Kobayashi length bounds on bordered surfaces and generalized integral points on abelian varieties

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Abstract

Let B be a compact Riemann surface and $B_0 \subset B$ a bordered hyperbolic subsurface obtained by removing finitely many disjoint closed disks. Fix a nontrivial loop α in B_0. For $s \geq 0$, let $L(\alpha, s)$ denote the supremum, over all finite subsets $S \subset B_0$ with $\#S \leq s$, of the minimal Kobayashi length of a loop in $B_0 \setminus S$ that is freely homotopic to α in B_0. Phung in [8] proved that $L(\alpha, s)$ grows at most linearly and at least as $\sqrt{s}/\log s$. We sharpen the upper bound to $O(\sqrt{s \log s})$, which determines $\lim_{s\to\infty} \frac{\log L(\alpha,s)}{\log s} = \frac{1}{2}$, answering a question raised in [8, Question 1.4]. As an application, we improve the counting bound for generalized integral points on abelian varieties over complex function fields: for an abelian variety of dimension n over $\mathbb{C}(B)$, Phung proved that the number of (s, B_0)-generalized integral points modulo the constant trace grows at most as s^{2nk}, where $k = \mathrm{rk}(\pi_1(B_0))$. We sharpen this to $s^{nk+\varepsilon}$ for every $\varepsilon > 0$, halving the exponent.

1 Introduction

1.1 Generalized integral points and the Lang–Vojta conjecture

Let k be an algebraically closed field and let B be a non-singular, irreducible algebraic curve over k with function field $K = k(B)$. Let X be a smooth, integral, projective K-variety and fix a reduced effective divisor $D \subset X$. A *model* of (X, D) is a pair $(\mathcal{X}, \mathcal{D})$ where $\mathcal{X}$ is a normal k-variety equipped with a proper, flat morphism $f\colon \mathcal{X} \to B$ satisfying $\mathcal{X}_K \cong X$, and $\mathcal{D} \subset \mathcal{X}$ is a horizontal divisor with $\mathcal{D}_K \cong D$. Such a model always exists. We also define

$$\mathrm{Sing}(f) := \{b \in B\colon \mathcal{X}_b \text{ is singular}\}.$$

The set of rational points $X(K)$ corresponds bijectively to the sections of f: for any $P \in X(K)$, let $\sigma_P\colon B \to \mathcal{X}$ be the section obtained as the Zariski closure of P in $\mathcal{X}$. By abuse of notation we will simply write $\sigma_P \in X(K)$. Given a finite set $S \subset B$, the set of $(S, \mathcal{D})$-*integral points* of $\mathcal{X}$ is

$$\mathcal{O}_{S,\mathcal{D}}(\mathcal{X}) := \{\sigma_P \in X(K)\colon f(\sigma_P(B) \cap \mathcal{D}) \subseteq S\},$$

that is, the set of sections whose image doesn't intersect the divisor $\mathcal{D}$ "above" $B \setminus S$.

Recall that the pair (X, D) is said to be of *log general type* when D is a normal crossings divisor and the log canonical bundle $\omega_X(D)$ is big. A special case is when $X = A$ is an abelian variety: the canonical bundle ω_A is trivial, so the log canonical bundle $\omega_A(D) \cong \mathcal{O}_A(D)$ is big whenever D is big.

The *geometric Lang–Vojta conjecture* predicts strong constraints for the integral points of pairs of log general type

Conjecture 1.1 (Geometric Lang–Vojta)**.** Let (X, D) be a pair of log general type and let $(\mathcal{X}, \mathcal{D})$ be a model. Then there exist a proper closed subset $Z \subset X \setminus D$ and a constant $m = m(\mathcal{X}, \mathcal{D}) > 0$ with the following property: for every $\sigma_P \in \mathcal{O}_{S,\mathcal{D}}(\mathcal{X})$ with $\sigma_P(B) \not\subset \mathcal{Z}$, where $\mathcal{Z}$ denotes the Zariski closure of Z in $\mathcal{X}$, we have

$$\deg_B \sigma_P^* \mathcal{D} \; \leq \; m\,(2g(B) - 2 + \#S)\,. \tag{1}$$

Roughly speaking it says that all integral points, apart from a subset of them having bounded height, can be confined in an *exceptional* (proper) closed set Z; moreover the height bound can be expressed in terms of the geometry of the base B.

The geometric Lang-Vojta conjecture has been settled in very few cases: when X is a curve; when $X = \mathbb{P}^2$ and D has a special shape; when $X = A$ is an abelian variety with trivial $K|k$ trace or when $k = \mathbb{C}$ and A is defined over $\mathbb{C}$ (constant case). For a quick review of the known cases we refer the reader to [8, Section 1] and we remark that for a general complex abelian variety — in particular, when $\mathrm{Tr}_{K/\mathbb{C}}(A) \neq 0$ and A is nonconstant — the conjecture remains open.

From now on we fix the base field $k = \mathbb{C}$, a model $(\mathcal{X}, \mathcal{D})$ of (X, D), and a Riemannian metric ρ on B inducing a path-length ℓ_ρ and a distance function $d_\rho(\cdot, \cdot)$. A *(closed) disk* in B is a set of the form $\overline{\Delta}(x_0, R) := \{b \in B \colon d_\rho(b, x_0) \leq R\}$ for $R > 0$ and $x_0 \in B$.

The notion of $(S, \mathcal{D})$-integral point can be generalized by allowing S to vary while keeping only its cardinality $s = \#S$ bounded, and by requiring the intersection condition to hold only on the complement of finitely many disjoint closed disks. This notion of generalized integral points was introduced in [8] and we recall it below.

Definition 1.2. Let $t \in \mathbb{Z}_{\geq 0}$. Consider t disjoint disks $\overline{\Delta}_1, \ldots, \overline{\Delta}_t$ in B such that $\mathrm{Sing}(f) \subset \bigsqcup_{i=1}^t \overline{\Delta}_i$ and distinct points of $\mathrm{Sing}(f)$ are contained in distinct disks. Set $B_0 := B \setminus \bigsqcup_{i=1}^t \overline{\Delta}_i$, for any $s \in \mathbb{Z}_{\geq 0}$

$$I(s, B_0) := \{\sigma_P \in X(K) \colon \#\left(f(\sigma_P(B_0) \cap \mathcal{D})\right) \leq s\}$$

is the set of *generalized* (s, B_0)*-integral points of* $(\mathcal{X}, \mathcal{D})$.

One immediately observes the inclusion

$$\bigcup_{\substack{S \subset B \\ \#S \leq s}} \mathcal{O}_{S,\mathcal{D}}(\mathcal{X}) \; \subseteq \; I(s, B_0)\,,$$

so $I(s, B_0)$ is potentially much larger than the integral points for any single fixed S. Understanding the growth of $\#I(s, B_0)$ as a function of s (with B_0 fixed) is the main object of this paper.

1.2 Main results

Building on Parshin's foundational work [7], Phung in [8] proved an important quantitative bound on $\#I(s, B_0)$ for abelian varieties, working in the following setting.

(P) Let A be an abelian variety of dimension n over $K = \mathbb{C}(B)$ and let $D \subset A$ be a reduced effective divisor. We fix a model $(\mathcal{A}, \mathcal{D})$, where the proper flat morphism is $f \colon \mathcal{A} \to B$ and $\sigma_O \colon B \to \mathcal{A}$ is a fixed section. We assume:

- (i) There exist $t \in \mathbb{Z}_{\geq 0}$ disjoint disks $\overline{\Delta}_1, \ldots, \overline{\Delta}_t$ in B such that $\mathrm{Sing}(f) \subset \bigsqcup_{i=1}^t \overline{\Delta}_i$ and distinct points of $\mathrm{Sing}(f)$ are contained in distinct disks. We define

$$B_0 := B \setminus \bigsqcup_{i=1}^{t} \overline{\Delta}_i$$

  so that $f \colon \mathcal{A}_{B_0} \to B_0$ is a family of abelian varieties $(A_b, \sigma_O(b))_{b \in B_0}$.

- (ii) B_0 is hyperbolic (this can always be achieved by taking $t \geq 3$).

- (iii) $D \subset A$ does not contain any translates of nonzero abelian subvarieties.

Phung's main result is the following.

Theorem 1.3 (Phung [8, Theorem A]). *Assume that the setting (P) holds. Then there exists $m := m(B_0) \in \mathbb{R}_{>0}$ such that*

$$\#\left({}^{I(s,B_0)}\!\big/\!{}_{\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})} \right) \leq m(s+1)^{2n\,\mathrm{rk}(\pi_1(B_0,b_0))} \tag{2}$$

Note that the bound behaves coherently: if B_0 shrinks then the set of generalized integral points grows, but the number of punctures of B also increases, making $\mathrm{rk}(\pi_1(B_0, b_0))$ larger.

