Finite translation orbits on double families of abelian varieties

Paolo Dolce Francesco Tropeano

March 27, 2024

Abstract

Consider two families of g-dimensional abelian varieties induced by two distinct rational maps on the same variety $\overline{\mathcal{A}}$ onto two bases $\overline{S}_1$ and $\overline{S}_2$ and having big common domain of definition. Two non-torsion sections of these families induce two (birational) fiberwise translations on $\overline{\mathcal{A}}$, respectively. We show that if $\dim \overline{S}_1 + \dim \overline{S}_2 \leq 2g$, the points with finite orbit under the action of a certain subset of the group generated by both translations lie in a proper Zariski closed subset that can be described to a certain extent. Our work is a higher dimensional generalization of a result of Corvaja, Tsimermann and Zannier.

0 Introduction

In the theory of unlikely intersections many interesting problems emerge when on an algebraic family of abelian varieties we consider the intersection between (the image of) a non-torsion section with the union of the N-torsion subschemes for $N \in \mathbb{N}$. Thanks to the work of André, Corvaja, Masser and Zannier in [8, 2, 10, 30], which exploits the properties of Betti map, we have now acquired a powerful systematic approach for the study of such particular unlikely intersection problems; even though the first indirect appearance of the Betti map in Diophantine problems dates back to Manin in [18]. Several important results have been proved on the back of such ideas. Of exceptional interest are the recent developments in [15, 7, 16, 12] on the geometric Bogomolov conjecture, the relative Manin-Mumford conjecture, and the uniform Mordell-Lang conjecture.

In the case of elliptic surfaces, Corvaja, Tsimerman and Zannier in [9] consider a related dynamical point of view: they take two dynamical systems defined by the fiberwise translations induced by two distinct sections, and they are able to find a proper Zariski closed subset which contains all the points which have finite orbit under the action of the group generated by both translations. Moreover, still in [9], they study some interesting applications. We mention that for a different perspective and related results on finite orbits of automorphisms of projective surfaces, the reader can check the article [6] of Cantat and Dujardin.

In the present paper we propose a generalization of [9] for families of g dimensional abelian varietes over a base of dimension at most g. In order to get the proof of our main result we also prove new auxiliary propositions which seem to be of independent interest and herald of potential applications. Let's now give a detailed account of our work.

General notations. We assume that *all* algebraic varieties and morphisms are defined over $\overline{\mathbb{Q}}$. An algebraic point p of a variety X will be simply denoted as $p \in X$ (or sometimes more explicitly as $p \in X(\overline{\mathbb{Q}})$) and in general we don't make use of schematic points. Moreover, we denote with $X(\mathbb{C})$ the analytification of X, which clearly carries the topology of complex manifold. With $\dim X$ we denote the dimension of X as complex manifold. In several proofs we need to deal with many real positive constants that in general are allocated inside some variables (denoted usually with $C, c_0, c_1, \ldots$). Our main convention is that these variables are "local in the paper", in the sense that their value/meaning holds only in the proof in which they are used, if not otherwise specified. In this paper we make use of concepts coming from transcendental Diophantine problems like: of o-minimal structures, definable sets and definable families, so for the main definitions and properties we remand the reader to the seminal works [25] and [24].

Let $\overline{S}$ be a smooth, irreducible projective variety and let $f : \overline{\mathcal{A}} \to \overline{S}$ be a *family of abelian varieties* with a section i.e. a proper flat morphism of finite type such that $\overline{\mathcal{A}}$ is a smooth variety and the generic fiber is an abelian variety of dimension g over $\overline{\mathbb{Q}}(\overline{S})$ with a rational point. After removing the singular fibers and their images we obtain a g-dimensional abelian scheme $f : \mathcal{A} \to S$ (the fiberwise group law extends uniquely to a global map that gives the structure of abelian scheme over S, see [22, Theorem 6.14]). The set of N-torsion points of $\mathcal{A}$ is denoted by $\mathcal{A}[N]$, and moreover we put $\mathcal{A}_{\text{tor}} = \bigcup_{N \geq 1} \mathcal{A}[N]$. We assume the existence of a non-torsion section $\sigma \colon S \to \mathcal{A}$ of f (i.e. the image of σ is not contained in any $\mathcal{A}[N]$) and that $\mathbb{Z}\sigma$ is Zariski dense in $\mathcal{A}$. We define the following automorphism:

$$\begin{aligned} t_\sigma : \mathcal{A}(\mathbb{C}) &\to \mathcal{A}(\mathbb{C}) \\ p &\mapsto p + \sigma(f(p)). \end{aligned}$$

Let Γ_σ be the group generated by t_σ that acts naturally on $\mathcal{A}(\mathbb{C})$, for any $p \in \mathcal{A}(\mathbb{C})$ we are interested in the orbit

$$\Gamma_\sigma(p) := \{t_\sigma^r(p) \colon r \in \mathbb{N}\}.$$

Clearly each orbit is contained in a single fiber of f, but it is important to study whether the locus $\mathfrak{F}^{(1)}$ of points $p \in \mathcal{A}(\mathbb{C})$ such that $\Gamma_\sigma(p)$ is finite can be confined in a subset lying over a proper closed subset of the base. We recall that a torsion value of σ is an element of $\sigma^{-1}(\mathcal{A}_{\text{tor}})$ and obviously $\Gamma_\sigma(p)$ is finite if and only if $f(p)$ is a torsion value. Therefore, such study of $\mathfrak{F}^{(1)}$ can be reduced to the study of the Zariski density of the torsion values of σ. But the last property depends on the values of $\dim S$ and g in the following way: if $\dim S \geq g$ then $\sigma^{-1}(\mathcal{A}_{\text{tor}})$ is Zariski dense in S if and only if the rank of the Betti map β_σ is $2g$ (see [16, Theorem 1.3]). Note that [2, Proposition 2.1.1] shows that $\operatorname{rk}_{\mathbb{R}} \beta_\sigma \geq 2g$ implies that $\sigma^{-1}(\mathcal{A}_{\text{tor}})$ is dense in $S(\mathbb{C})$ with respect to the analytic topology. On the other hand if $\dim S < g$ then $\sigma^{-1}(\mathcal{A}_{\text{tor}})$ is not Zariski dense in S. This is a special case of the relative Manin-Mumford conjecture that has been recently proved in [16, Theorem 1.1].

We are interested in a variation of the above problem in the case of a variety $\overline{\mathcal{A}}$ endowed with a *double abelian rational fibration*: there exists two surjective rational maps $f_1 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_1$ and $f_2 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_2$ such that the induced morphisms are families of abelian varieties with zero sections; in particular, for each of them the generic fiber is an abelian variety over $k_{\overline{S}_1} := \overline{\mathbb{Q}}(\overline{S}_1)$ and $k_{\overline{S}_2} := \overline{\mathbb{Q}}(\overline{S}_2)$ respectively. We always denote by f_i the restrictions of the introduced rational maps to families of abelian varieties and abelian schemes: precisely, after removing the loci where f_1 and f_2 are not defined, we get two families of abelian varieties $f_i : \overline{\mathcal{A}}_i \to \overline{S}_i$; moreover, after removing the respective singular fibers and discriminant loci we obtain two abelian schemes $f_i : \mathcal{A}_i \to S_i$. We assume the existence of non-torsion sections $\sigma_i : S_i \to \mathcal{A}_i$ of f_i. In addition we impose the following rather standard conditions on these abelian schemes:

1) The two abelian families are "distinct", in the sense that their common fibers (if any) lie over a proper Zariski closed subset E either of $\overline{S}_1$ or of $\overline{S}_2$.

2) The intersection $\mathcal{A}_1 \cap \mathcal{A}_2$ is a $2g$-dimensional variety.

3) $\mathbb{Z}\sigma_i$ is Zariski dense in $\mathcal{A}_i$.

