular points that are not elliptic points, we obtain $C'_a \cdot C'_{a'} = kp^{n-2a'}$ when $p \equiv 1 \pmod{12}$. In a similar way we get all the intersection numbers: Let ξ and k be defined respectively in Equations (3) and (4) and let

$$\begin{aligned}
\mu(a,a') &:= \min(|n-2a|, |n-2a'|), \\
I(a,a') &:= \begin{cases} 0, & \text{if } (n-2a)(n-2a') \le 0, \\ 1, & \text{if } (n-2a)(n-2a') > 0. \end{cases}
\end{aligned}$$

Then the intersection numbers between the components of $\mathcal{X}'_p$ are the following:

$$\begin{aligned}
C'_a \cdot C'_{a'} &= kp^{\mu(a,a')I(a,a')} + \frac{1}{2}\xi(-1)\varepsilon_2(M)(p^{\mu(a,a')} - 1)I(a,a') + \frac{1}{3}\xi(-3)\varepsilon_3(M)\left(p^{\mu(a,a')} - \frac{3-(-1)^n}{2}\right)I(a,a'), \\
&\qquad \text{for } a \neq a',\ a, a' \in \{0,\dots,n\}, \\
C'_a \cdot E'_{a',i} &= \begin{cases} 1, & \text{if } a = a' \text{ or } a' = \infty, \\ 0, & \text{otherwise,} \end{cases} \\
&\qquad \text{for } a \in \{0,\dots,n\},\ a' \in \{1,\dots,n-1,\infty\},\ i \in \{1,\dots,\varepsilon_2(M)\},
\end{aligned}$$

$$
C'_a \cdot F'_{a',i} = \begin{cases} 1, & \begin{cases} \text{if } a = a' \\ \text{or } a' = \infty \\ \text{or } a' = 0 \text{ and } n-2a > 0 \\ \text{or } a' = n \text{ and } n-2a < 0, \end{cases} \\ 0, & \text{otherwise,} \end{cases}
$$

$$
\text{for } a \in \{0,\dots,n\},\ a' \in \{0,\dots,n,\infty\},\ i \in \{1,\dots,\varepsilon_3(M)\},
$$

$$
E'_{a,i} \cdot E'_{a',i'} = 0, \quad \text{for } (a,i) \neq (a',i'),\ a,a' \in \{1,\dots,n-1,\infty\},\ i,i' \in \{1,\dots,\varepsilon_2(M)\},
$$

$$
E'_{a,i} \cdot F'_{a',i'} = 0, \quad \text{for } a \in \{1,\dots,n-1,\infty\},\ a' \in \{0,\dots,n,\infty\},\ i \in \{1,\dots,\varepsilon_2(M)\},\ i' \in \{1,\dots,\varepsilon_3(M)\},
$$

$$
F'_{a,i} \cdot F'_{a',i'} = \begin{cases} 1, & \text{if } (a,a') \in \{(0,n),(n,0)\} \text{ and } i = i', \\ 0, & \text{otherwise,} \end{cases}
$$

$$
\text{for } (a,i) \neq (a',i'),\ a,a' \in \{0,\dots,n,\infty\},\ i,i' \in \{1,\dots,\varepsilon_3(M)\}.
$$

By [Liu06, Proposition 1.21 (a), Chapter 9], for each prime divisor C we have $C \cdot \mathcal{X}'_p = 0$. Hence, from the previous intersection numbers we get the following self-intersection numbers:

$$
\begin{aligned}
(C'_0)^2 = (C'_n)^2 &= -\frac{1}{12} d(M) p^{n-1}(p-1) - \frac{1}{2}\xi(-1)\varepsilon_2(M) - \frac{3-(-1)^n}{6}\xi(-3)\varepsilon_3(M), \\
(C'_a)^2 &= -\frac{d(M)p^{|n-2a|}}{6} - \frac{\varepsilon_2(M)}{2} - \frac{\varepsilon_3(M)}{3} - \frac{1-(-1)^n}{6}\xi(-3)\varepsilon_3(M), \quad \text{for } a \in \{1,\dots,n-1\}, \\
(E'_{a,i})^2 &= -2, \quad \text{for } a \in \{1,\dots,n-1,\infty\},\ i \in \{1,\dots,\varepsilon_2(M)\}, \\
(F'_{a,i})^2 &= \begin{cases} -2, & \text{if } a \in \{0,n\}, \\ -3, & \text{if } a \in \{1,\dots,n-1,\infty\}, \end{cases} \quad \text{for } i \in \{1,\dots,\varepsilon_3(M)\}.
\end{aligned}
$$

For the reader's convenience we show how to obtain for instance $(E'_{a,i})^2$, for $p \equiv 1 \pmod{12}$:

$$
\begin{aligned}
&E'_{a,i} \cdot \mathcal{X}'_p = 0, \\
&E'_{a,i} \cdot \left[C'_0 + C'_n + \sum_{a'=1}^{n-1} \phi(p^{\min(a',n-a')}) \left(C'_{a'} + \frac{1}{2}\sum_{i'=1}^{\varepsilon_2(M)} E'_{a',i'} + \frac{1}{3}\sum_{i'=1}^{\varepsilon_3(M)} F'_{a',i'} \right) \right] = 0, \\
&E'_{a,i} \cdot C'_0 + E'_{a,i} \cdot C'_n + \sum_{a'=1}^{n-1} \phi(p^{\min(a',n-a')}) \left(E'_{a,i} \cdot C'_{a'} + \frac{1}{2}\sum_{i'=1}^{\varepsilon_2(M)} E'_{a,i} \cdot E'_{a',i'} + \frac{1}{3}\sum_{i'=1}^{\varepsilon_3(M)} E'_{a,i} \cdot F'_{a',i'} \right) = 0, \\
&0 + 0 + \phi(p^{\min(a,n-a)}) \left(1 + \frac{1}{2}(E'_{a,i})^2 + \frac{1}{3} \cdot 0 \right) = 0, \\
&(E'_{a,i})^2 = -2.
\end{aligned}
$$

Remark 3.1. When n is even, we observe that

$$
(C'_{n/2})^2 = -\frac{d(M) + 3\varepsilon_2(M) + 2\varepsilon_3(M)}{6},
$$

hence it does not depend on p^n, but only on M.

3.2 Non-minimal Edixhoven's models

The self-intersections computed in the end of Section 3.1 are always different from -1 except for:

- if $n = M = 1$ and $p \in \{5,7,13\}$, in these cases $(C'_0)^2 = (C'_1)^2 = -1$ and $C'_0 \cong C'_1 \cong X_0(M) \cong \mathbb{P}^1$;
- if $M = 1$ and n is even (it follows by Remark 3.1), in these cases $(C'_{n/2})^2 = -1$ and $C'_{n/2} \cong X_0(M) \cong \mathbb{P}^1$.

So in all these cases the Edixhoven's model is not minimal. Since we are interested in the asymptotic behaviour of the intersection numbers, we don't discuss the *ad hoc* blow downs necessary when $N \in \{5, 7, 13\}$ and for simplicity we make the following assumption:

From now on until the end of Section 3 we fix $N > 1$, $(N, 6) = 1$ and $N \notin \{5, 7, 13\}$.

If $M = 1$ and n is even, we need three blow downs to get the minimal regular model. We denote the composition of these three blow downs by $\pi\colon \mathcal{X}' \to \mathcal{X}$. If $\mathcal{X}'$ is already minimal, π is just the identity map. Let C be a prime divisor of $\mathcal{X}$, let $\pi^*(C)$ be the pullback of C, let C' be the corresponding divisor of C in $\mathcal{X}'$ and, for $i = 1, \dots, c$, let D_i be the prime divisors contracted by π. Repeatedly applying [Liu06, Proposition 2.23, Chapter 9], we get

$$\pi^*(C) = C' + \sum_{i=1}^{c} d_i D_i,$$

with $d_1, \dots, d_c \in \mathbb{Z}$. So, we can compute the d_i's using the fact that $D_i \cdot \pi^*(C) = 0$ for every $i = 1, \dots, c$, see [Liu06, Theorem 2.12 (a), Chapter 9]. Now we list the results of the pullbacks when $M = 1$ and n is even. Since $d(1) = \varepsilon_2(1) = \varepsilon_3(1) = 1$ (see Remark 2.1 for the notation), we have:

Case $p \equiv 1 \pmod{12}$. We have to contract $C'_{n/2}$, $E'_{n/2,1}$ and $F'_{n/2,1}$ obtaining

$$\begin{aligned}
\pi^*(C_a) &= C'_a + 6kC'_{n/2} + 3kE'_{n/2,1} + 2kF'_{n/2,1}, \quad \text{for } a \neq n/2,\\
\pi^*(E_{a,1}) &= E'_{a,1}, \quad \text{for } a \neq n/2,\\
\pi^*(F_{a,1}) &= F'_{a,1}, \quad \text{for } a \neq n/2.
\end{aligned}$$

Case $p \equiv 5 \pmod{12}$. We have to contract $C'_{n/2}$, $E'_{n/2,1}$ and F'_{∞} obtaining

$$\begin{aligned}
\pi^*(C_a) &= C'_a + (6k+2)C'_{n/2} + (3k+1)E'_{n/2,1} + (2k+1)F'_{\infty,1}, \quad \text{for } a \neq n/2,\\
\pi^*(E_{a,1}) &= E'_{a,1}, \quad \text{for } a \neq n/2.
\end{aligned}$$

Case $p \equiv 7 \pmod{12}$. We have to contract $C'_{n/2}$, $E'_{\infty,1}$ and $F'_{n/2,1}$ obtaining

$$\begin{aligned}
\pi^*(C_a) &= C'_a + (6k+3)C'_{n/2} + (3k+2)E'_{\infty,1} + (2k+1)F'_{n/2,1}, \quad \text{for } a \neq n/2,\\
\pi^*(F_{a,1}) &= F'_{a,1}, \quad \text{for } a \neq n/2.
\end{aligned}$$

Case $p \equiv 11 \pmod{12}$. We have to contract $C'_{n/2}$, $E'_{\infty,1}$ and $F'_{\infty,1}$ obtaining

$$\pi^*(C_a) = C'_a + (6k+5)C'_{n/2} + (3k+3)E'_{\infty,1} + (2k+2)F'_{\infty,1}, \quad \text{for } a \neq n/2.$$

3.3 Minimal regular models and intersection matrices

Here we describe the minimal regular model for $X_0(N)$ with $N = p^n M > 1$, coprime to 6 and $N \notin \{5, 7, 13\}$. As explained in Section 3.2 above, if either $M > 1$ or n is odd, the Edixhoven's model is already a minimal regular model. In every case we use the same notation as Section 3.1 for model and components just dropping the dash when we refer to the minimal regular model. In the cases where blow downs are necessary, $N = p^n$ and the only fiber that change is $\mathcal{X}_p$, hence we write it below for convenience. Since $C \cdot D = \pi^*(C) \cdot \pi^*(D)$ for every prime divisor C and D (see [Liu06, Theorem 2.12 (c), Chapter 9]), using Section 3.2 we compute all the intersection numbers of the components of $\mathcal{X}_p$. Finally, we compute the self-intersection numbers, as above, using the intersection numbers and the fact that for each prime divisor C we have $C \cdot \mathcal{X}_p = 0$ ([Liu06, Proposition 1.21 (a), Chapter 9]).

$$\mathcal{X}_p = \begin{cases} \sum_{\substack{a=0\\a\neq n/2}}^{n} \phi(p^{\min(a,n-a)})C_a + \sum_{\substack{a=1\\a\neq n/2}}^{n-1} \phi(p^{\min(a,n-a)})\left(\frac{1}{2}E_{a,1} + \frac{1}{3}F_{a,1}\right), & \text{if } p \equiv 1 \pmod{12}, \\ \sum_{\substack{a=0\\a\neq n/2}}^{n} \phi(p^{\min(a,n-a)})C_a + \frac{1}{2}\sum_{\substack{a=1\\a\neq n/2}}^{n-1} \phi(p^{\min(a,n-a)})E_{a,1}, & \text{if } p \equiv 5 \pmod{12}, \\ \sum_{\substack{a=0\\a\neq n/2}}^{n} \phi(p^{\min(a,n-a)})C_a + \frac{1}{3}\sum_{\substack{a=1\\a\neq n/2}}^{n-1} \phi(p^{\min(a,n-a)})F_{a,1}, & \text{if } p \equiv 7 \pmod{12}, \\ \sum_{\substack{a=0\\a\neq n/2}}^{n} \phi(p^{\min(a,n-a)})C_a, & \text{if } p \equiv 11 \pmod{12}, \end{cases}$$

$$C_a \cdot C_{a'} = \begin{cases} C'_a \cdot C'_{a'} + 6k^2, & \text{if } p \equiv 1 \pmod{12}, \\ C'_a \cdot C'_{a'} + 6k^2 + 4k + 1, & \text{if } p \equiv 5 \pmod{12}, \\ C'_a \cdot C'_{a'} + 6k^2 + 6k + 2, & \text{if } p \equiv 7 \pmod{12}, \\ C'_a \cdot C'_{a'} + 6k^2 + 10k + 5, & \text{if } p \equiv 11 \pmod{12}, \end{cases} \quad \text{for } a, a' \in \{0, \ldots, n\} - \{n/2\},$$

$$C_a \cdot E_{a',1} = \begin{cases} 1, & \text{if } a = a', \\ 0, & \text{if } a \neq a', \end{cases} \quad \text{for } a \in \{0, \ldots, n\} - \{n/2\},\ a' \in \{1, \ldots, n-1\} - \{n/2\},$$

$$C_a \cdot F_{a',1} = \begin{cases} 1, & \text{if } a = a', \\ 0, & \text{if } a \neq a', \end{cases} \quad \text{for } a \in \{0, \ldots, n\} - \{n/2\},\ a' \in \{1, \ldots, n-1\} - \{n/2\},$$

$$E_{a,1} \cdot E_{a',1} = \begin{cases} -2, & \text{if } a = a', \\ 0, & \text{if } a \neq a', \end{cases} \quad \text{for } a, a' \in \{1, \ldots, n-1\} - \{n/2\},$$

$$E_{a,1} \cdot F_{a',1} = 0, \quad \text{for } a, a' \in \{1, \ldots, n-1\} - \{n/2\},$$

$$F_{a,1} \cdot F_{a',1} = \begin{cases} -3, & \text{if } a = a', \\ 0, & \text{if } a \neq a', \end{cases} \quad \text{for } a, a' \in \{1, \ldots, n-1\} - \{n/2\}.$$

4 Asymptotics for the self-intersection of $\overline{\omega}$

The proof of Theorem 1.1 is performed in several steps. We continue with the same notation as the previous sections, so $f\colon \mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is the minimal regular model of $X_0(N)$. The geometric genus of $X_0(N)$ is g and for each prime $\ell \nmid N$ the fiber $\mathcal{X}_\ell$ is irreducible with genus g. It is well known that the genus of $X_0(N)$ is 0 if and only if $N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25\}$, so in these cases the canonical Kähler form Ω^{can} is not well defined, therefore we make the following assumption:

From now on we fix $N > 1$, $(N, 6) = 1$ and $N \notin \{5, 7, 13, 25\}$.

In Section 4.1 we prove the following formula for the self-intersection of the canonical Arakelov divisor:

$$\langle \overline{\omega}, \overline{\omega} \rangle = \underbrace{-4g(g-1)\langle H_0, H_\infty \rangle}_{(a)} \underbrace{- \frac{\langle V_0, V_0 \rangle + \langle V_\infty, V_\infty \rangle}{2g-2} + \frac{g\langle V_0, V_\infty \rangle}{g-1}}_{(b)} \underbrace{+ \frac{h_0 + h_\infty}{2}}_{(c)}, \tag{5}$$

where H_0 and H_∞ are the horizontal divisors on $\mathcal{X}$ induced by the cusps 0 and ∞ of $X_0(N)$, V_0 and V_∞ are two carefully chosen vertical divisors supported over the primes dividing N, h_0 and h_∞ are certain Néron-Tate's heights of some points in the Jacobian of $X_0(N)$. We stress that the crucial point is the *constructive proof* of the existence of V_0 and V_∞ (see Proposition 4.3). In fact assume that $\mathcal{P}$ is the set of primes dividing N and that $V_m^{(p)}$ is the part of V_m supported over p, for $p \in \mathcal{P}$ and $m \in \{0, \infty\}$; then the vectors made of the (rational) multiplicities of the components of $V_0^{(p)}$ and $V_\infty^{(p)}$ are the solutions of two linear systems. These systems are described and solved in Section 4.2. The last step of the proof consists in computing the asymptotics for $N \to +\infty$ for all summands of Equation (5). The estimates of the pieces (a) and (c) of the formula are already known in the literature, so in Section 4.3 we *exactly compute* (b) and we study its asymptotics. Finally, we put all together to conclude the proof.