The proof of Theorem 1.3 rests on a Parshin cocycle argument. Let $k := \mathrm{rk}(\pi_1(B_0, b_0))$ and fix generators $\alpha_1, \dots, \alpha_k$. By Ehresmann's theorem, the fibration $\mathcal{A}_{B_0} \to B_0$ gives rise to a short exact sequence $1 \to \Gamma \to \pi_1(\mathcal{A}_{B_0}, w_0) \to G \to 1$, where $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \cong \mathbb{Z}^{2n}$ and $G = \pi_1(B_0, b_0)$. Each section σ_P induces a splitting of this sequence, and the difference between the splittings of σ_P and σ_O defines a 1-cocycle $c_P\colon G \to \Gamma$ whose cohomology class determines P modulo the trace and torsion. To count the possible cocycle classes, one bounds the lattice element $c_P(\alpha_j) \in \Gamma \cong \mathbb{Z}^{2n}$ for each generator by the displacement in the universal cover, which in turn is controlled by the h-length of the loop $\sigma_P(\gamma_j)$ in $\mathcal{A}_{B_0}$. A theorem of Green provides the comparison $\ell_h(\sigma_P(\gamma_j)) \leq c^{-1}\ell_S(\gamma_j)$, so the problem reduces to bounding the Kobayashi length of loops γ_j in $B_0 \setminus S$ representing the generators α_j. Phung's linear bound $\ell_S(\gamma_j) = O(s)$ then yields the displacement bound $H(s) = O(s)$, and lattice counting in $\Gamma \cong \mathbb{Z}^{2n}$ gives at most $O(H(s)^{2n}) = O(s^{2n})$ possibilities per generator, hence the exponent $2nk$ in (2).

When D is moreover ample, Theorem 1.3 implies that for every $\sigma_P \in I(s, B_0)$ there exists a constant $M = M(\mathcal{A}, \mathcal{D}, B_0, s) > 0$ such that $\deg_B \sigma_P^* \mathcal{D} < M$ (see [8, Corollary 1.3]). This is a weak form of the geometric Lang–Vojta conjecture for generalized integral points: the bound M is indeed allowed to depend on s, whereas Conjecture 1.1 requires a bound that is linear in s.

The exponent $2n\,\mathrm{rk}(\pi_1(B_0))$ in (2) can be compared with known results for the smaller set

$$J(s) := \{\sigma_P \in A(K)\colon \#\left(f(\sigma_P(B) \cap \mathcal{D})\right) \leq s\} \;\subseteq\; I(s, B_0)\,.$$

When $n = 1$ (elliptic curves), classical height-theoretic bounds of Hindry–Silverman [3, Theorem 8.5] together with Shioda's bound [10, Theorem 2.5] for the Mordell–Weil rank show that $\#J(s)$ is bounded by a polynomial in s of degree at most $\mathrm{rk}(\pi_1(B_0))$. This is half the exponent appearing in Phung's bound for the much larger set $I(s, B_0)$.

The main result of this paper eliminates this discrepancy:

Theorem 1.4. *Assume that the setting (P) holds. Then for every $\varepsilon > 0$ there exists $m := m(B_0, \varepsilon) \in \mathbb{R}_{>0}$ such that*

$$\#\left({}^{I(s,B_0)}\!\big/\!{}_{\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})} \right) \leq m(s+1)^{n\;\mathrm{rk}(\pi_1(B_0,b_0))+\varepsilon} \tag{3}$$

The key ingredient behind Theorem 1.4 is a new, essentially optimal, upper bound for the Kobayashi length of loops on punctured bordered surfaces (Theorem 1.5). Let us briefly describe the context. Let B_0 be the bordered hyperbolic surface introduced above. For a finite set $S \subset B_0$, the punctured surface $B_0 \setminus S$ carries its own Kobayashi metric, and the induced length ℓ_S of paths in $B_0 \setminus S$ is larger than the length of the same paths measured in the Kobayashi metric of B_0: the new cusps created by the punctures stretch the metric, so loops become longer as the cardinality of S grows. Given a loop α representing a nontrivial class in $\pi_1(B_0, b_0)$, it is natural to ask how the minimal Kobayashi length of a representative of α in $B_0 \setminus S$ grows with the number of punctures. This is captured by the quantity

$$L(\alpha, s) := \sup_{\substack{S \subset B_0 \\ \#S \leq s}} \inf \{\ell_S(\gamma)\colon \gamma \subset B_0 \setminus S \text{ is a loop freely homotopic to } \alpha \text{ in } B_0\}\,,$$

which measures the worst-case minimal Kobayashi length over all configurations of at most s punctures. Phung [8, Theorems B, C] proved that $L(\alpha, s)$ grows at least as $c\sqrt{s}/\log(s+2)$ and at most linearly in s. As explained above, the linear upper bound $L(\alpha, s) = O(s)$ is what produces the factor of 2 in the exponent of (2): replacing $H(s) = O(s)$ by $H(s) = O(\sqrt{s \log s})$ in the lattice counting halves the exponent from $2nk$ to nk. We prove:

Theorem 1.5. *Fix a loop $\alpha \subset B_0$ representing a nontrivial class in $\pi_1(B_0, b_0)$. There exists $C > 0$, depending only on α, B_0, and ρ, such that*

$$L(\alpha, s) \leq C\sqrt{(s+1)\log(s+2)} \quad \textit{for all } s \geq 0\,. \tag{4}$$

Combined with Phung's lower bound, this determines the exact growth rate (Corollary 2.4):

$$\lim_{s\to+\infty} \frac{\log L(\alpha, s)}{\log s} = \frac{1}{2}\,. \tag{5}$$

This answers [8, Question 1.4] which asked about the asymptotic behavior of $\frac{\log L(\alpha,s)}{\log s}$.

The proof of Theorem 1.5 proceeds as follows. Fix a smooth representative γ of α in B_0 and consider the Fermi strip $T_\delta(\gamma)$, a thin tubular neighborhood of γ foliated by parallel curves γ_u. Phung's linear upper bound $L(\alpha, s) = O(s)$ in [8] is obtained by estimating the Kobayashi length of a single loop directly; our improvement replaces this pointwise approach with an integral averaging argument. The key idea is to bound the L^p-norm of the distortion function $\lambda_S = \kappa_S/\sqrt{\rho}$ (the ratio of the Kobayashi metric of $B_0 \setminus S$ to the background metric ρ) over the entire strip, for a variable exponent $1 < p < 2$. This is achieved by a Voronoi decomposition of the strip into cells centered at the punctures: near each puncture, λ_S blows up like the inverse of the distance, and the L^p-integrability for $p < 2$ (but not for $p = 2$) controls the singularity. The crucial gain over the linear bound is that summing s Voronoi contributions of size $O(1/(2-p))$ produces a factor of $s/(2-p)$ rather than s, and Hölder's inequality then converts this L^p-bound into an upper bound on the Kobayashi length of a typical parallel curve γ_{u_0}, which is freely homotopic to α in B_0 and avoids S. Optimising the exponent as $p = 2 - 1/\log(s+2)$ balances the divergence of the L^p-norm as $p \to 2^-$ against the sharpness of the Hölder estimate, yielding the $O(\sqrt{(s+1)\log(s+2)})$ bound.

Theorem 1.4 follows from Theorem 1.5 by substituting the improved length estimate into the Parshin cocycle argument outlined above. The cocycle construction requires loops representing the generators $\alpha_1, \ldots, \alpha_k$ that share a *common base point* and are conjugated to the generators by a *single* path. Theorem 1.5, applied independently to each generator, produces loops with the optimal Kobayashi length bound but with potentially different base points. In Proposition 2.7 we show that, by choosing smooth representatives of $\alpha_1, \ldots, \alpha_k$ that share a common tangent direction at b_0 (so that their Fermi strips can be controlled by a single averaging parameter), a common base point can be arranged without degrading the asymptotic bound. The improved displacement bound $H(s) = O(\sqrt{s \log s})$ then propagates through the lattice counting argument unchanged, halving the exponent.

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2 Kobayashi lengths estimates

2.1 Kobayashi pseudo metric

Let X be a connected complex manifold and let TX be its tangent bundle.

Definition 2.1. A *Finsler pseudo-metric* on X is a function

$$\begin{aligned} F\colon TX &\to \mathbb{R}_{\geq 0} \\ (x, v) &\mapsto F(x, v) \end{aligned}$$

satisfying the homogeneity condition $F(x, \lambda v) = |\lambda| F(x, v)$ for $\lambda \in \mathbb{C}$. If F is positive definite, i.e. $F(x, v) > 0$ for all $x \in X$ and $v \in T_{X,x} \setminus \{0\}$, then we say that F is a *Finsler metric.*

Given a Borel measurable Finsler pseudo-metric F, for any piecewise C^1 path$\gamma\colon [0,1] \to X$ we define the length of γ as:

$$\ell_F(\gamma) := \int_0^1 F(\gamma(t), \gamma'(t))dt < +\infty$$

Without loss of generality from now on we can assume that any path on X is piecewise C_1, and given a set of paths Γ we define

$$\ell_F(\Gamma) := \inf_{\gamma\in\Gamma} \ell_F(\gamma)\,.$$

We can endow X with a structure of pseudo-metric space by setting

$$d_F(x, y) := \inf_{\gamma} \ell_F(\gamma)\,, \quad \text{where the inf runs over all curves } \gamma \text{ joining } x \text{ and } y\,.$$

Moreover, if Y, Z are two subsets of X, we put

$$\mathrm{diam}_F(Y) := \sup_{y,y' \in Y} d_F(y,y')\,, \quad d_F(Y,Z) := \inf_{y\in Y, z\in Z} d_F(y,z)\,.$$

Any Riemannian metric ρ on X obviously induces a Finsler metric, so a topological metric on X which is simply denoted by d_ρ.