4) The abelian schemes $\mathcal{A}_i \to S_i$ have no fixed part, i.e. the respective generic fibers have trivial $\overline{k_{\overline{S}_i}}/\overline{\mathbb{Q}}$-trace.

The fiber of a point $s \in S_i(\mathbb{C})$ with respect to the morphism f_i will be denoted by $\mathcal{A}_{i,s}$ and the discriminant locus of f_i is $\Delta_i = \overline{S}_i \setminus S_i$. We denote with β_i the Betti map associated to the section σ_i.

There exists unique birational transformations t_i of $\overline{\mathcal{A}}(\mathbb{C})$ acting by translation along the general fiber of f_i and mapping the zero section to σ_i:

$$\begin{array}{rcl} t_i : \overline{\mathcal{A}}(\mathbb{C}) & \dashrightarrow & \overline{\mathcal{A}}(\mathbb{C}) \\ p & \mapsto & p + \sigma_i(f_i(p)). \end{array}$$

We study the action of the subgroup $\Gamma_{\sigma_1,\sigma_2} := \langle t_1, t_2 \rangle$ generated by t_1 and t_2 in the group of birational transformations $\mathrm{Bir}(\overline{\mathcal{A}}(\mathbb{C}))$; in particular we want to confine the points with finite orbits. First of all, since t_1 and t_2 are not defined everywhere on $\overline{\mathcal{A}}(\mathbb{C})$ we have to be careful with the notion of orbit. For $p \in \overline{\mathcal{A}}(\mathbb{C})$ we put:

$$\Gamma_{\sigma_1,\sigma_2}(p) := \{\gamma(p) \colon \gamma \in \Gamma_{\sigma_1,\sigma_2} \text{ and } \gamma(p) \text{ is well defined at } p\}\,.$$

In fact, we shall focus on a subset of the orbit showing that already the points with finite orbits under the action of a "small subset" of $\Gamma_{\sigma_1,\sigma_2}$ lie in a proper Zariski closed subset of $\overline{\mathcal{A}}(\mathbb{C})$. This small subset of $\Gamma_{\sigma_1,\sigma_2}$ will be precisely the following:

$$O = O_{\sigma_1,\sigma_2} := \{t_1^{r_1} \circ t_2^{r_2} \colon r_1, r_2 \in \mathbb{N}\}\,.$$

For any $p \in \overline{\mathcal{A}}(\mathbb{C})$ we clearly have $O(p) \subseteq \Gamma_{\sigma_1,\sigma_2}(p)$ and moreover we define

$$\mathfrak{F} = \mathfrak{F}^{(2)} := \{p \in \overline{\mathcal{A}}(\mathbb{C}) : O(p) \text{ is finite}\}.$$

We adopt the convention that points where the rational map f_2 is undefined are not points with finite orbit. Therefore, we forget about them in counting points of $\mathfrak{F}$.

Remark 0.1. Note that if $p \in \mathfrak{F}$ then both $f_1(p)$ and $f_2(p)$ are torsion values for the relative sections, and in particular $p \in \overline{\mathcal{A}}(\overline{\mathbb{Q}})$. In other words $\mathfrak{F}$ is contained in the intersection between the f_1-fibers and the f_2-fibers of the torsion values, which form a dense subset of $\overline{\mathcal{A}}$.

The case $g = 1$ has been already treated in [9, Theorem 1.1] where it is shown that $\mathfrak{F}$ lies over finitely many fibers of f_2. The following theorem is the main result of this paper and generalizes [9, Theorem 1.1]:

Theorem 0.2. *Let $f_1 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_1$ and $f_2 : \overline{\mathcal{A}} \dashrightarrow \overline{S}_2$ be a double abelian rational fibration of the variety $\overline{\mathcal{A}}$ satisfying the above conditions $1) - 4)$. If $\dim \overline{S}_1 + \dim \overline{S}_2 \leq 2g$, then there exist two proper Zariski closed subsets $Z_1 \subset \overline{S}_1(\mathbb{C})$ and $Z_2 \subset \overline{S}_2(\mathbb{C})$ such that*

$$\mathfrak{F} \subseteq f_1^{-1}(Z_1) \cup f_2^{-1}(Z_2)\,. \tag{1}$$

Our result can be seen as a generalization of the relative Manin-Mumford claim for sections in the following way: in the case of a single family of abelian varieties [16, Theorem 1.1] says that the relative locus $\mathfrak{F}^{(1)}$ is not Zariski dense for $\dim S \leq g-1$. On the other hand, in the case of two families of abelian varieties with same base S, Theorem 0.2 implies that $\mathfrak{F}^{(2)}$ is not Zariski dense for $\dim S \leq g$.

Remark 0.3. If any of the sets $\sigma_i^{-1}(\mathcal{A}_{i,\mathrm{tor}})$ is not Zariski dense then the theorem is obviously true thanks to Remark 0.1. Therefore if either $\dim S_1 < g$ or $\dim S_2 < g$ then Theorem 0.2 follows directly from [16, Theorem 1.1]. For the same reason, thanks to [16, Theorem 1.3] we can restrict ourselves to prove just the case:

$$2\dim S_1 = 2\dim S_2 = 2g = \mathrm{rk}_{\mathbb{R}}\,\beta_1 = \mathrm{rk}_{\mathbb{R}}\,\beta_2\,. \tag{2}$$

Observe that Equation (2) is crucial for the application of the so called "height inequality" of [12, Theorem 1.6] that relates the projective height of the base to the fiberwise Neron-Tate height. In our proof this result appears several times, and on different abelian schemes, to ensure that the height of "most of" the torsion values can be uniformly bounded. On the other hand, it is well known that the height inequality fails in general without assumptions on the rank of the Betti map. See also [29, Theorem 5.3.5] for a generalization of height inequality which nevertheless requires the same hypotheses in the case of abelian schemes.

Remark 0.4. At first glance it might seem that in the case $1 = \dim S_1 = \dim S_2 = g$, Theorem 0.2 is slightly weaker than [9, Theorem 1.1] where the claim is just $\mathfrak{F} \subseteq f_2^{-1}(Z)$ for a proper closed subset Z. However, Proposition 2.7 shows that the two statements are actually equivalent.

Remark 0.5. Let Z be a Zariski closed subset of $\overline{\mathcal{A}}$ which is not horizontal with respect to either f_1 or f_2 (i.e. either $f_1(Z) \neq S_1$ or $f_2(Z) \neq S_2$). If Theorem 0.2 holds for $\mathfrak{F} \cap (\overline{\mathcal{A}} \setminus Z)$, then it also holds for $\mathfrak{F}$. This follows from the fact that the morphisms f_1 and f_2 are proper: if $f_i(Z) \neq S_i$ for any $i = 1, 2$ then the points with finite orbit inside Z lie in $f_i^{-1}(f_i(Z))$ and $f_i(Z)$ is closed.

Our proof follows the same general strategy employed in the low dimensional situation of [9]: after some preliminary considerations we are eventually reduced to show that the points of the type $\sigma_2(b)$ for $b \in f_2(\mathfrak{F})$ have uniformly bounded torsion order. Denote this order with $m := m(b)$, then by using the properties of the Betti map we are able to see a collection of conjugates of certain torsion values as rational points inside a definable family of $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ in the sense of [25]. Some considerations that relate the Weil heights, the torsion orders and the conjugates of algebraic points allow us to give a lower bound on the number of such rational points and an upper bound on their height. The crucial point is that these bounds depend on m. On the other hand, the result [25, Theorem 1.9] of Pila and Wilkie gives an upper bound on the number of rational points with bounded height of the transcendental part of such definable family. But, after using the independence result [1, Theorem 3] of André we prove that the definable family has actually empty algebraic part. It means that we can compare the aforementioned bounds on the number of rational points and deduce a uniform upper bound for m.