4.1 A formula for $\langle \overline{\omega}, \overline{\omega} \rangle$

We denote by H_0 and H_∞ the closures in $\mathcal{X}$ of the two cusps 0 and ∞. By [Liu06, Proposition 1.30, Chapter 9], we know that $H_m \cdot \mathcal{X}_q = 1$, for $m \in \{0, \infty\}$, and every prime q. Moreover, by the components labelling of [KM85, page 296], we can assume that H_0 and H_∞ respectively meet transversally the components C_0 and C_n of each special fiber. Let

$$G_m := \mathcal{K} - (2g-2)H_m, \quad \text{for } m \in \{0, \infty\}.$$

Proposition 4.1. *Let F be a vertical divisor of $\mathcal{X}$ which is not supported over the primes diving N, then*

$$G_m \cdot F = 0, \quad \text{for } m \in \{0, \infty\}.$$

Proof. Since the primes dividing N are the only prime whose fiber can contain more than one component, we write $F = \sum_{\ell \nmid N} n_\ell \mathcal{X}_\ell$, where $n_\ell \in \mathbb{Z}$. By the adjunction formula (see [Liu06, Theorem 1.37, Chapter 9]), we have $\mathcal{K} \cdot \mathcal{X}_\ell = 2g-2$. Moreover, by [Liu06, Proposition 1.30, Chapter 9], we have $H_m \cdot \mathcal{X}_\ell = 1$. ☐

Definition 4.2. A divisor D of $\mathcal{X}$ is called *f-numerically trivial* if $D \cdot F = 0$ for every vertical divisor F.

In the following proposition we explain how we can modify the divisors E_m in order to get a f-numerically trivial $\mathbb{Q}$-divisor. It is a special case of [Mor13, Lemma 2.2.2], but we prefer write a full proof since it is constructive and it is crucial in order to carry out the computations of Section 4.2.

Proposition 4.3. *There are two vertical $\mathbb{Q}$-divisors V_0 and V_∞ supported over the primes dividing N such that*

$$D_m := G_m + V_m, \quad \text{for } m \in \{0, \infty\},$$

is f-numerically trivial.

Proof. We explain how to effectively construct infinitely many V_0 and V_∞ that satisfy the proposition. Let $\mathcal{P}$ be the set of primes dividing N and let ν_p be the number of distinct components of X_p. For simplicity of presentation we relabel all the components and the relative multiplicities so that we can write:

$$\mathcal{X}_p = \sum_{i=1}^{\nu_p} m_i^{(p)} \Gamma_i^{(p)}, \quad \text{for } p \in \mathcal{P}.$$

Then by using [Liu06, Remark 1.31, Chapter 9] we get:

$$\sum_{i=1}^{\nu_p} m_i^{(p)} (G_m \cdot \Gamma_i^{(p)}) = G_m \cdot \mathcal{X}_p = \deg_{\mathbb{Q}} G_{m,\mathbb{Q}} = 0.$$

By [Mor13, Lemma 2.2.1], it follows that the vector $(G_m \cdot \Gamma_1^{(p)}, \ldots, G_m \cdot \Gamma_{\nu_p}^{(p)}) \in \mathbb{Q}^{\nu_p}$ lies in the image of the linear function induced by the matrix $(\Gamma_i^{(p)} \cdot \Gamma_j^{(p)})_{ij}$. So we can find a solution for the linear system:

$$\begin{pmatrix} \Gamma_1^{(p)} \cdot \Gamma_1^{(p)} & \ldots & \Gamma_1^{(p)} \cdot \Gamma_{\nu_p}^{(p)} \\ \vdots & & \vdots \\ \Gamma_{\nu_p}^{(p)} \cdot \Gamma_1^{(p)} & \ldots & \Gamma_{\nu_p}^{(p)} \cdot \Gamma_{\nu_p}^{(p)} \end{pmatrix} \begin{pmatrix} x_1^{(p)} \\ \vdots \\ x_{\nu_p}^{(p)} \end{pmatrix} = \begin{pmatrix} -G_m \cdot \Gamma_1^{(p)} \\ \vdots \\ -G_m \cdot \Gamma_{\nu_p}^{(p)} \end{pmatrix}. \tag{6}$$

Actually [Mor13, Lemma 2.2.1] shows that the dimension of the affine subspace of the solutions of system (6) is one. If $(x_1^{(p)}, \dots x_{\nu_p}^{(p)})$ is any solution, we put:

$$V_m^{(p)} := \sum_{i=1}^{\nu_p} x_i^{(p)} \Gamma_i^{(p)}, \quad \text{for } m \in \{0, \infty\}. \tag{7}$$

We define

$$V_m := \sum_{p \in \mathcal{P}} V_m^{(p)}, \quad \text{for } m \in \{0, \infty\}.$$

By Proposition 4.1 and the fact that V_m is supported over $\mathcal{P}$, it follows that $(G_m + V_m) \cdot \mathcal{X}_\ell = 0$, for every $\ell \nmid N$. Moreover, by the construction of the vector $(x_1^{(p)}, \dots, x_{\nu_p}^{(p)})$ for $p \in \mathcal{P}$, we have that $(G_m + V_m) \cdot F = 0$ for every divisor F supported over $\mathcal{P}$. □

From now on we fix a choice of V_0 and V_∞ that satisfy Proposition 4.3, so the divisors D_0 and D_∞ are fixed as well. We recall that $\langle \cdot, \cdot \rangle$ denotes the Arakelov intersection pairing. If one of the two Arakelov divisors is an ordinary vertical divisor supported over a prime p, then their Arakelov intersection differs from the usual intersection pairing (denoted by a dot) by a $\log p$ factor. We define the numbers:

$$h_m := \langle D_m, D_m \rangle, \quad \text{for } m \in \{0, \infty\}.$$

In the next proposition we show how the self-intersection of the Arakelov canonical divisor can be expressed in terms of h_m and some intersection numbers involving only H_m and V_m.

Proposition 4.4. *With the notations fixed above and for any V_0, V_∞ satisfying Proposition 4.3 we have that:*

$$\langle \overline{\omega}, \overline{\omega} \rangle = -4g(g-1)\langle H_0, H_\infty \rangle - \frac{\langle V_0, V_0 \rangle + \langle V_\infty, V_\infty \rangle}{2g-2} + \frac{g\langle V_0, V_\infty \rangle}{g-1} + \frac{h_0 + h_\infty}{2}.$$

Proof. As explained in Section 2.1 we know that $\langle \overline{\omega}, \overline{\omega} \rangle = \langle \mathcal{K}, \mathcal{K} \rangle$. By [Fal84, Theorem 4(c)], we have that $h_m = -2\widehat{h}(D_{m,\mathbb{Q}})$ where $D_{m,\mathbb{Q}}$ can be seen as a point in the Jacobian of $X_0(N)$ and $\widehat{h}$ is the Néron-Tate's height. But $\langle D_m, V_m \rangle = 0$ by Proposition 4.3, so:

$$h_m = \langle D_m, D_m - V_m \rangle = \langle \mathcal{K} - (2g-2)H_m + V_m, \mathcal{K} - (2g-2)H_m \rangle. \tag{8}$$

Expanding Equation (8) we obtain:

$$\langle \mathcal{K}, \mathcal{K} \rangle = 2(2g-2)\langle \mathcal{K}, H_m \rangle - (2g-2)^2 \langle H_m, H_m \rangle - \langle V_m, \mathcal{K} \rangle + (2g-2)\langle V_m, H_m \rangle + h_m. \tag{9}$$

We now expand $\langle D_m, V_m \rangle = 0$ to get the equality

$$-\langle V_m, \mathcal{K} \rangle + (2g-2)\langle V_m, H_m \rangle = \langle V_m, V_m \rangle. \tag{10}$$

Moreover the Arakelov adjunction formula (see [Lan88, Corollary 5.6]) says that

$$\langle \mathcal{K}, H_m \rangle = -\langle H_m, H_m \rangle. \tag{11}$$

By substituting Equations (10) and (11) inside Equation (9) we get:

$$\langle \mathcal{K}, \mathcal{K} \rangle = -4g(g-1)\langle H_m, H_m \rangle + \langle V_m, V_m \rangle + h_m.$$

Now summing for $m = 0, \infty$, gives:

$$\langle \mathcal{K}, \mathcal{K} \rangle = -2g(g-1)\left(\langle H_0, H_0 \rangle + \langle H_\infty, H_\infty \rangle\right) + \frac{\langle V_0, V_0 \rangle + \langle V_\infty, V_\infty \rangle + h_0 + h_\infty}{2}. \tag{12}$$