A very important example of Finsler pseudo-metric is the *Kobayashi-Royden pseudo-metric* defined as

$$\kappa_X(x,v) := \inf\left\{\frac{2}{R} \in \mathbb{R}_{>0} \colon \exists f \in \mathrm{Hol}(\Delta(0,R), X) \text{ such that } f(0)=x,\ f'(0)=v\right\}.$$

Royden in [9, proposition 3] shows that the Kobayashi-Royden pseudo-metric is upper semi-continuous and hence Borel measurable. Note that κ_X is intrinsic in the geometry of X, so the induced topological pseudo-metric is denoted by d_X and it is called the *Kobayashi pseudo-metric*. We say that X is *Kobayashi hyperbolic* if d_X is a metric. If X is Riemannian hyperbolic with respect to a Riemannian metric h then X is Kobayashi hyperbolic. The vice versa is in general false. However, if X is a (Riemannian) hyperbolic Riemann surface, then X is isometric to $\Gamma\backslash\mathbb{H}$ where $\mathbb{H}$ is the Poincaré half-plane endowed with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and Γ is a Fuchsian group without elliptic elements. One can show that the induced quotient metric on X is exactly the Kobayashi metric. In this case the two notions of hyperbolicity coincide, and we don't need to distinguish.

From the definition one can easily deduce the decreasing property of the Kobayashi-Royden metric: if $\phi\colon (X,\kappa_X) \to (Y,\kappa_Y)$ is a holomorphic map, then

$$\kappa_Y(\phi(x), d_x\phi(v)) \le \kappa_X(x,v)\,, \quad \forall (x,v) \in TX\,.$$

This in turn implies the decreasing property of the Kobayashi distance

$$d_Y(\phi(x),\phi(x')) \le d_X(x,x')\,, \quad \forall x,x' \in X\,.$$

2.2 Growth of loop lengths in terms of the cusps

We fix a compact Riemann surface B of genus g endowed with a Riemannian metric ρ. Let $\overline{\Delta}_1,\dots,\overline{\Delta}_t$ be disjoint disks on B and define $B_0 := B \smallsetminus \sqcup_{i=1}^t \overline{\Delta}_i$. We assume that $t \ge 3$ so that B_0 is hyperbolic. Now, let $S \subseteq B_0$ be a finite set (possibly empty), and put $s := \#S$. The Riemann surface $B_0 \smallsetminus S$ is hyperbolic and κ_S is its Kobayashi-Royden metric. The induced lengths and distance are denoted by ℓ_S and d_S, respectively. For any local chart (U,z) we have

$$\kappa_S(b,v) = \kappa_S\left(b, |v|\,\partial z|_b\right) = |v|\kappa_S\left(b,\partial z|_b\right)\,, \quad \sqrt{\rho_b(v,v)} = |v|\sqrt{\rho_b\left(\partial z|_b, \partial z|_b\right)}$$

so the *distortion function*:

$$\lambda_S(b) := \frac{\kappa_S(b,v)}{\sqrt{\rho_b(v,v)}}\,, \quad \forall b \in B_0 \smallsetminus S$$

is well defined (independent of the choice of $v \ne 0$ and of the chart) and smooth on $B_0 \smallsetminus S$. Fix a loop α in B_0 that doesn't represent a trivial class in $\pi_1(B_0,b_0)$, and let $\gamma \Subset B_0$ be a smooth loop freely homotopic to α (it exists by [4, Theorem 6.26]). Parametrize γ by its ρ-arc-length and let $\eta(\tau)$ be the normal unit vector of γ at τ. Then for $\delta \in \mathbb{R}_{>0}$ sufficiently small we have the diffeomorphism:

$$\begin{aligned}[0,\ell_\rho(\gamma)] \times [-\delta,\delta] &\to \{b \in B_0 \colon d_\rho(b,\gamma) \le \delta\} =: T_\delta(\gamma) \Subset B_0\\ (\tau,u) &\mapsto \exp_{\gamma(\tau)}(u\eta(\tau))\end{aligned}$$

The coordinates (τ,u) are called *Fermi coordinates*, and $T_\delta(\gamma)$ is the *Fermi strip* with respect to γ. For any $u \in [-\delta,\delta]$ we define the *parallel curve* $\gamma_u\colon [0,\ell_\rho(\gamma)] \to \exp_{\gamma(\tau)}(u\eta(\tau))$.
In Fermi coordinates the induced metric on each parallel curve γ_u is $\rho|_{\gamma_u} = \rho_{\tau\tau}\,d\tau^2$, so its ρ-length is

$$\ell_\rho(\gamma_u) = \int_0^{\ell_\rho(\gamma)} \sqrt{\rho_{\tau\tau}(\tau,u)}\,d\tau\,.$$

Write the Riemannian area form in Fermi coordinates as $dA_\rho = J(\tau,u)\,d\tau\,du$, where

$$J(\tau,u) := \sqrt{\rho_{\tau\tau}\rho_{uu} - \rho_{\tau u}^2}$$

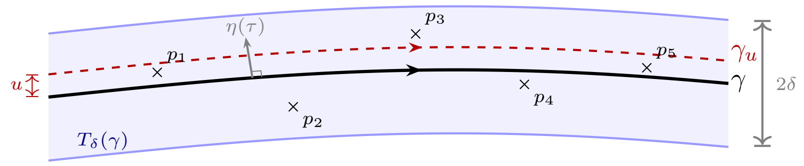

Figure 1: The Fermi strip $T_\delta(\gamma)$ around γ, with punctures $p_1, \dots, p_5 \in S$ and the parallel curve γ_u.

is the square root of the determinant of the metric ρ in the coordinates (τ, u). We claim that for δ small enough, J and $\sqrt{\rho_{\tau\tau}}$ are uniformly comparable on the strip. Indeed, by the properties of Fermi coordinates ([5, Proposition 5.26]), the metric at $u = 0$ satisfies $\rho_{\tau u}(\tau, 0) = 0$ and $\rho_{uu}(\tau, 0) = 1$, so $J(\tau, 0) = \sqrt{\rho_{\tau\tau}(\tau, 0)}$. Both J and $\sqrt{\rho_{\tau\tau}}$ are continuous and strictly positive on $[0, \ell_\rho(\gamma)] \times \{0\}$. By continuity, both functions remain strictly positive on $[0, \ell_\rho(\gamma)] \times [-\delta, \delta]$ for δ small enough, so their ratio $J/\sqrt{\rho_{\tau\tau}}$ is a continuous positive function on a compact set that equals 1 on $u = 0$. It follows that there exists $\kappa_0 = \kappa_0(\rho, \gamma, \delta) \geq 1$ such that

$$\kappa_0^{-1}\sqrt{\rho_{\tau\tau}(\tau, u)} \leq J(\tau, u) \leq \kappa_0\sqrt{\rho_{\tau\tau}(\tau, u)} \qquad \forall\, (\tau, u) \in [0, \ell_\rho(\gamma)] \times [-\delta, \delta]\,. \tag{6}$$

In particular, for any measurable $f \geq 0$:

$$\kappa_0^{-1}\int_{-\delta}^{\delta}\int_0^{\ell_\rho(\gamma)} f(\tau, u)\,\sqrt{\rho_{\tau\tau}(\tau, u)}\,d\tau\,du \;\leq\; \int_{T_\delta(\gamma)} f\,dA_\rho \;\leq\; \kappa_0\int_{-\delta}^{\delta}\int_0^{\ell_\rho(\gamma)} f(\tau, u)\,\sqrt{\rho_{\tau\tau}(\tau, u)}\,d\tau\,du\,. \tag{7}$$

Lemma 2.2. *For every $1 \leq p < 2$ and any finite set $S \subset B_0$ there exists a constant $A := A(\rho, B_0, \alpha) \in \mathbb{R}_{>0}$ such that*

$$\int_{T_\delta(\gamma)} \lambda_S^p\,dA_\rho \leq \frac{A(s+1)}{2-p}$$

Proof. Assume $S = \{p_1, \dots, p_s\} \neq \emptyset$. Consider the Voronoi decomposition of $T_\delta(\gamma)$ induced by S: for each $p_j \in S$, define

$$V_j := \{b \in T_\delta(\gamma) \colon d_\rho(b, p_j) \leq d_\rho(b, p_i)\ \forall i \neq j\}\,.$$

Note that the punctures p_j need not lie inside $T_\delta(\gamma)$. Nevertheless, we have $T_\delta(\gamma) = \bigcup_{j=1}^s V_j$, the interiors of the V_j are pairwise disjoint, and the boundaries (where ties occur) have ρ-measure zero. Some cells V_j may be empty.

