Sheet 3: differentiability and directional derivatives

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1. Let $U \subseteq \mathbb{R}^n$ be an open set, let $\xi \in U$, and consider a function $f\colon U \to \mathbb{R}$ satisfying the property described below. For every $v \in \mathbb{R}^n$ and every curve $\gamma\colon]-\varepsilon, \varepsilon[\to \mathbb{R}^n$ such that $\gamma(0) = \xi$ and $\gamma'(0) = v$, the composition $(f \circ \gamma)\colon]-\varepsilon, \varepsilon[\to \mathbb{R}$ satisfies $(f \circ \gamma)'(0) = L(v)$, where $L\colon \mathbb{R}^n \to \mathbb{R}$ is fixed linear map. Prove that f is differentiable at ξ.

2. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined in the following way[1].:

$$f(x, y) = \begin{cases} \dfrac{y^3}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

(a) Show that f is continuous on its domain.

(b) Show that for any curve $\gamma\colon]-\varepsilon, \varepsilon[\to \mathbb{R}^2$ such that $\gamma(0) = 0$ and $\gamma'(0) \neq 0$ the compositon $f \circ \gamma$ is derivable at 0.

(c) Compare the item 2.b with the exercise 1. Why cannot we apply exercise 1. to conclude that f is differentiable at $(0, 0)$?

(d) Compute the directional derivative of f along any direction $(h, k) \in \mathbb{R}^2 \setminus (0, 0)$ and compare the result with the item 2.b. What is then the geometrical interpretation of the derivative $f \circ \gamma$?

3. Determine whether the function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

is continuous on its domain.

4. Let X, Y be two normed vector spaces and let $U \subseteq X$ be an open set. Prove that a function $f\colon U \to Y$ is in $\mathcal{C}^1(U, Y)$ if and only if the partial derivatives of f exist and are conitnuous in U[2].

5. Show that the following function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable on its domain:

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

[1] Recall that we proved during the lectures that such function is not differentiable at $(0, 0)$

[2] During the lectures we only showed that the existence and the continuity of the partial derivatives imply the differentiability.