Consider the divisor

$$D_\infty - D_0 = (2g-2)(H_0 - H_\infty) + V_\infty - V_0.$$

It satisfies the hypotheses of [Fal84, Theorem 4(c)]; but $(D_\infty - D_0)_{\mathbb{Q}}$ is supported on the cusps of $X_0(N)$, therefore by the Manin-Drinfeld's theorem (see [Man72, Corollary 3.6] and [Dri73, Theorem 1]) it is a torsion element in the Jacobian. It follows that $\langle D_\infty - D_0, D_\infty - D_0\rangle = 0$, which means:

$$\langle H_0, H_0\rangle + \langle H_\infty, H_\infty\rangle = 2\langle H_0, H_\infty\rangle + \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle - 2\langle V_0, V_\infty\rangle}{(2g-2)^2}. \tag{13}$$

Substituting Equation (13) inside Equation (12) we finally get:

$$\langle \mathcal{K}, \mathcal{K}\rangle = -4g(g-1)\langle H_0, H_\infty\rangle - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} + \frac{g\langle V_0, V_\infty\rangle}{g-1} + \frac{h_0 + h_\infty}{2}.$$

□

We point out that the strategy we used to obtain the above formula for $\langle \overline{\omega}, \overline{\omega}\rangle$ cannot be applied to the minimal model of a general algebraic curve. In fact the proof of Proposition 4.4 depends heavily on the Manin-Drinfeld's theorem which is a specific property of modular curves.

4.2 Computation of V_0 and V_∞ when $N = p^n$

From now on we fix $N = p^n$, $p > 3$ and $p^n \notin \{5, 7, 13, 25\}$.

In this section we compute V_0 and V_∞ for $N = p^n$, with $p > 3$ and $p^n \notin \{5, 7, 13, 25\}$, since in this case we can express the coefficients of V_0 and V_∞ as polynomials in p. Notice that under this assumption on N the divisors V_0 and V_∞ are supported only over p. Keeping the notation of Section 3, we denote by $C_a, E_{a,i}, F_{a,i}$ the components of the special fiber $\mathcal{X}_p$ in $\mathcal{X}$ and by $C'_a, E'_{a,i}, F'_{a,i}$ components of the special fiber $\mathcal{X}'_p$ in $\mathcal{X}'$. Moreover $\mathcal{K}$ is a canonical divisor of $\mathcal{X}$ and $\mathcal{K}'$ is a canonical divisor of $\mathcal{X}'$. Let C be one of the components of $\mathcal{X}_p$ of genus g_C, by the adjunction formula we know that $C \cdot (C + \mathcal{K}) = 2g_C - 2$ (see for instance [Liu06, Theorem 1.37, Chapter 9]). From this, we can compute the intersection $C \cdot \mathcal{K}$ because:

$$\begin{aligned} C \cdot (C + \mathcal{K}) &= 2g_C - 2, \\ C \cdot C + C \cdot \mathcal{K} &= 2g_C - 2, \\ C \cdot \mathcal{K} &= 2g_C - 2 - C^2. \end{aligned} \tag{14}$$

If $M = 1$ and n is even, i.e., if we need to blow down, we have

$$\pi^*(\mathcal{K}) = \begin{cases} \mathcal{K}' - 4C'_{n/2} - 2E'_{n/2,1} - F'_{n/2,1}, & \text{if } p \equiv 1 \pmod{12}, \\ \mathcal{K}' - 4C'_{n/2} - 2E'_{n/2,1} - F'_{\infty,1}, & \text{if } p \equiv 5 \pmod{12}, \\ \mathcal{K}' - 4C'_{n/2} - 2E'_{\infty,1} - F'_{n/2,1}, & \text{if } p \equiv 7 \pmod{12}, \\ \mathcal{K}' - 4C'_{n/2} - 2E'_{\infty,1} - F'_{\infty,1}, & \text{if } p \equiv 11 \pmod{12}, \end{cases}$$

where the computations are done as explained in Section 3.2. Since when $M = 1$ we have that

$$g_{C'_a} = g_{E'_{a',1}} = g_{F'_{a',1}} = 0, \quad \text{for every } a \in \{0, \dots, n\} \text{ and } a' \in \{1, \dots, n-1, \infty\}, \tag{15}$$

it follows that

$$\begin{aligned} C_a \cdot \mathcal{K} &= -2 - (C'_a)^2 - 4k, \quad \text{for } a \in \{0, \dots, n\} - \{n/2\}, \\ E_{a,1} \cdot \mathcal{K} &= 0, \quad \text{for } a \in \{1, \dots, n-1, \infty\} - \{n/2\}, \\ F_{a,1} \cdot \mathcal{K} &= 1, \quad \text{for } a \in \{1, \dots, n-1, \infty\} - \{n/2\}. \end{aligned} \tag{16}$$

We explain how to compute, for instance, $C_a \cdot \mathcal{K}$ with $a \neq n/2$ and $p \equiv 1 \pmod{12}$:

$$\begin{aligned} C_a \cdot \mathcal{K} = &\pi^*(C_a) \cdot \pi^*(\mathcal{K}) = \\ &= (C'_a + 6kC'_{n/2} + 3kE'_{n/2,1} + 2kF'_{n/2,1}) \cdot (\mathcal{K}' - 4C'_{n/2} - 2E'_{n/2,1} - F'_{n/2,1}) = \\ &= C'_a \cdot \mathcal{K}' - 4C'_a \cdot C'_{n/2} + 6kC'_{n/2} \cdot \mathcal{K}' - 24k(C'_{n/2})^2 - 24kC'_{n/2} \cdot E'_{n/2,1} - 14kC'_{n/2} \cdot F'_{n/2,1} + \\ &+ 3kE'_{n/2,1} \cdot \mathcal{K}' - 6k(E'_{n/2,1})^2 + 2kF'_{n/2,1} \cdot \mathcal{K}' - 2k(F'_{n/2,1})^2 = \\ &= 2g_{C'_a} - 2 - (C'_a)^2 - 4k + 6k(2g_{C'_{n/2}} - 2 - (C'_{n/2})^2) + 24k - 24k - 14k + \\ &+ 3k(2g_{E'_{n/2,1}} - 2 - (E'_{n/2,1})^2) + 12k + 2k(2g_{F'_{n/2,1}} - 2 - (F'_{n/2,1})^2) + 6k = \\ &= 2g_{C'_a} - 2 - (C'_a)^2 - 4k - 6k - 14k + 12k + 2k + 6k = -2 - (C'_a)^2 - 4k. \end{aligned}$$

Remark 4.5. We do not need this in the following, but we remark that by the previous computations and using Equation (14), we find that the blow downs change the genera of the prime divisors in the following way:

$$\begin{aligned} g_{C_a} &= g_{C'_a} + 3k^2 - 2k, \quad \text{for } a \in \{0, \dots, n\} - \{n/2\}, \\ g_{E_{a,1}} &= g_{E'_{a,1}}, \quad \text{for } a \in \{1, \dots, n-1, \infty\} - \{n/2\}, \\ g_{F_{a,1}} &= g_{F'_{a,1}}, \quad \text{for } a \in \{1, \dots, n-1, \infty\} - \{n/2\}. \end{aligned}$$

For example, again for C_a with $a \neq n/2$ and $p \equiv 1 \pmod{12}$, we have that

$$2g_{C_a} - 2 - C_a^2 = C_a \cdot \mathcal{K} = 2g_{C'_a} - 2 - (C'_a)^2 - 4k,$$

and then

$$g_{C_a} = g_{C'_a} + \frac{C_a^2 - (C'_a)^2}{2} - 2k = g_{C'_a} + 3k^2 - 2k.$$

Moreover, by Equation (15), we have

$$\begin{aligned} g_{C_a} &= 3k^2 - 2k, \quad \text{for } a \in \{0, \dots, n\} - \{n/2\}, \\ g_{E_{a,1}} &= 0, \quad \text{for } a \in \{1, \dots, n-1, \infty\} - \{n/2\}, \\ g_{F_{a,1}} &= 0, \quad \text{for } a \in \{1, \dots, n-1, \infty\} - \{n/2\}. \end{aligned}$$

By Equations (15) and (16) and using the intersection numbers computed in Sections 3.1 and 3.3, now we can explicitly write down the linear systems in Equation (6) for any $p \mid N$ and $m \in \{0, \infty\}$. We remark that the constant terms of the systems

$$-G_m \cdot C = (2g-2)H_m \cdot C - \mathcal{K} \cdot C, \quad \text{for } C \text{ a component},$$

are given by Equation (14) and

$$\begin{aligned} H_m \cdot C_0 &= \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m = \infty, \end{cases} \\ H_m \cdot C_n &= \begin{cases} 1, & \text{if } m = \infty, \\ 0, & \text{if } m = 0, \end{cases} \\ H_m \cdot C_a &= H_m \cdot E_{a,i} = H_m \cdot F_{a,i} = 0, \quad \text{for every } m, a, i. \end{aligned}$$

Below we write these systems and their solutions for the different cases when $M = 1$.