Step 1: Pointwise and area bounds. Put $T := T_\delta(\gamma)$ and fix $b \in T \setminus S$. Set $D_b := d_\rho(b, S) > 0$. By [8, Lemma 3.3], there exist constants $c_1, r_1 > 0$ (depending only on B and ρ) such that for every $b \in B$, every $v \in T_bB \setminus \{0\}$, and every $0 < r < r_1$:

$$\kappa_{\Delta(b,r)}(b, v) \;\leq\; \frac{c_1}{r}\,\sqrt{\rho_b(v, v)}\,.$$

By [8, Lemma 3.5], there exist constants $c_2, r_2 > 0$ such that $\mathrm{Area}_\rho(\Delta(q, r)) \leq c_2\,r^2$ for all $q \in B$ and $0 < r \leq r_2$. Set $D := \min\{d_\rho(T, \partial B_0),\, r_1,\, r_2\} > 0$ and $R_b := \min\{D_b, D\}$. Then $\Delta(b, R_b) \subset B_0 \setminus S$ (it lies in B_0 because $R_b \leq D \leq d_\rho(T, \partial B_0)$, and avoids S because $R_b \leq D_b$), so the decreasing property of the Kobayashi–Royden metric (applied to the inclusion $\Delta(b, R_b) \hookrightarrow B_0 \setminus S$) gives:

$$\lambda_S(b) \;=\; \frac{\kappa_{B_0\setminus S}(b, v)}{\sqrt{\rho_b(v, v)}} \;\leq\; \frac{\kappa_{\Delta(b,R_b)}(b, v)}{\sqrt{\rho_b(v, v)}} \;\leq\; \frac{c_1}{R_b} \;=\; \frac{c_1}{\min\{D_b,\, D\}}\,. \tag{8}$$

Moreover, since $D \leq r_2$:

$$\mathrm{Area}_\rho(\Delta(q, r)) \;\leq\; c_2\,r^2 \qquad \text{for all } q \in B \text{ and } r \leq D\,. \tag{9}$$

Step 2: Integral over Voronoi cells. For $b \in V_j$, the point p_j is the nearest element of S to b, so $D_b = d_\rho(b, p_j)$. We split V_j into two regions:

$$V_j^{\mathrm{near}} := \{b \in V_j \colon d_\rho(b, p_j) \leq D\}\,, \qquad V_j^{\mathrm{far}} := \{b \in V_j \colon d_\rho(b, p_j) > D\}\,.$$

On V_j^{near} we have $R_b = D_b = d_\rho(b, p_j)$, so (8) gives $\lambda_S(b) \le c_1/d_\rho(b,p_j)$. By the "layer cake representation" in measure theory ($\int f^p\,d\mu = p\int_0^\infty t^{p-1}\mu(\{f>t\})\,dt$, see [6, Theorem 1.13]) and the substitution $r = 1/t$:

$$
\begin{aligned}
\int_{V_j^{\text{near}}} \lambda_S^p \, dA_\rho &\le c_1^p \int_{V_j^{\text{near}}} \frac{dA_\rho}{d_\rho(b,p_j)^p} \\
&= p\, c_1^p \int_0^{+\infty} t^{p-1}\, \mathrm{Area}_\rho\big\{b \in V_j^{\text{near}} \colon d_\rho(b,p_j)^{-1} > t\big\}\, dt \\
&= p\, c_1^p \int_0^{+\infty} \frac{\mathrm{Area}_\rho(V_j^{\text{near}} \cap \Delta(p_j,r))}{r^{p+1}}\, dr \\
&\le p\, c_1^p \int_0^{D} \frac{\mathrm{Area}_\rho(\Delta(p_j,r))}{r^{p+1}}\, dr + p\, c_1^p \int_D^{+\infty} \frac{\mathrm{Area}_\rho(T)}{r^{p+1}}\, dr \\
&\le \frac{p\, c_1^p\, c_2\, D^{2-p}}{2-p} + \frac{c_1^p\, \mathrm{Area}_\rho(T)}{D^p}\,,
\end{aligned}
\tag{10}
$$

where the fourth line uses $V_j^{\text{near}} \cap \Delta(p_j,r) \subset \Delta(p_j,r)$ for $r \le D$, and $\mathrm{Area}_\rho(V_j^{\text{near}} \cap \Delta(p_j,r)) \le \mathrm{Area}_\rho(T)$ for $r \ge D$. Moreover the fifth line uses the area bound (9) and $\int_D^{+\infty} r^{-(p+1)}\,dr = D^{-p}/p$. Note that the integral $\int_0^D r^{1-p}\,dr$ converges since $p<2$.

On V_j^{far} we have $R_b = D$, so $\lambda_S(b) \le c_1/D$ and

$$
\int_{V_j^{\text{far}}} \lambda_S^p \, dA_\rho \le \frac{c_1^p}{D^p}\, \mathrm{Area}_\rho(T)\,. \tag{11}
$$

Set $c_3 := \max\big\{2\max\{1,c_1^2\}\, c_2 \max\{1,D\},\ 2\max\{c_1/D,\, c_1^2/D^2\}\,\mathrm{Area}_\rho(T)\big\}$. Since $1 \le p < 2$ we have $c_1^p \le \max\{1, c_1^2\}$, $D^{2-p} \le \max\{1,D\}$, and $c_1^p/D^p \le \max\{c_1/D,\, c_1^2/D^2\}$, so combining (10) and (11):

$$
\int_{V_j} \lambda_S^p \, dA_\rho \le \frac{c_3}{2-p} + c_3 \le \frac{2c_3}{2-p}\,, \tag{12}
$$

where the last inequality uses $1 \le 1/(2-p)$ for $p \ge 1$.

Step 3: Summing over cells. Summing over all $j = 1,\dots,s$ (cells with $V_j = \emptyset$ contribute zero):

$$
\int_T \lambda_S^p \, dA_\rho = \sum_{j=1}^{s} \int_{V_j} \lambda_S^p\, dA_\rho \le \frac{2c_3 s}{2-p}\,.
$$

If $s = 0$ (i.e. $S = \emptyset$) on T we have

$$
\lambda_S(b) = \frac{\kappa_{B_0}(b,v)}{\sqrt{\rho_b(v,v)}} \le M := \sup_{b\in T} \frac{\kappa_{B_0}(b,v)}{\sqrt{\rho_b(v,v)}} < +\infty\,;
$$

where the finiteness follows from $T \Subset B_0$. In this case we have

$$
\int_T \lambda_S^p\, dA_\rho \le M^p\, \mathrm{Area}_\rho(T) \le \max\{1, M^2\}\,\mathrm{Area}_\rho(T)\,.
$$

Setting $A := \max\{2c_3,\, \max\{1,M^2\}\,\mathrm{Area}_\rho(T)\}$ the claim is proved. □

Let us now introduce the main object of this section and then prove the main result.

Definition 2.3. Fix a loop α in B_0 that doesn't represent a trivial class in $\pi_1(B_0,b_0)$. Let Γ_S^α denote the set of loops in $B_0 \setminus S$ that are freely homotopic to α in B_0. We define

$$
L(\alpha,s) := \sup_{\substack{S\subset B_0\\ \#S\le s}} \ell_S(\Gamma_S^\alpha)\,.
$$

Note that since α is chosen (non-trivially) in B_0 and the free homotopies are checked in the non-cuspidal hyperbolic surface B_0, we have $\ell_S(\Gamma_S^\alpha) > 0$ by [1, Proposition 2.24].

Proof of Theorem 1.5. Consider the smooth loop $\gamma \Subset B_0$ freely homotopic to α, introduced at the beginning of the section. Put $T := T_\delta(\gamma)$ parametrized with Fermi coordinates (τ, u) as above. Set

$$L_0 := \sup_{|u|\le\delta} \ell_\rho(\gamma_u) < +\infty\,.$$

Fix an arbitrary $S \subset B_0$ with $\#S = s$ and a parameter $1 < p < 2$. Define $q := \frac{p}{p-1}$ (the conjugate exponent, so $1/p + 1/q = 1$). For each $u \in]-\delta,\delta[$ with $\gamma_u \cap S = \emptyset$, Hölder's inequality with respect to the measure $d\mu = \sqrt{\rho_{\tau\tau}}\,d\tau$ gives:

$$\begin{aligned} \ell_S(\gamma_u) &= \int_0^{\ell_\rho(\gamma)} \kappa_S(\gamma_u(\tau),\gamma_u'(\tau))\,d\tau \\ &= \int_0^{\ell_\rho(\gamma)} \lambda_S(\tau,u)\sqrt{\rho_{\tau\tau}(\tau,u)}\,d\tau \\ &\le \left(\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau,u)^p\sqrt{\rho_{\tau\tau}}\,d\tau\right)^{1/p}\left(\int_0^{\ell_\rho(\gamma)}\sqrt{\rho_{\tau\tau}}\,d\tau\right)^{1/q} \\ &= \left(\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau,u)^p\sqrt{\rho_{\tau\tau}}\,d\tau\right)^{1/p}\ell_\rho(\gamma_u)^{1/q}\,, \end{aligned} \tag{13}$$

By Equation (7) and Lemma 2.2 we get:

$$\int_{-\delta}^{\delta}\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau,u)^p\sqrt{\rho_{\tau\tau}}\,d\tau\,du \;\le\; \kappa_0\int_T \lambda_S^p\,dA_\rho \;\le\; \frac{\kappa_0 A(s+1)}{2-p}\,. \tag{14}$$

The set $E := \{u \in]-\delta,\delta[: \gamma_u \cap S \ne \emptyset\}$ has at most s points, hence has measure zero. Define $g(u) := \int_0^{\ell_\rho(\gamma)} \lambda_S(\tau,u)^p\sqrt{\rho_{\tau\tau}}\,d\tau$ and $M := \frac{\kappa_0 A(s+1)}{2\delta(2-p)}$. By (14),

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} g(u)\,du \;\le\; M\,. \tag{15}$$

We claim there exists $u_0 \in]-\delta,\delta[\setminus E$ with $g(u_0) \le M$. If not, then $g(u) > M$ for all $u \in]-\delta,\delta[\setminus E$. Since $g \ge 0$ and E has measure zero:

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} g(u)\,du = \frac{1}{2\delta}\int_{]-\delta,\delta[\setminus E} g(u)\,du > \frac{1}{2\delta}\cdot M\cdot 2\delta = M\,,$$

contradicting (15). Hence such u_0 exists, and since $u_0 \notin E$ we have $\gamma_{u_0} \cap S = \emptyset$ and

$$\int_0^{\ell_\rho(\gamma)} \lambda_S(\tau,u_0)^p\sqrt{\rho_{\tau\tau}(\tau,u_0)}\,d\tau \;=\; g(u_0) \;\le\; M \;=\; \frac{\kappa_0 A(s+1)}{2\delta(2-p)}\,. \tag{16}$$

Substituting Equation (16) into Equation (13) and using $\ell_\rho(\gamma_{u_0}) \le L_0$ we get

$$\ell_S(\gamma_{u_0}) \le \left(\frac{\kappa_0 A}{2\delta}\right)^{1/p} L_0^{1/q}\left(\frac{s+1}{2-p}\right)^{1/p}\,. \tag{17}$$

Since the Fermi coordinate map is a diffeomorphism, the family $\{\gamma_u\}_{u\in[0,u_0]}$ is a free homotopy in $T_\delta(\gamma) \subset B_0$ from $\gamma_0 = \gamma$ to γ_{u_0}. As γ is freely homotopic to α in B_0 (by construction), γ_{u_0} is freely homotopic to α in B_0 by transitivity. Moreover $\gamma_{u_0} \subset B_0 \setminus S$ (since $u_0 \notin E$), so $\gamma_{u_0} \in \Gamma_S^\alpha$.