Albeit, our higher dimensional setting unravels several subtle complications as opposed to [9]. Below we summarize the new technical ingredients introduced in this paper:

(i) The height inequality of Dimitrov, Gao and Habegger gives a uniform height bound only for the torsion values contained in an open dense subset (see Corollary 1.4). Note that when the base is a curve there is no problem because having a uniform bound on a Zariski open dense subset is clearly equivalent to a uniform bound for all torsion values. Therefore, in each step of our proof we have to be very careful in keeping track of the closed subset excluded by the height inequality. In addition, we need to apply the height inequality to an abelian scheme having a f_2-fiber as base, thus the open dense subset with uniformly bounded height is not closed with respect to the sum (of the base). We fix this issue by considering some ad hoc arguments involving the properties of Néron-Tate height.

(ii) We need an upper bound on the torsion order of (the image of) torsion values that depends only on the heights and the degree of the points. Thus we prove the following:

Proposition 0.6 (See Proposition 1.7)**.** *Let $f: \mathcal{A} \to S$ be a g-dimensional abelian scheme (induced by a morphism of varieties) admitting a non-torsion section $\sigma: S \to \mathcal{A}$. Let K be the field of definition of S, let s be a torsion value for σ and put $d(s) := [K(s) : \mathbb{Q}]$. Then there exists real constant $C = C(g)$ (so independent from the point s) such that*

$$\operatorname{ord}(\sigma(s)) \leq \left((14g)^{64g^2} d(s) \max\left(1,\, h(s) + C,\, \log d(s)\right)^2\right)^{\frac{35840 g^3}{16}}.$$

The proof is a combination of a similar result for abelian varieties due to Rémond in [27][1] with some modular properties of the Faltings height.

(iii) We prove the following result which is essential in several steps of the proof of Theorem 0.2:

Proposition 0.7 (See Proposition 1.10)**.** *Let's fix the following data: X is a projective variety; B is a closed subvariety of X; K is a number field containing the fields of definition of X and B. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{1}{2}[K(\alpha) : K]$ different K-embeddings $\tau\colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C_δ.*

Roughly speaking it says that fixed a uniform constant C and a subvariety B, there is a lower bound on the number of Galois conjugates, that don't lie "near" B, of a point $\alpha \notin B$ having height at most C; where the important fact is that such bound should depend only on the degree of α. It is a generalization of a well known result for $\mathbb{P}^1$, appeared in several articles of Masser and Zannier (precisely cited in the main text) and which turned out to be very useful for proving some results of Zilber-Pink type. This tool seems to be very interesting since it allows to move torsion points in a "comfort zone" of the variety, where many arguments can be carried on with enough uniformity.

(iv) In the proof of Theorem 0.2 we need to remove a Zariski closed subset from each f_2-fiber, but it must be shown that it is possible to do it "with no harm". In other words we need to rule out the occurrence that "too many" points of the type $t_2^r(p)$ lie in this closed subset. In [9] this can be done rather easily since the proper closed sets of the fibers are made of finitely many points, so it is possible

[1] We mention that Masser and Zannier obtained also a similar, but less sharp, bound in [21].

to encircle each of them with an arbitrarily small open disk. On the other hand, in the general case the intersection is higher dimensional, hence we need more sophisticated techniques (see Section 2.1.2).

Finally, we point out that the present work motivates the following very natural question that might be addressed with similar techniques:

Question 0.8. What is a generalization of Theorem 0.2 in the case of $n > 2$ abelian rational fibrations $f_i : \overline{\mathcal{A}} \dashrightarrow \overline{S}_i$, for $i = 1, \dots, n$? In particular, what is the best relationship between the dimension of the bases and g in this case?

The outline of the paper is the following: in Section 1 we collect the preliminary results, whereas the full proof of Theorem 0.2 is carried out in Section 2. The same section contains also a description of the Zariski closed subsets Z_1 and Z_2 that confine the fibers containing the points with finite orbit.

Acknowledgements Both authors express their gratitude to *G. Dill*, *D. Masser* and *R. Pengo* for their useful replies to some questions they were asked during the drafting of the present paper.