4.2.1 Case n odd and $p \equiv 1 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix} C_0^2 & C_0 \cdot C_1 & \dots & C_0 \cdot F_{n-1,1} \\ C_1 \cdot C_0 & C_1^2 & \dots & C_1 \cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ C_n \cdot C_0 & C_n \cdot C_1 & \dots & C_n \cdot F_{n-1,1} \\ E_{1,1} \cdot C_0 & E_{1,1} \cdot C_1 & \dots & E_{1,1} \cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ E_{n-1,1} \cdot C_0 & E_{n-1,1} \cdot C_1 & \dots & E_{n-1,1} \cdot F_{n-1,1} \\ F_{1,1} \cdot C_0 & F_{1,1} \cdot C_1 & \dots & F_{1,1} \cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ F_{n-1,1} \cdot C_0 & F_{n-1,1} \cdot C_1 & \dots & F_{n-1,1}^2 \end{pmatrix} \begin{pmatrix} x_{C_0} \\ x_{C_1} \\ \vdots \\ x_{C_n} \\ x_{E_{1,1}} \\ \vdots \\ x_{E_{n-1,1}} \\ x_{F_{1,1}} \\ \vdots \\ x_{F_{n-1,1}} \end{pmatrix} = \begin{pmatrix} -G_m \cdot C_0 \\ -G_m \cdot C_1 \\ \vdots \\ -G_m \cdot C_n \\ -G_m \cdot E_{1,1} \\ \vdots \\ -G_m \cdot E_{n-1,1} \\ -G_m \cdot F_{1,1} \\ \vdots \\ -G_m \cdot F_{n-1,1} \end{pmatrix}.$$

In the following propositions we write the solutions of the previous linear systems.

Proposition 4.6. *Let $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ and $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for $a \in \{1, \dots, n-1\}$ we have:*

$$\begin{aligned}
u_{C_0} &= t, \\
u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 12\min(2a,n)p^{\frac{n+1}{2}} + \\
&\quad + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 14ap + 14(a-1)\Big), \\
u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 12np^{\frac{n+1}{2}} + \\
&\quad + 12(n-2)p^{\frac{n-1}{2}} - 14np + 14(n-2)\Big), \\
u_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 12\min(2a,n)p^{\frac{n+1}{2}} + \\
&\quad + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 14ap + 14(a-1)\Big), \\
u_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,n-a)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} + \\
&\quad + p^{\max(a,n-a)}(p+1) - 12\min(2a,n)p^{\frac{n+1}{2}} + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 14ap + 14(a-1)\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$ *and*

$$\begin{aligned}
v_{C_0} &= t,\\
v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 14ap - 14(a-1)\Big),\\
v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(-np^{n+1} - 2p^n + (n-2)p^{n-1} + 12np^{\frac{n+1}{2}} - 12(n-2)p^{\frac{n-1}{2}} +\\
&\quad + 14np - 14(n-2)\Big),\\
v_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 14ap - 14(a-1)\Big),\\
v_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,n-a)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + p^{\max(a,n-a)}(p+1) + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 14ap - 14(a-1)\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. □

4.2.2 Case n even and $p \equiv 1 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix}
C_0^2 & C_0 \cdot C_1 & \dots & C_0 \cdot F_{n-1,1}\\
C_1 \cdot C_0 & C_1^2 & \dots & C_1 \cdot F_{n-1,1}\\
\vdots & \vdots & & \vdots\\
C_n \cdot C_0 & C_n \cdot C_1 & \dots & C_n \cdot F_{n-1,1}\\
E_{1,1} \cdot C_0 & E_{1,1} \cdot C_1 & \dots & E_{1,1} \cdot F_{n-1,1}\\
\vdots & \vdots & & \vdots\\
E_{n-1,1} \cdot C_0 & E_{n-1,1} \cdot C_1 & \dots & E_{n-1,1} \cdot F_{n-1,1}\\
F_{1,1} \cdot C_0 & F_{1,1} \cdot C_1 & \dots & F_{1,1} \cdot F_{n-1,1}\\
\vdots & \vdots & & \vdots\\
F_{n-1,1} \cdot C_0 & F_{n-1,1} \cdot C_1 & \dots & F_{n-1,1}^2
\end{pmatrix}
\begin{pmatrix}
x_{C_0}\\ x_{C_1}\\ \vdots\\ x_{C_n}\\ x_{E_{1,1}}\\ \vdots\\ x_{E_{n-1,1}}\\ x_{F_{1,1}}\\ \vdots\\ x_{F_{n-1,1}}
\end{pmatrix}
=
\begin{pmatrix}
-G_m \cdot C_0\\ -G_m \cdot C_1\\ \vdots\\ -G_m \cdot C_n\\ -G_m \cdot E_{1,1}\\ \vdots\\ -G_m \cdot E_{n-1,1}\\ -G_m \cdot F_{1,1}\\ \vdots\\ -G_m \cdot F_{n-1,1}
\end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.7. *Let* $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *and* $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *be as defined in Equation* (7). *Then the solutions of the linear systems in Equation* (6) *are the following: for* $a \in \{1, \dots, n-$

$1\} - \left\{\frac{n}{2}\right\}$ *we have:*

$$\begin{aligned}
u_{C_0} &= t,\\
u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} +\\
&\quad + 6\min(2a-2,n-2)p^{\frac{n-2}{2}} - 14ap + 14(a-1)\Big),\\
u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 6np^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} +\\
&\quad + 6(n-2)p^{\frac{n-2}{2}} - 14np + 14(n-2)\Big),\\
u_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} +\\
&\quad + 6\min(2a-2,n-2)p^{\frac{n-2}{2}} - 14ap + 14(a-1)\Big),\\
u_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,n-a)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} + p^{\max(a,n-a)}(p+1) +\\
&\quad - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + 6\min(2a-2,n-2)p^{\frac{n-2}{2}} - 14ap + 14(a-1)\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$ *and*

$$\begin{aligned}
v_{C_0} &= t,\\
v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 6\max(0,2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 14ap - 14(a-1)\Big),\\
v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(-np^{n+1} - 2p^n + (n-2)p^{n-1} + 6np^{\frac{n+2}{2}} + 12p^{\frac{n}{2}} - 6(n-2)p^{\frac{n-2}{2}} +\\
&\quad + 14np - 14(n-2)\Big),\\
v_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 6\max(0,2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 14ap - 14(a-1)\Big),\\
v_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,b)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} + p^{\max(a,n-a)}(p+1) +\\
&\quad + 6\max(0,2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 14ap - 14(a-1)\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. □

4.2.3 Case n odd and $p \equiv 5 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix}
C_0^2 & C_0\cdot C_1 & \dots & C_0\cdot F_{n,1}\\
C_1\cdot C_0 & C_1^2 & \dots & C_1\cdot F_{n,1}\\
\vdots & \vdots & & \vdots\\
C_n\cdot C_0 & C_n\cdot C_1 & \dots & C_n\cdot F_{n,1}\\
E_{1,1}\cdot C_0 & E_{1,1}\cdot C_1 & \dots & E_{1,1}\cdot F_{n,1}\\
\vdots & \vdots & & \vdots\\
E_{n-1,1}\cdot C_0 & E_{n-1,1}\cdot C_1 & \dots & E_{n-1,1}\cdot F_{n,1}\\
F_{0,1}\cdot C_0 & F_{0,1}\cdot C_1 & \dots & F_{0,1}\cdot F_{n,1}\\
F_{n,1}\cdot C_0 & F_{n,1}\cdot C_1 & \dots & F_{n,1}^2
\end{pmatrix}
\begin{pmatrix}
x_{C_0}\\ x_{C_1}\\ \vdots\\ x_{C_n}\\ x_{E_{1,1}}\\ \vdots\\ x_{E_{n-1,1}}\\ x_{F_{0,1}}\\ x_{F_{n,1}}
\end{pmatrix}
=
\begin{pmatrix}
-G_m\cdot C_0\\ -G_m\cdot C_1\\ \vdots\\ -G_m\cdot C_n\\ -G_m\cdot E_{1,1}\\ \vdots\\ -G_m\cdot E_{n-1,1}\\ -G_m\cdot F_{0,1}\\ -G_m\cdot F_{n,1}
\end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.8. *Let* $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *and* $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for* $a \in \{1, \dots, n-1\}$ *we have:*