We now assume $s \ge 1$ (the case $s = 0$ is handled below) and set

$$p := 2 - \frac{1}{\log(s+2)}\,, \quad \text{so that} \quad 2 - p = \frac{1}{\log(s+2)}\,, \quad p > 1\,.$$

Then

$$\frac{s+1}{2-p} = (s+1)\log(s+2)\,, \qquad \frac{1}{p} = \frac{\log(s+2)}{2\log(s+2)-1}\,.$$

We split

$$\left(\frac{s+1}{2-p}\right)^{1/p} = \big((s+1)\log(s+2)\big)^{1/2} \cdot \big((s+1)\log(s+2)\big)^{1/p-1/2}$$

and claim the second factor is $O(1)$. Indeed, $1/p - 1/2 = O(1/\log s)$ and

$$\log\big((s+1)\log(s+2)\big) = \log(s+1) + \log\log(s+2) = O(\log s)\,,$$

so

$$\big((s+1)\log(s+2)\big)^{1/p-1/2} = \exp\!\big(O(1/\log s)\cdot O(\log s)\big) = \exp(O(1)) = O(1)\,.$$

For the first factor of Equation (17) $C_0(p) := (\kappa_0 A/(2\delta))^{1/p} L_0^{1/q}$: as $p \to 2^-$ we have $1/p \to 1/2$ and $1/q \to 1/2$, so $C_0(p) \to (\kappa_0 A/(2\delta))^{1/2} L_0^{1/2}$. Since C_0 is continuous on $[2-1/\log 3, 2[$ and has a finite limit at $p=2$, it is bounded.

Combining, we conclude that there exists $C > 0$, depending only on α, B_0, and ρ, such that for every $S \subset B_0$ with $\#S = s \geq 1$:

$$\ell_S(\Gamma_S^\alpha) \leq \ell_S(\gamma_{u_0}) \leq C\sqrt{(s+1)\log(s+2)}\,.$$

Since this holds for every such S, taking the supremum gives $L(\alpha, s) \leq C\sqrt{(s+1)\log(s+2)}$. For $s=0$, $L(\alpha, 0) = \inf_\beta \ell_{\kappa_{B_0}}(\beta)$ is a fixed positive constant. □

Corollary 2.4. *Fix a loop $\alpha \subset B_0$ representing a nontrivial class in $\pi_1(B_0, b_0)$, then*

$$\lim_{s\to+\infty} \frac{\log L(\alpha, s)}{\log s} = \frac{1}{2}\,.$$

Proof. Combining Theorem 1.5 with Phung's lower bound [8, Theorem C] gives

$$\frac{c\sqrt{s}}{\log(s+2)} \leq L(\alpha, s) \leq C\sqrt{(s+1)\log(s+2)}$$

so the result follows immediately. □

Remark 2.5. Corollary 2.4 doesn't determine the precise asymptotic growth of $L(\alpha, s)$. In particular, it gives $L(\alpha, s) = O(s^{1/2+\varepsilon})$ for every $\varepsilon > 0$, but the logarithmic factors in the upper and lower bounds do not match: the upper bound contains $\sqrt{\log(s+2)}$ while the lower bound contains $1/\log(s+2)$. It remains an open question whether $L(\alpha, s) = O(\sqrt{s})$, or whether a logarithmic correction is necessary.

For the application to generalized integral points, the Parshin cocycle argument requires loops representing the generators of $\pi_1(B_0, b_0)$ that share a *common base point* and are conjugated to the generators by a *single* path. Theorem 1.5, applied independently to each generator, produces loops with the optimal Kobayashi length bound but with potentially different base points. The next proposition shows that a common base point can be arranged without degrading the asymptotic bound.

Lemma 2.6. *Let M be a smooth orientable Riemannian surface without boundary, let $p \in M$, and let $w \in T_pM$ be a unit tangent vector. Then for every smooth simple loop γ at p with $\gamma \Subset M$, there exists a smooth simple loop $\tilde\gamma$ at p with $\tilde\gamma \Subset M$, homotopic to γ relative to p, and satisfying*

$$\frac{\tilde\gamma'(0)}{|\tilde\gamma'(0)|} = w\,.$$

Proof. Let $w_0 := \gamma'(0)/|\gamma'(0)|$ be the unit tangent vector of γ at p. If $w_0 = w$ there is nothing to prove, so assume $w_0 \neq w$. Let $\theta \in]0, 2\pi[$ be the angle from w_0 to w and for each $t \in [0,1]$ let $R_t \in \mathrm{SO}(2)$ denote the rotation of T_pM by angle $t\theta$, so that $R_0 = \mathrm{Id}$ and $R_1(w_0) = w$.

Choose $\epsilon > 0$ small enough that the exponential map $\exp_p$ is a diffeomorphism from $B(0,\epsilon) \subset T_pM$ onto an open neighborhood U of p in M with $U \Subset M$, and that γ crosses U in a single connected branch. Choose a smooth cutoff function $\chi\colon [0,\infty) \to [0,1]$ with $\chi \equiv 1$ on $[0, \epsilon/3]$ and $\chi \equiv 0$ on $[2\epsilon/3, \infty)$. Define the map $\phi\colon M \to M$ by

$$\phi(x) := \begin{cases} \exp_p\!\big(R_{\chi(|y|)}\, y\big), & \text{if } x = \exp_p(y) \in U, \\ x, & \text{if } x \notin U, \end{cases}$$

where $|y|$ denotes the norm in T_pM. Since $\chi \equiv 1$ near $|y| = 0$, the map ϕ equals $\exp_p \circ R_1 \circ \exp_p^{-1}$ in a neighborhood of p, and since $\chi \equiv 0$ for $|y| \geq 2\epsilon/3$, ϕ is the identity outside U. In normal coordinates centered at p, the map ϕ acts as $y \mapsto R_{\chi(|y|)}\, y$, which is a pointwise rotation. Since $R_t \in \mathrm{SO}(2)$ for every t, a direct computation shows that the Jacobian determinant equals 1 everywhere. In particular, ϕ is a smooth diffeomorphism of M. Moreover:

(i) $\phi(p) = p$ and $d_p\phi = R_1$, so $(\phi \circ \gamma)'(0) = R_1(\gamma'(0)) = |\gamma'(0)|\, w$, which has unit tangent direction w;

(ii) ϕ is the identity outside U, so $\phi \circ \gamma$ coincides with γ outside U and in particular $\phi \circ \gamma \Subset M$;

(iii) ϕ is a diffeomorphism, so $\phi \circ \gamma$ is simple;

(iv) the family ϕ_t (defined by replacing χ with $t\chi$ for $t \in [0,1]$) satisfies $\phi_0 = \mathrm{Id}$ and $\phi_1 = \phi$, and $\phi_t(p) = p$ for all t. Hence $t \mapsto \phi_t \circ \gamma$ is a homotopy of loops based at p from γ to $\tilde{\gamma}$, so $[\tilde{\gamma}] = [\gamma]$ in $\pi_1(M, p)$.

Setting $\tilde{\gamma} := \phi \circ \gamma$ completes the proof. □

Proposition 2.7. *Let B_0 be as above and fix a simple basis $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$. There exist constants $C, \delta' > 0$, depending only on $\alpha_1, \dots, \alpha_k$, B_0, and ρ, such that for every finite set $S \subset B_0$ with $\#S = s \geq 1$, there exist a point $q \in B_0 \setminus S$, a path σ from b_0 to q in B_0 with $\ell_\rho(\sigma) \leq \delta'$, and loops $\hat{\gamma}_1, \dots, \hat{\gamma}_k \subset B_0 \setminus S$ based at q such that*

$$[\sigma^{-1} \circ \hat{\gamma}_j \circ \sigma] = \alpha_j \quad \text{in } \pi_1(B_0, b_0) \qquad \text{for every } j = 1, \dots, k,$$

and

$$\ell_S(\hat{\gamma}_j) \;\leq\; C\sqrt{(s+1)\log(s+2)} \qquad \text{for all } j = 1, \dots, k.$$

Proof. Fix a unit tangent vector $w \in T_{b_0}B_0$ and let η_0 be the unit vector orthogonal to w with respect to ρ. By Lemma 2.6, for each $j = 1, \dots, k$ we can choose a smooth simple loop $\gamma^{(j)}$ at b_0, compactly contained in B_0, freely homotopic to α_j, and whose unit tangent vector at b_0 equals w. Since all loops share the tangent direction w at b_0, their unit normal vectors at b_0 all equal η_0. For each j, construct the Fermi strip $T_\delta(\gamma^{(j)})$ as in the proof of Theorem 1.5, with a common width $\delta > 0$ small enough for all k strips. The parallel curve $\gamma_u^{(j)}$ has base point

$$q(u) \;:=\; \exp_{b_0}(u\, \eta_0),$$

which is independent of j. For $1 < p < 2$, define

$$g_j(u) \;:=\; \int_0^{\ell_\rho(\gamma^{(j)})} \lambda_S(\tau, u)^p \,\sqrt{\rho_{\tau\tau}}\, d\tau$$

and set $g := \sum_{j=1}^k g_j$. Let $E_j := \{u \in]-\delta, \delta[\, :\ \gamma_u^{(j)} \cap S \neq \emptyset\}$; then $\#E_j \leq s$, so the set $E := \bigcup_{j=1}^k E_j$ has at most ks points and in particular has measure zero. By (7) and Lemma 2.2 applied to each Fermi strip,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} g(u)\, du \;\leq\; \sum_{j=1}^{k} \frac{\kappa_0^{(j)} A_j (s+1)}{2\delta(2-p)} \quad =:\; M\,,$$

where $\kappa_0^{(j)}$ and A_j are the constants from (7) and Lemma 2.2 corresponding to the strip $T_\delta(\gamma^{(j)})$. By the mean-value argument used in the proof of Theorem 1.5, there exists $u_0 \in]-\delta, \delta[\setminus E$ with $g(u_0) \leq M$. In particular, $g_j(u_0) \leq M$ for every j.

Since $u_0 \notin E$, the parallel curve $\gamma_{u_0}^{(j)}$ avoids S for every j. Applying Hölder's inequality (13) with $g_j(u_0) \leq M$ in place of the individual bound, and the optimisation $p = 2 - 1/\log(s+2)$ exactly as in the proof of Theorem 1.5, we obtain

$$\ell_S(\gamma_{u_0}^{(j)}) \;\leq\; C_j' \sqrt{(s+1)\log(s+2)}$$

for every $j = 1, \dots, k$, where $C_j' > 0$ depends only on $\alpha_1, \dots, \alpha_k$, B_0, and ρ.

Set $\hat{\gamma}_j := \gamma_{u_0}^{(j)}$ and $q := q(u_0) = \exp_{b_0}(u_0\, \eta_0) \in B_0 \setminus S$. Define the path $\sigma\colon [0, |u_0|] \to B_0$ by $\sigma(u) := \exp_{b_0}(u\, \eta_0)$; it connects b_0 to q with $\ell_\rho(\sigma) = |u_0| \leq \delta$. For each j, the family $\{\gamma_u^{(j)}\}$ for u between 0 and u_0 is a free homotopy in $T_\delta(\gamma^{(j)}) \subset B_0$ from $\gamma_0^{(j)} = \gamma^{(j)}$ (based at b_0) to $\gamma_{u_0}^{(j)} = \hat{\gamma}_j$ (based at q), and this homotopy moves the base point along σ. It follows that $[\sigma^{-1} \circ \hat{\gamma}_j \circ \sigma] = [\gamma^{(j)}] = \alpha_j$ in $\pi_1(B_0, b_0)$ for every j. Setting $C := \max_j C_j'$ and $\delta' := \delta$ completes the proof. □

3 Bounding generalized integral points

3.1 Phung's bound

In this section, we summarize the proof of Theorem 1.3 of Phung. Our goal is not to reprove any part of it—Phung's argument is already complete—but rather to recall its structure in sufficient detail so that we can later identify precisely where our improvement is applied.

Proof of Theorem 1.3. Fix a hermitian metric h on $\mathcal{A}$. Put $k := \mathrm{rk}(\pi_1(B_0))$, where we have implicitly fixed the base point $b_0 \in B_0$ for $\pi_1(B_0)$.

Step 1: Reduction to the hyperbolic locus. Consider the non-hyperbolic locus

$$V := \{b \in B \colon \mathcal{D}_b \text{ is not hyperbolic}\}$$

It is well known that V is an analytic closed subset of B and moreover by [8] it is at most countable. Then, thanks to [8] it is possible to enlarge the disks $\overline{\Delta}_i$, keeping them disjoint, in a way that $V \subseteq \bigsqcup_{i=1}^t \overline{\Delta}_i$. In this way B_0 becomes smaller but the fundamental group $\pi_1(B_0)$ doesn't change. It means that for our purposes we can always assume $V \subseteq \bigsqcup_{i=1}^t \overline{\Delta}_i$, so that $\mathcal{D}_b$ is Kobayashi hyperbolic for any $b \in B_0$. By a result of Green ([8, Theorem 7.6]) it follows that $\mathcal{A}_b \setminus \mathcal{D}_b$ is complete and Kobayashi hyperbolic for any $b \in B_0$. This is the point where one has to use the assumption (P)(iii): it is in the hypotheses of Green's theorem. In addition, the (common) radius of the disks can be taken in a way $\overline{\Delta}_i$ is geodesically convex. As a consequence any geodesic of B can cross any $\overline{\Delta}_i$ at most once. Then, for any two points $x, y \in B_0$ we can always find a path $c_{x,y} \subset B_0$ joining them and such that:

$$\ell(c_{x,y}) \leq \mathrm{diam}(B) + tM \qquad \text{where } M > \max_i \ell(\partial\overline{\Delta}_i)$$

We denote $\mathcal{C}_{b_0} := \{c_{b_0,p}\}_{p \in B_0}$.

Step 2: Bounding loop lengths in the base. Let $\sigma_P \in I(s, B_0)$ and let $\alpha_1, \ldots, \alpha_k$ a simple basis of $\pi_1(B_0, b_0)$. By [8, Theorem 5.1] there exists $b \in B_0$ and simple loops $\gamma_j \in \Gamma_S^{\alpha_j}$ such that $\alpha_j = c_{b_0,b}^{-1}\, \gamma_j\, c_{b_0,b}$ in $\pi_1(B_0, b_0)$ where $c_{b_0,b} \in \mathcal{C}_{b_0}$ and the following bound holds:

$$\ell_S(\gamma_j) \leq L(s+1) \quad \forall j = 1, \ldots, k \tag{18}$$

for $L \in \mathbb{R}_{>0}$ independent of s, S, b. Let $g \colon \mathbb{C} \to (\mathcal{A} \smallsetminus \mathcal{D})_{|B_0}$ be a holomorphic map, then the composition $f \circ g \colon \mathbb{C} \to B_0$ is constant, since B_0 is hyperbolic. Then the image of g is in $\mathcal{A}_b \smallsetminus \mathcal{D}_b$ for some $b \in B_0$, so by the previous step g is constant. So $(\mathcal{A} \smallsetminus \mathcal{D})|_{B_0}$ is Brody hyperbolic. Up to enlarging the discs slightly, so are the analytic closures of $(\mathcal{A} \smallsetminus \mathcal{D})|_{B_0}$ and $\mathcal{D}|_{B_0}$. At this point a theorem of Green (see [8, Theorem 7.5]) says that there exists $c \in \mathbb{R}_{>0}$ such that

$$d_{(\mathcal{A} \smallsetminus \mathcal{D})|_{B_0}} \geq c\, h \tag{19}$$

Now notice that

$$\sigma_P(\gamma_j) \subset B_0 \smallsetminus S \subseteq (\mathcal{A} \smallsetminus \mathcal{D})|_{B_0 \smallsetminus S}$$

So by Equations (18) to (19) and the decreasing property of the Kobayashi-Royden metric we deduce

$$\ell_h(\sigma_P(\gamma_j)) \leq c^{-1} L(s+1)\,. \tag{20}$$

Step 3: The Parshin cocycle. Since $f \colon \mathcal{A}_{B_0} \to B_0$ is a proper smooth submersion, Ehresmann's theorem gives a fiber bundle in the differentiable category. Let σ_O be the zero section and fix $w_0 := \sigma_O(b_0) \in \mathcal{A}_{b_0}$. Since B_0 is a $K(\pi, 1)$-space (i.e. $\pi_i(B_0) = 0$ for $i \geq 2$, which holds because its universal cover is the disc $\mathbb{D}$), the long exact sequence of homotopy groups for the fibration $\mathcal{A}_{b_0} \to \mathcal{A}_{B_0} \to B_0$ yields the short exact sequence

$$1 \longrightarrow \pi_1(\mathcal{A}_{b_0}, w_0) \longrightarrow \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{f_*} \pi_1(B_0, b_0) \longrightarrow 1\,. \tag{21}$$

As $\mathcal{A}_{b_0}$ is a complex torus of dimension n, we have $\pi_1(\mathcal{A}_{b_0}, w_0) = H_1(\mathcal{A}_{b_0}, \mathbb{Z}) =: \Gamma \cong \mathbb{Z}^{2n}$. The zero section σ_O splits (21) via $i_O(\alpha) := [\sigma_O(\gamma)]$ where $\alpha = [\gamma] \in G := \pi_1(B_0, b_0)$. Each rational point $P \in A(K)$ with section $\sigma_P \colon B_0 \to \mathcal{A}_{B_0}$ gives a second splitting

$$i_P(\alpha) := [\eta_{w_0,\sigma_P(b_0)}^{-1} \circ \sigma_P(\gamma) \circ \eta_{w_0,\sigma_P(b_0)}]\,,$$

where $\eta_{w_0,\sigma_P(b_0)}$ is a path in the fiber $\mathcal{A}_{b_0}$ from w_0 to $\sigma_P(b_0)$. Since both i_P and i_O are sections of f_*, for every $\alpha \in G$:

$$f_*\big(i_P(\alpha) \cdot i_O(\alpha)^{-1}\big) = \alpha \cdot \alpha^{-1} = 1\,,$$

so $i_P(\alpha) \cdot i_O(\alpha)^{-1} \in \ker(f_*) = \Gamma$. We define the *Parshin cocycle*

$$c_P \colon G \to \Gamma\,, \quad c_P(\alpha) := i_P(\alpha) \cdot i_O(\alpha)^{-1}\,.$$