$$
\begin{aligned}
u_{C_0} &= t,\\
u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} +\\
&\quad - 12\min(2a,n)p^{\frac{n+1}{2}} + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 6ap + 6(a-1)\Big),\\
u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 12np^{\frac{n+1}{2}} +\\
&\quad + 12(n-2)p^{\frac{n-1}{2}} - 6np + 6(n-2)\Big),\\
u_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} +\\
&\quad - 12\min(2a,n)p^{\frac{n+1}{2}} + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 6ap + 6(a-1)\Big),\\
u_{F_{0,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(\tfrac{1}{2}(3n-1)p^{n+1} + 8p^n - \tfrac{1}{2}(3n-17)p^{n-1} +\\
&\quad - 18(2n-1)p^{\frac{n+1}{2}} + 18(2n-3)p^{\frac{n-1}{2}} - 3(3n-1)p + 3(3n-5)\Big),\\
u_{F_{n,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(\tfrac{1}{2}(3n+1)p^{n+1} + 10p^n - \tfrac{1}{2}(3n-19)p^{n-1} +\\
&\quad - 6(6n-1)p^{\frac{n+1}{2}} + 6(6n-11)p^{\frac{n-1}{2}} - 3(3n+1)p + 3(3n-7)\Big),
\end{aligned}
$$

for $t \in \mathbb{Q}$ *and*

$$
\begin{aligned}
v_{C_0} &= t,\\
v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 6ap + 6(1-a)\Big),\\
v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(-np^{n+1} - 2p^n + (n-2)p^{n-1} + 12np^{\frac{n+1}{2}} +\\
&\quad - 12(n-2)p^{\frac{n-1}{2}} + 6np - 6(n-2)\Big),\\
v_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 6ap - 6(a-1)\Big),\\
v_{F_{0,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(-\tfrac{1}{2}(3n-1)p^{n+1} + 4p^n + \tfrac{1}{2}(3n+7)p^{n-1} +\\
&\quad + 6p^{\frac{n+1}{2}} + 6p^{\frac{n-1}{2}} + 3(3n-1)p - 3(3n-5)\Big),\\
v_{F_{n,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(-\tfrac{1}{2}(3n+1)p^{n+1} + 2p^n + \tfrac{1}{2}(3n+5)p^{n-1} +\\
&\quad + 18p^{\frac{n+1}{2}} + 18p^{\frac{n-1}{2}} + 3(3n+1)p - 3(3n-7)\Big),
\end{aligned}
$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. □

4.2.4 Case n even and $p \equiv 5 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix} C_0^2 & C_0 \cdot C_1 & \dots & C_0 \cdot E_{n-1,1} \\ C_1 \cdot C_0 & C_1^2 & \dots & C_1 \cdot E_{n-1,1} \\ \vdots & \vdots & & \vdots \\ C_n \cdot C_0 & C_n \cdot C_1 & \dots & C_n \cdot E_{n-1,1} \\ E_{1,1} \cdot C_0 & E_{1,1} \cdot C_1 & \dots & E_{1,1} \cdot E_{n-1,1} \\ \vdots & \vdots & & \vdots \\ E_{n-1,1} \cdot C_0 & E_{n-1,1} \cdot C_1 & \dots & E_{n-1,1}^2 \end{pmatrix} \begin{pmatrix} x_{C_0} \\ x_{C_1} \\ \vdots \\ x_{C_n} \\ x_{E_{1,1}} \\ \vdots \\ x_{E_{n-1,1}} \end{pmatrix} = \begin{pmatrix} -G_m \cdot C_0 \\ -G_m \cdot C_1 \\ \vdots \\ -G_m \cdot C_n \\ -G_m \cdot E_{1,1} \\ \vdots \\ -G_m \cdot E_{n-1,1} \end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.9. *Let $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ and $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for $a \in \{1, \dots, n-1\} - \left\{\frac{n}{2}\right\}$ we have:*

$$\begin{aligned} u_{C_0} &= t, \\ u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + \\ &\quad + 6\min(2a-2,n-2)p^{\frac{n-2}{2}} - 6ap + 6(a-1)\Big), \\ u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 6np^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + \\ &\quad + 6(n-2)p^{\frac{n-2}{2}} - 6np + 6(n-2)\Big), \\ u_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + \\ &\quad + 6\min(2a-2,n-2)p^{\frac{n-2}{2}} - 6ap + 6(a-1)\Big), \end{aligned}$$

for $t \in \mathbb{Q}$ and

$$\begin{aligned} v_{C_0} &= t, \\ v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} + \\ &\quad + 6\max(0,2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 6ap - 6(a-1)\Big), \\ v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(-np^{n+1} - 2p^n + (n-2)p^{n-1} + 6np^{\frac{n+2}{2}} + \\ &\quad + 12p^{\frac{n}{2}} - 6(n-2)p^{\frac{n-2}{2}} + 6np - 6(n-2)\Big), \\ v_{E_{a,1}} &= \frac{t}{2}\phi(p^{\min(a,n-a)}) + \frac{1}{2p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} + \\ &\quad + 6\max(0,2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 6ap + 6(1-a)\Big), \end{aligned}$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. □

4.2.5 Case n odd and $p \equiv 7 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix} C_0^2 & C_0\cdot C_1 & \dots & C_0\cdot F_{n-1,1} \\ C_1\cdot C_0 & C_1^2 & \dots & C_1\cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ C_n\cdot C_0 & C_n\cdot C_1 & \dots & C_n\cdot F_{n-1,1} \\ E_{\infty,1}\cdot C_0 & E_{\infty,1}\cdot C_1 & \dots & E_{\infty,1}\cdot F_{n-1,1} \\ F_{1,1}\cdot C_0 & F_{1,1}\cdot C_1 & \dots & F_{1,1}\cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ F_{n-1,1}\cdot C_0 & F_{n-1,1}\cdot C_1 & \dots & F_{n-1,1}^2 \end{pmatrix} \begin{pmatrix} x_{C_0} \\ x_{C_1} \\ \vdots \\ x_{C_n} \\ x_{E_{\infty,1}} \\ x_{F_{1,1}} \\ \vdots \\ x_{F_{n-1,1}} \end{pmatrix} = \begin{pmatrix} -G_m\cdot C_0 \\ -G_m\cdot C_1 \\ \vdots \\ -G_m\cdot C_n \\ -G_m\cdot E_{\infty,1} \\ -G_m\cdot F_{1,1} \\ \vdots \\ -G_m\cdot F_{n-1,1} \end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.10. *Let $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}}\Gamma_i^{(p)}$ and $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}}\Gamma_i^{(p)}$ be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for $a \in \{1,\dots,n-1\}$ we have:*

$$\begin{aligned}
u_{C_0} &= t,\\
u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} +\\
&\quad - 12\min(2a,n)p^{\frac{n+1}{2}} + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 8ap + 8(a-1)\Big),\\
u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 12np^{\frac{n+1}{2}} +\\
&\quad + 12(n-2)p^{\frac{n-1}{2}} - 8np + 8(n-2)\Big),\\
u_{E_{\infty,1}} &= \frac{t}{2}\phi(p^{\frac{n-1}{2}}) + \frac{1}{2p^{\frac{n-1}{2}}(p^2-1)}\Big(np^{n+1} + 6p^n - (n-6)p^{n-1} +\\
&\quad - 8(3n-1)p^{\frac{n+1}{2}} + 8(3n-5)p^{\frac{n-1}{2}} - 8np + 8(n-2)\Big),\\
u_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,n-a)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} +\\
&\quad + p^{\max(a,n-a)}(p+1) - 12\min(2a,n)p^{\frac{n+1}{2}} + 12\min(2a-2,n-2)p^{\frac{n-1}{2}} - 8ap + 8(a-1)\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$ and

$$\begin{aligned}
v_{C_0} &= t,\\
v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 8ap - 8(a-1)\Big),\\
v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(-np^{n+1} - 2p^n + (n-2)p^{n-1} + 12np^{\frac{n+1}{2}} +\\
&\quad + 12(2-n)p^{\frac{n-1}{2}} + 8np - 8(n-2)\Big),\\
v_{E_{\infty,1}} &= \frac{t}{2}\phi(p^{\frac{n-1}{2}}) + \frac{1}{2p^{\frac{n-1}{2}}(p^2-1)}\Big(-np^{n+1} + 2p^n + (n+2)p^{n-1} +\\
&\quad + 8p^{\frac{n+1}{2}} + 8p^{\frac{n-1}{2}} + 8np - 8(n-2)\Big),\\
v_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,n-a)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} +\\
&\quad + p^{\max(a,n-a)}(p+1) + 12\max(0,2a-n)p^{\frac{n+1}{2}} - 12\max(0,2a-n)p^{\frac{n-1}{2}} + 8ap - 8(a-1)\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. ☐

4.2.6 Case n even and $p \equiv 7 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix} C_0^2 & C_0 \cdot C_1 & \dots & C_0 \cdot F_{n-1,1} \\ C_1 \cdot C_0 & C_1^2 & \dots & C_1 \cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ C_n \cdot C_0 & C_n \cdot C_1 & \dots & C_n \cdot F_{n-1,1} \\ F_{1,1} \cdot C_0 & F_{1,1} \cdot C_1 & \dots & F_{1,1} \cdot F_{n-1,1} \\ \vdots & \vdots & & \vdots \\ F_{n-1,1} \cdot C_0 & F_{n-1,1} \cdot C_1 & \dots & F_{n-1,1}^2 \end{pmatrix} \begin{pmatrix} x_{C_0} \\ x_{C_1} \\ \vdots \\ x_{C_n} \\ x_{F_{1,1}} \\ \vdots \\ x_{F_{n-1,1}} \end{pmatrix} = \begin{pmatrix} -G_m \cdot C_0 \\ -G_m \cdot C_1 \\ \vdots \\ -G_m \cdot C_n \\ -G_m \cdot F_{1,1} \\ \vdots \\ -G_m \cdot F_{n-1,1} \end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.11. *Let* $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *and* $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for* $a \in \{1, \dots, n-1\} - \left\{\frac{n}{2}\right\}$ *we have:*

$$\begin{aligned} u_{C_0} &= t, \\ u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + \\ &\quad + 6\min(2a-2, n-2)p^{\frac{n-2}{2}} - 8ap + 8(a-1)\Big), \\ u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 6np^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + \\ &\quad + 6(n-2)p^{\frac{n-2}{2}} - 8np + 8(n-2)\Big), \\ u_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,n-a)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} + \\ &\quad + p^{\max(a,n-a)}(p+1) - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + 6\min(2a-2,n-2)p^{\frac{n-2}{2}} - 8ap + 8(a-1)\Big), \end{aligned}$$

for $t \in \mathbb{Q}$ *and*

$$\begin{aligned} v_{C_0} &= t, \\ v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} + \\ &\quad + 6\max(0, 2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 8ap - 8(a-1)\Big), \\ v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(-np^{n+1} - 2p^n + (n-2)p^{n-1} + 6np^{\frac{n+2}{2}} + \\ &\quad + 12p^{\frac{n}{2}} - 6(n-2)p^{\frac{n-2}{2}} + 8np - 8(n-2)\Big), \\ v_{F_{a,1}} &= \frac{t}{3}\phi(p^{\min(a,b)}) + \frac{1}{3p^{\max(a,n-a)}(p+1)}\Big(-ap^{n+1} + p^n + (a+1)p^{n-1} + \\ &\quad + p^{\max(a,n-a)}(p+1) + 6\max(0,2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}} + 8ap - 8(a-1)\Big), \end{aligned}$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. ☐

4.2.7 Case n odd and $p \equiv 11 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix} C_0^2 & C_0 \cdot C_1 & \dots & C_0 \cdot F_{n,1} \\ C_1 \cdot C_0 & C_1^2 & \dots & C_1 \cdot F_{n,1} \\ \vdots & \vdots & & \vdots \\ C_n \cdot C_0 & C_n \cdot C_1 & \dots & C_n \cdot F_{n,1} \\ E_{\infty,1} \cdot C_0 & E_{\infty,1} \cdot C_1 & \dots & E_{\infty,1} \cdot F_{n,1} \\ F_{0,1} \cdot C_0 & F_{0,1} \cdot C_1 & \dots & F_{0,1} \cdot F_{n,1} \\ F_{n,1} \cdot C_0 & F_{n,1} \cdot C_1 & \dots & F_{n,1}^2 \end{pmatrix} \begin{pmatrix} x_{C_0} \\ x_{C_1} \\ \vdots \\ x_{C_n} \\ x_{E_{\infty,1}} \\ x_{F_{0,1}} \\ x_{F_{n,1}} \end{pmatrix} = \begin{pmatrix} -G_m \cdot C_0 \\ -G_m \cdot C_1 \\ \vdots \\ -G_m \cdot C_n \\ -G_m \cdot E_{\infty,1} \\ -G_m \cdot F_{0,1} \\ -G_m \cdot F_{n,1} \end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.12. *Let $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ and $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for $a \in \{1, \dots, n-1\}$ we have:*

$$\begin{aligned} u_{C_0} &= t, \\ u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} + \\ &\quad - 12\min(2a,n)p^{\frac{n+1}{2}} + 12\min(2a-2,n-2)p^{\frac{n-1}{2}}\Big), \\ u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 12np^{\frac{n+1}{2}} + \\ &\quad + 12(n-2)p^{\frac{n-1}{2}}\Big), \\ u_{E_{\infty,1}} &= \frac{t}{2}\phi(p^{\frac{n-1}{2}}) + \frac{1}{2p^{\frac{n-1}{2}}(p^2-1)}\Big(np^{n+1} + 6p^n - (n-6)p^{n-1} + \\ &\quad - 8(3n-1)p^{\frac{n+1}{2}} + 8(3n-5)p^{\frac{n-1}{2}}\Big), \\ u_{F_{0,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(\tfrac{1}{2}(3n-1)p^{n+1} + 8p^n - \tfrac{1}{2}(3n-17)p^{n-1} + \\ &\quad - 18(2n-1)p^{\frac{n+1}{2}} + 18(2n-3)p^{\frac{n-1}{2}}\Big), \\ u_{F_{n,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(\tfrac{1}{2}(3n+1)p^{n+1} + 10p^n - \tfrac{1}{2}(3n-19)p^{n-1} + \\ &\quad - 6(6n-1)p^{\frac{n+1}{2}} + 6(6n-11)p^{\frac{n-1}{2}}\Big), \end{aligned}$$

for $t \in \mathbb{Q}$ *and*

$$\begin{aligned}
v_{C_0} &= t,\\
v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(- ap^{n+1} + p^n + (a+1)p^{n-1} + \\
&\quad + 12\max(0, 2a-n)p^{\frac{n+1}{2}} - 12\max(0, 2a-n)p^{\frac{n-1}{2}}\Big),\\
v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(- np^{n+1} - 2p^n + (n-2)p^{n-1} + 12np^{\frac{n+1}{2}} + \\
&\quad - 12(n-2)p^{\frac{n-1}{2}}\Big),\\
v_{E_{\infty,1}} &= \frac{t}{2}\phi(p^{\frac{n-1}{2}}) + \frac{1}{2p^{\frac{n-1}{2}}(p^2-1)}\Big(- np^{n+1} + 2p^n + (n+2)p^{n-1} + \\
&\quad + 8p^{\frac{n+1}{2}} + 8p^{\frac{n-1}{2}}\Big),\\
v_{F_{0,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(- \tfrac{1}{2}(3n-1)p^{n+1} + 4p^n + \\
&\quad + \tfrac{1}{2}(3n+7)p^{n-1} + 6p^{\frac{n+1}{2}} + 6p^{\frac{n-1}{2}}\Big),\\
v_{F_{n,1}} &= \frac{t}{3}\phi(p^{\frac{n-1}{2}}) + \frac{1}{3p^{\frac{n-1}{2}}(p^2-1)}\Big(- \tfrac{1}{2}(3n+1)p^{n+1} + 2p^n + \\
&\quad + \tfrac{1}{2}(3n+5)p^{n-1} + 18p^{\frac{n+1}{2}} + 18p^{\frac{n-1}{2}}\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$*.*

Proof. We checked these solutions using the software Mathematica. □

4.2.8 Case n even and $p \equiv 11 \mod 12$

Let $G_m = \mathcal{K} - (2g-2)H_m$. The linear systems in Equation (6) are

$$\begin{pmatrix} C_0^2 & C_0 \cdot C_1 & \dots & C_0 \cdot C_n \\ C_1 \cdot C_0 & C_1^2 & \dots & C_1 \cdot C_n \\ \vdots & \vdots & & \vdots \\ C_n \cdot C_0 & C_n \cdot C_1 & \dots & C_n^2 \end{pmatrix} \begin{pmatrix} x_{C_0} \\ x_{C_1} \\ \vdots \\ x_{C_n} \end{pmatrix} = \begin{pmatrix} -G_m \cdot C_0 \\ -G_m \cdot C_1 \\ \vdots \\ -G_m \cdot C_n \end{pmatrix}.$$

In the following proposition we write the solutions of the previous linear systems.