One verifies that c_P is a 1-cocycle of G with coefficients in the G-module Γ, where the G-action on Γ is by conjugation

$$\begin{aligned} \varphi \colon G &\to \mathrm{Aut}(\Gamma) \\ \alpha &\mapsto \varphi_\alpha \colon \gamma \mapsto i_O(\alpha)\,\gamma\, i_O(\alpha)^{-1} \end{aligned}$$

Note that the action φ doesn't depend on the choice of the splitting i_O, in fact for another splitting i_P, we have $i_P(\alpha) = c_P(\alpha) \cdot i_O(\alpha)$ with $c_P(\alpha) \in \Gamma$, and

$$i_P(\alpha)\,\gamma\, i_P(\alpha)^{-1} = c_P(\alpha) \cdot \varphi_\alpha(\gamma) \cdot c_P(\alpha)^{-1} = \varphi_\alpha(\gamma)\,,$$

where the last equality uses the commutativity of Γ. The cohomology class $[c_P] \in H^1(G,\Gamma)$ is independent of the choice of path $\eta_{w_0,\sigma_P(b_0)}$ (a different choice changes c_P by a coboundary). The following result proved in [7, Proposition 1] and [8, Proposition 7.2] will be crucial:

Proposition 3.1 (Parshin-Phung)**.** *The homomorphism*

$$\begin{aligned} \Psi \colon A(K) &\to H^1(G,\Gamma) \\ P &\mapsto [c_P] \end{aligned}$$

factors through $A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ and has the following injectivity property: if $\Psi(P) = \Psi(Q)$, then $P - Q \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) + A(K)_{\mathrm{tors}}$. In particular, each element $[c] \in H^1(G,\Gamma)$ in the image of Ψ arises from at most $t_{\mathcal{A}} := \#\big(A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big)_{\mathrm{tors}} < \infty$ rational points modulo the trace (finite by the Lang–Néron theorem).

Remark 3.2. It would be interesting to understand the relationship between the Parshin cocycle and the "analytic cocycle" induced by the logarithm of the section σ_P, as defined in [2, Equation (13)].

To extract a quantitative bound from the cocycle c_P, we work in the semidirect product coordinates provided by i_O. The splitting i_O exhibits

$$\pi_1(\mathcal{A}_{B_0}, w_0) \;\cong\; \Gamma \rtimes_\varphi G\,, \tag{22}$$

In these coordinates, $i_O(\alpha_j) = (0,\alpha_j)$ and $i_P(\alpha_j) = (\beta_j,\alpha_j)$ for some $\beta_j \in \Gamma$, so the Parshin cocycle reads $c_P(\alpha_j) = \beta_j$. Since G is free, the k-tuple $(\beta_1,\ldots,\beta_k) \in \Gamma^k$ determines c_P completely. The displacement bound in Step 4 will give a norm bound on each β_j.

Step 4: Bounding loop lengths in $\mathcal{A}$. Let us now go back to the special loops (based in b) $\gamma_j \in \Gamma_S^{\alpha_j}$ for $j = 1,\ldots,k$ considered in Step 2. Consider the following loop based at w_0:

$$\sigma_P(\gamma_j)^\# := (v_b \circ \sigma_O(c_{b_0,b}))^{-1} \circ \sigma_P(\gamma_j) \circ (v_b \circ \sigma_O(c_{b_0,b}))$$

where $v_b \subset \mathcal{A}_b$ is a h-geodesic from $\sigma_O(b)$ to $\sigma_P(b)$. Since $\sigma_P(\gamma_j) = \sigma_P(c_{b_0,b}) \circ \sigma_P(\alpha_j) \circ \sigma_P(c_{b_0,b})^{-1}$, $\sigma_P(\gamma_j)^\#$ is free homotopic to $\sigma_P(\alpha_j)$ and each of its components satisfies the following uniform bounds:

- the loop $\sigma_P(\gamma_j)$ satisfies:

  $$\ell_h(\sigma_P(\gamma_j)) \le c^{-1}L(s+1)$$

  by Equation (20);
- the fiber path $v_b \subset \mathcal{A}_b$ satisfies:

  $$\ell_h(v_b) \le \delta_0 := \mathrm{diam}_h(\overline{\mathcal{A}}^{\mathrm{an}}_{B_0}) < +\infty\;;$$
- since B is compact $\|d\sigma_O\|_\infty := \sup_{b\in B}\|d_b\sigma_O\| < +\infty$, so

  $$\ell_h(\sigma_O(c_{b_0,b})) \le \|d\sigma_O\|_\infty\, \ell(c_{b_0,b}) \le \|d\sigma_O\|_\infty(\mathrm{diam}(B) + tM) =: \delta_0'$$

In other words for any element of the simple basis α_j we can construct a loop $\left[\sigma_P(\gamma_j)^{\#}\right] \in \pi_1(\mathcal{A}_{B_0}, w_0)$ that is free homotopic to $\sigma_P(\alpha_j)$ and satisfies

$$\ell_h\left(\sigma_P(\gamma_j)^{\#}\right) \leq H(s) := c^{-1}L(s+1) + 2(\delta_0 + \delta_0')$$

where the constants L, δ_0, δ_0' don't depend on S and P.

The loop $\sigma_P(\gamma_j)^{\#}$ is based at w_0, so it defines an element $[\sigma_P(\gamma_j)^{\#}] \in \pi_1(\mathcal{A}_{B_0}, w_0)$. This element need not equal $i_P(\alpha_j) = (\beta_j, \alpha_j)$, because $\sigma_P(\gamma_j)^{\#}$ and the loop defining $i_P(\alpha_j)$ use different conjugating paths from w_0 to the fibre above b: the former uses $v_b \circ \sigma_O(c_{b_0,b})$, while the latter uses $\sigma_P(c_{b_0,b}) \circ \eta_{w_0,\sigma_P(b_0)}$. Both are paths from w_0 to $\sigma_P(b)$, so their concatenation

$$\omega \;:=\; \left(v_b \circ \sigma_O(c_{b_0,b})\right)^{-1} \circ\; \sigma_P(c_{b_0,b}) \circ\; \eta_{w_0,\sigma_P(b_0)}$$

is a loop at w_0, and a direct verification gives

$$[\sigma_P(\gamma_j)^{\#}] \;=\; [\omega]^{-1} \cdot i_P(\alpha_j) \cdot [\omega] \qquad \text{in } \pi_1(\mathcal{A}_{B_0}, w_0)\,.$$

In other words, $[\sigma_P(\gamma_j)^{\#}]$ and $i_P(\alpha_j)$ are *conjugate* in $\pi_1(\mathcal{A}_{B_0}, w_0)$. The map $i'_P\colon G \to \pi_1(\mathcal{A}_{B_0}, w_0)$ defined by $i'_P(\alpha_j) := [\sigma_P(\gamma_j)^{\#}]$ is therefore a section of (21) in the same conjugacy class as i_P, so the associated cocycles are cohomologous: $[c'] = [c_P] \in H^1(G, \Gamma)$. By Proposition 3.1, we may freely replace i_P by i'_P for counting purposes. We do so and, to lighten notation, simply write β_j for the components of the new cocycle.

Step 5: Displacement bound via the universal cover. We now explain how the h-length bound on $\sigma_P(\gamma_j)^{\#}$ constrains the lattice element $\beta_j \in \Gamma$. Let $\pi\colon \widetilde{\mathcal{A}}_0 \to \mathcal{A}_{B_0}$ be the universal cover. Since $\mathcal{A}_{B_0} \to B_0$ is a fibre bundle (by Ehresmann's theorem) with fibre $\mathcal{A}_{b_0}$ (a complex torus of dimension n, so $\widetilde{\mathcal{A}_{b_0}} \cong \mathbb{R}^{2n}$) and base B_0 (a hyperbolic Riemann surface, so $\widetilde{B_0} \cong \Delta$, the unit disk), pulling back to Δ trivialises the bundle and gives $\widetilde{\mathcal{A}}_0 \cong \mathbb{R}^{2n} \times \Delta$.

Fix a point $\tilde{w} = (\tilde{x}_0, \tilde{y}_0) \in \pi^{-1}(w_0)$. Pull back the hermitian metric h to a Riemannian metric $\tilde{h}$ on $\widetilde{\mathcal{A}}_0$. The group of deck transformations of π is $\pi_1(\mathcal{A}_{B_0}, w_0) \cong \Gamma \rtimes_\varphi G$. An element $(\beta, \alpha) \in \Gamma \rtimes_\varphi G$ acts on $\widetilde{\mathcal{A}}_0 \cong \mathbb{R}^{2n} \times \Delta$ by:

$$(\beta, \alpha) \cdot (\tilde{x}_0, \tilde{y}_0) \;=\; \left(\varphi_\alpha(\beta) \cdot \tilde{x}_0,\;\; \alpha \cdot \tilde{y}_0\right). \tag{23}$$

Here $\alpha \cdot \tilde{y}_0$ is the deck transformation of $v\colon \Delta \to B_0$ by $\alpha \in G = \pi_1(B_0, b_0)$ (acting on the base), and $\varphi_\alpha(\beta) \cdot \tilde{x}_0$ is the deck transformation of $u\colon \mathbb{R}^{2n} \to \mathcal{A}_{b_0}$ by the element $\varphi_\alpha(\beta) \in \Gamma$ (acting on the fibre). The monodromy φ_α appears because the fibre component "twists" as one transports along the base loop α: the semidirect product structure of $\Gamma \rtimes_\varphi G$ encodes precisely this twisting.

The loop $\sigma_P(\gamma_j)^{\#}$ is based at w_0 and has homotopy class (β_j, α_j). Its lift to $\widetilde{\mathcal{A}}_0$ from $\tilde{w}$ is a path from $\tilde{w}$ to $(\beta_j, \alpha_j) \cdot \tilde{w}$. Since π is a local isometry for the metrics $\tilde{h}$ and h, the $\tilde{h}$-length of the lift equals $\ell_h(\sigma_P(\gamma_j)^{\#}) \leq H(s)$, so:

$$d_{\tilde{h}}\big(\tilde{w},\; (\beta_j, \alpha_j) \cdot \tilde{w}\big) \;\leq\; H(s) \;:=\; c^{-1}L(s+1) + 2(\delta_0 + \delta_0') \;=\; O(s)\,, \tag{24}$$

where L, δ_0, δ_0' are independent of S and P. By (23), the point $(\beta_j, \alpha_j) \cdot \tilde{w}$ has fibre component $\varphi_{\alpha_j}(\beta_j) \cdot \tilde{x}_0$ in the slice $\mathbb{R}^{2n} \times \{\alpha_j \cdot \tilde{y}_0\}$. Let $d_j := \tilde{h}\big|_{\mathbb{R}^{2n} \times \{\alpha_j \cdot \tilde{y}_0\}}$ be the induced Riemannian metric on this slice. Since the geodesic distance within a slice is at most the ambient distance:

$$d_j\big(\varphi_{\alpha_j}(\beta_j) \cdot \tilde{x}_0,\; \tilde{x}_0\big) \;\leq\; d_{\tilde{h}}\big(\tilde{w},\; (\beta_j, \alpha_j) \cdot \tilde{w}\big) \;\leq\; H(s)\,. \tag{25}$$

Remark 3.3. The intermediate bound (24) controls the displacement in the *full* universal cover $\widetilde{\mathcal{A}}_0 \cong \mathbb{R}^{2n} \times \Delta$, involving both fibre and base directions (and the metric $\tilde{h}$ is not a product metric due to the monodromy). The projection step (25) extracts a purely fibre-theoretic bound, which is what we need for the lattice count in the next step.

Step 6: Counting. For each $j = 1, \ldots, k$, we count the elements $\beta \in \Gamma$ compatible with (25):

$$N_j(s) \;:=\; \#\Big\{\beta \in \Gamma :\; d_j\big(\varphi_{\alpha_j}(\beta) \cdot \tilde{x}_0,\; \tilde{x}_0\big) \leq H(s)\Big\}\,. \tag{26}$$

Via the universal covering $u\colon \mathbb{R}^{2n} \to \mathcal{A}_{b_0}$, the torus A_{b_0} becomes a compact geodesic Riemannian manifold with the metric d_j, and $\Gamma \cong \mathbb{Z}^{2n}$ acts on $(\mathbb{R}^{2n}, d_j)$ by deck transformations. Since $\varphi_{\alpha_j} \in \mathrm{Aut}(\Gamma)$ is a bijection, the set in (26) has the same cardinality as $\{\gamma \in \Gamma : d_j(\gamma \cdot \tilde{x}_0, \tilde{x}_0) \leq H(s)\}$, i.e., the number of Γ-translates of $\tilde{x}_0$ in a d_j-ball of radius $H(s)$.

By the fundamental lemma of the geometry of groups (cf. [8, Proposition A.8 and Lemma A.9]), since $\Gamma \cong \mathbb{Z}^{2n}$ is a lattice of rank $2n$ acting cocompactly on $(\mathbb{R}^{2n}, d_j)$, there exists $m_j > 0$ (depending on $\mathcal{A}$, h, α_j, but not on s) such that

$$N_j(s) \;\leq\; m_j\,(H(s)+1)^{2n}\,. \tag{27}$$

The exponent $2n$ reflects the rank of $\Gamma \cong \mathbb{Z}^{2n}$: in a ball of radius R in the universal cover of the $2n$-dimensional real torus $\mathcal{A}_{b_0}$, the number of Γ-translates of $\tilde{x}_0$ grows as R^{2n}.

Since G is free on $\alpha_1, \dots, \alpha_k$, the cohomology class $[c_P] \in H^1(G,\Gamma)$ is completely determined by the k-tuple $(\beta_1, \dots, \beta_k) \in \Gamma^k$. Bounding each component independently via (27) with $R = H(s)$:

$$\#\{\text{possible cohomology classes } [c_P]\} \;\leq\; \prod_{j=1}^{k} m_j\,(H(s)+1)^{2n} \;=\; m_0\,(H(s)+1)^{2nk}\,, \tag{28}$$

where $m_0 := \prod_{j=1}^{k} m_j$.

By Proposition 3.1, each class $[c_P] \in H^1(G,\Gamma)$ arises from at most $t_{\mathcal{A}} := \#\big(A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big)_{\mathrm{tors}} < \infty$ rational points modulo the trace (finite by the Lang–Néron theorem). Therefore:

$$\#\left(I(s,B_0)\big/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) \;\leq\; t_{\mathcal{A}}\cdot m_0\,(H(s)+1)^{2nk} \;=\; m\,(s+1)^{2nk}\,,$$

where $m := t_{\mathcal{A}} \cdot m_0 \cdot (c^{-1}L + 2(\delta_0+\delta_0')+1)^{2nk}$ depends only on B_0, $\mathcal{A}$, D, and h. □

3.2 Improved bound

We now prove a sharper version of Theorem 1.3. The general strategy is unchanged: we follow the same six-step scheme (reduction to the hyperbolic locus, loop construction, Parshin cocycle, lifting to $\mathcal{A}$, displacement in the universal cover, lattice counting). The only modification occurs in Steps 2 and 4, where we replace Phung's linear bound $\ell_S(\gamma_j) = O(s)$ with the sublinear bound $O(\sqrt{(s+1)\log(s+2)})$ provided by Proposition 2.7. This propagates through the counting argument and halves the exponent.

Proof of Theorem 1.4. The proof follows the same six-step scheme as the proof of Theorem 1.3. The only modification is in Step 2: we replace the appeal to [8, Theorem 5.1], which provides loops of Kobayashi length $O(s)$, with Proposition 2.7, which provides a point $b \in B_0 \setminus S$, a path σ from b_0 to b in B_0 with $\ell_\rho(\sigma) \leq \delta'$, and loops $\gamma_1, \dots, \gamma_k \subset B_0 \setminus S$ based at b with $[\sigma^{-1} \circ \gamma_j \circ \sigma] = \alpha_j$ in $\pi_1(B_0, b_0)$ and

$$\ell_S(\gamma_j) \;\leq\; C\sqrt{(s+1)\log(s+2)} \qquad \text{for all } j = 1, \dots, k\,.$$

Since the loops share a common base point b and the conjugation uses the single path σ, Steps 3–5 apply verbatim with the conjugating path $v_b \circ \sigma_O(\sigma)$ in $\mathcal{A}_{B_0}$ (in place of $v_b \circ \sigma_O(c_{b_0,b})$ in Theorem 1.3), whose h-length is bounded by $\delta_0 + \|d\sigma_O\|_\infty \delta'$, independently of s. The displacement bound becomes

$$d_j\big(\varphi_{\alpha_j}(\beta_j)\cdot\tilde{x}_0,\ \tilde{x}_0\big) \;\leq\; H(s) \;:=\; c^{-1}C\sqrt{(s+1)\log(s+2)} + 2(\delta_0 + \|d\sigma_O\|_\infty\delta')\,.$$

The lattice counting in Step 6 then gives

$$\#\left(I(s,B_0)\big/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) \;\leq\; m_0\,(H(s)+1)^{2nk}\,.$$

It remains to extract the exponent. For s large enough:

$$H(s)+1 \;\leq\; c^{-1}(C+1)\sqrt{(s+1)\log(s+2)}\,,$$

hence $(H(s)+1)^{2n} \leq \big(c^{-1}(C+1)\big)^{2n}\big((s+1)\log(s+2)\big)^n$. Fix $\varepsilon' > 0$. Since $\log(s+2) \leq (s+1)^{\varepsilon'/n}$ for s sufficiently large, there exists $s_0 = s_0(\varepsilon') \geq 0$ such that for all $s \geq s_0$:

$$\big((s+1)\log(s+2)\big)^n \;\leq\; (s+1)^{n+\varepsilon'}\,.$$

Over k basis elements, $(H(s)+1)^{2nk} \leq \mathrm{const}\cdot(s+1)^{nk+k\varepsilon'}$ for $s \geq s_0$. Setting $\varepsilon := k\varepsilon'$ and absorbing all constants into m, we obtain

$$\#\left(I(s,B_0)\big/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) \;\leq\; m\,(s+1)^{nk+\varepsilon} \qquad \text{for all } s \geq 0\,,$$

where $m = m(B_0, \mathcal{A}, \mathcal{D}, h, \varepsilon) > 0$ is independent of s. □

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