Proposition 4.13. *Let* $V_0 = V_0^{(p)} = \sum_{i=1}^{\nu_p} u_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *and* $V_\infty = V_\infty^{(p)} = \sum_{i=1}^{\nu_p} v_{\Gamma_i^{(p)}} \Gamma_i^{(p)}$ *be as defined in Equation (7). Then the solutions of the linear systems in Equation (6) are the following: for* $a \in \{1, \dots, n-1\} - \left\{\frac{n}{2}\right\}$ *we have:*

$$\begin{aligned}
u_{C_0} &= t,\\
u_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(ap^{n+1} + 3p^n - (a-3)p^{n-1} - 6\min(2a,n)p^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + \\
&\quad + 6\min(2a-2, n-2)p^{\frac{n-2}{2}}\Big),\\
u_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(np^{n+1} + 2p^n - (n-2)p^{n-1} - 6np^{\frac{n+2}{2}} - 12p^{\frac{n}{2}} + 6(n-2)p^{\frac{n-2}{2}}\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$ and

$$\begin{aligned}
v_{C_0} &= t, \\
v_{C_a} &= t\phi(p^{\min(a,n-a)}) + \frac{1}{p^{\max(a,n-a)}(p+1)}\Big(- ap^{n+1} + p^n + (a+1)p^{n-1} + \\
&\quad + 6\max(0, 2a-n)p^{\frac{n+2}{2}} - 6\max(0,2a-n)p^{\frac{n-2}{2}}\Big), \\
v_{C_n} &= t + \frac{1}{p^{n-1}(p^2-1)}\Big(- np^{n+1} - 2p^n + (n-2)p^{n-1} + 6np^{\frac{n+2}{2}} + 12p^{\frac{n}{2}} - 6(n-2)p^{\frac{n-2}{2}}\Big),
\end{aligned}$$

for $t \in \mathbb{Q}$.

Proof. We checked these solutions using the software Mathematica. □

4.3 Conclusion of the proof

In this section we use the relation $2\langle H_0, H_\infty\rangle = -\mathcal{G}(0,\infty)$ explained in the last paragraph of Section 2.1.

Proof of Theorem 1.1. First of all we recall that $g = \frac{p^n}{12} + o(p^n)$ for $p^n \to +\infty$ (see Equation (2)). We now write down the asymptotics for the summands of Equation (5). For the summand (a) [MvP22, Theorem 1.2] says that $2g(1-g)\mathcal{G}(0,\infty) = 2g\log(p^n) + o(g\log(p^n))$, so

$$-4g(g-1)\langle H_0, H_\infty\rangle = 2g\log(p^n) + o(g\log(p^n)).$$

In [MU98, Section 6] it is shown that

$$h_m = \begin{cases} O((n+1)^2\log(p^n)), & \text{if } p \not\equiv 11 \pmod{12}, \\ 0, & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Therefore regarding the summand (c) we have the following:

$$\frac{h_0 + h_\infty}{2} = o(g\log p^n).$$

Finally we use Propositions 4.6 to 4.13 in order to estimate the summand (b).

Case $p \equiv 1 \pmod{12}$. For n odd, we have

$$\begin{aligned}
&\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\
&- 3p^{2n-2} - 6np^{\frac{3n+1}{2}} + 24p^{\frac{3n-1}{2}} + 6(n+4)p^{\frac{3n-3}{2}} - (3n-2)p^{n+1} + \\
&+ 24p^n + (3n+22)p^{n-1} + 24np^{\frac{n+1}{2}} - 24(n-2)p^{\frac{n-1}{2}} - 70np + 70(n-2)\Big) = \\
&= g\log(p^n) + o(g\log p),
\end{aligned}$$

and for n even, we have

$$\begin{aligned}
&\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\
&- 3p^{2n-2} - 3np^{\frac{3n+2}{2}} - 3(n-4)p^{\frac{3n}{2}} + 3(n+8)p^{\frac{3n-2}{2}} + 3(n+4)p^{\frac{3n-4}{2}} + \\
&- (3n-20)p^{n+1} + 24p^n + (3n+4)p^{n-1} + 12np^{\frac{n+2}{2}} + 24p^{\frac{n}{2}} - 12(n-2)p^{\frac{n-2}{2}} + \\
&- 70np + 70(n-2)\Big) = g\log(p^n) + o(g\log p).
\end{aligned}$$

Case $p \equiv 5 \pmod{12}$**.** For n odd, we have

$$\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\ - 3p^{2n-2} - 6np^{\frac{3n+1}{2}} + 24p^{\frac{3n-1}{2}} + 6(n+4)p^{\frac{3n-3}{2}} + 3np^{n+1} + \\ + 40p^n - (3n-40)p^{n-1} - 72np^{\frac{n+1}{2}} + 72(n-2)p^{\frac{n-1}{2}} - 54np + 54(n-2)\Big) = \\ = g\log(p^n) + o(g\log p),$$

and for n even, we have

$$\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\ - 3p^{2n-2} - 3np^{\frac{3n+2}{2}} - 3(n-4)p^{\frac{3n}{2}} + 3(n+8)p^{\frac{3n-2}{2}} + 3(n+4)p^{\frac{3n-4}{2}} + \\ + (3n+16)p^{n+1} + 40p^n - (n-8)p^{n-1} - 36np^{\frac{n+2}{2}} - 72p^{\frac{n}{2}} + 36(n-2)p^{\frac{n-2}{2}} + \\ - 54np + 54(n-2)\Big) = g\log(p^n) + o(g\log p).$$

Case $p \equiv 7 \pmod{12}$**.** For n odd, we have

$$\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\ - 3p^{2n-2} - 6np^{\frac{3n+1}{2}} + 24p^{\frac{3n-1}{2}} + 6(n+4)p^{\frac{3n-3}{2}} + 2p^{n+1} + \\ + 36p^n + 34p^{n-1} + 48np^{\frac{n+1}{2}} + 48(n-2)p^{\frac{n-1}{2}} - 64np + 64(n-2)\Big) = \\ = g\log(p^n) + o(g\log p),$$

and for n even, we have

$$\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\ - 3p^{2n-2} - 3np^{\frac{3n+2}{2}} - 3(n-4)p^{\frac{3n}{2}} + 3(n+8)p^{\frac{3n-2}{2}} + 3(n+4)p^{\frac{3n-4}{2}} + \\ + 20p^{n+1} + 36p^n + 16p^{n-1} - 24np^{\frac{n+2}{2}} - 48p^{\frac{n}{2}} + 24(n-2)p^{\frac{n-2}{2}} + \\ - 64np + 64(n-2)\Big) = g\log(p^n) + o(g\log p).$$

Case $p \equiv 11 \pmod{12}$**.** For n odd, we have

$$\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\ - 3p^{2n-2} - 6np^{\frac{3n+1}{2}} + 24p^{\frac{3n-1}{2}} + 6(n+4)p^{\frac{3n-3}{2}} + 6np^{n+1} + \\ + 52p^n - 2(3n-26)p^{n-1} - 144np^{\frac{n+1}{2}} + 144(n-2)p^{\frac{n-1}{2}}\Big) = \\ = g\log(p^n) + o(g\log p),$$

and for n even, we have

$$\frac{g\langle V_0, V_\infty\rangle}{g-1} - \frac{\langle V_0, V_0\rangle + \langle V_\infty, V_\infty\rangle}{2g-2} = ng\log p + \frac{\log p}{6p^{n-1}(p^2-1)}\Big(p^{2n} - 2p^{2n-1} + \\ - 3p^{2n-2} - 3np^{\frac{3n+2}{2}} - 3(n-4)p^{\frac{3n}{2}} + 3(n+8)p^{\frac{3n-2}{2}} + 3(n+4)p^{\frac{3n-4}{2}} + \\ + 2(3n+8)p^{n+1} + 52p^n + 6(n-6)p^{n-1} - 72np^{\frac{n+2}{2}} - 144p^{\frac{n}{2}} + 72(n-2)p^{\frac{n-2}{2}}\Big) = \\ = g\log(p^n) + o(g\log p).$$

□

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