1 Auxiliary results

In this section we present all the tools needed for the proof of Theorem 0.2. We describe the results in the most general setting, so for each topic we need to reshape the notations.

1.1 Betti map

Let S be a smooth, irreducible quasi-projective variety and let $f : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$ with "a zero section" σ_0. Moreover we assume that $\sigma : S \to \mathcal{A}$ is a non-torsion section. Each fiber $\mathcal{A}_s(\mathbb{C})$ is analytically isomorphic to a complex torus $\mathbb{C}^g/\Lambda_s$ and for any subset $T \subseteq S(\mathbb{C})$ we denote $\Lambda_T := \bigsqcup_{s\in T} \Lambda_s$. The space $\mathrm{Lie}(\mathcal{A}) := \bigsqcup_{s\in S(\mathbb{C})} \mathrm{Lie}(\mathcal{A}_s)$ has a natural structure of g-dimensional holomorphic vector bundle $\pi\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ (it is actually a complex Lie algebra bundle). By using the fiberwise exponential maps one can define a global map $\exp\colon \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}$. Let $\Sigma_0 \subset \mathcal{A}$ be the image of the zero section of the abelian scheme, then obviously $\exp^{-1}(\Sigma_0) = \Lambda_{S(\mathbb{C})}$. Clearly $S(\mathbb{C})$ can be covered by finitely many open simply connected subsets where the holomorphic vector bundle $\pi\colon \mathrm{Lie}(\mathcal{A}) \to S(\mathbb{C})$ trivializes. Let $U \subseteq S(\mathbb{C})$ be any of such subsets and consider the induced holomorphic map $\pi\colon \Lambda_U \to U$; it is actually a fiber bundle with structure group $\mathrm{GL}(n,\mathbb{Z})$. Since U is simply connected, by [11, Lemma 4.7] we conclude that $\pi\colon \Lambda_U \to U$ is a topologically trivial fiber bundle. Thus we can find $2g$ continuous sections of π:

$$\omega_i : U \to \Lambda_U\,, \quad i = 1, \dots 2g \tag{3}$$

such that $\{\omega_1(s), \dots, \omega_{2g}(s)\}$ is a set of periods for Λ_s for any $s \in U$. Since $\Lambda_U \subseteq \mathrm{Lie}(\mathcal{A})_{|U}$, we can put periods into the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Lie}(\mathcal{A})_{|U} \arrow[d, "\exp_{|U}"] \\ S(\mathbb{C}) \supset U \arrow[ur, "\omega_i"] \arrow[r, "\sigma_0|U"'] & \mathcal{A}_{|U}, \end{tikzcd}$$

where σ_0 is the zero section. Since σ_0 is holomorphic and exp is a local biolomorphism, then the period functions defined in Equation (3) are holomorphic. The map $\mathcal{P} = (\omega_1, \dots, \omega_{2g})$ is called a *period map*; roughly speaking it selects a $\mathbb{Z}$-basis for Λ_s which varies holomorphically for $s \in U$. The set $U \subseteq S(\mathbb{C})$ is simply connected therefore we can choose a holomorphic lifting $\ell_\sigma : U \to \mathrm{Lie}(\mathcal{A})$ of the restriction $\sigma_{|U}$; ℓ_σ is often called an *abelian logarithm*. Thus for any $s \in U$ we can write uniquely

$$\ell_\sigma(s) = \beta_1(s)\omega_1(s) + \ldots + \beta_{2g}\omega_{2g}(s) \tag{4}$$

where $\beta_i : U \to \mathbb{R}$ is a real analytic function for $i = 1, \dots, 2g$. The map $\beta_\sigma : U \to \mathbb{R}^{2g}$ defined as $\beta_\sigma = (\beta_1, \dots, \beta_{2g})$ is called the *Betti map associated to the section* σ, whereas the β_i's are the *Betti coordinates*. Observe that the Betti map depends both on the choice of period map $\mathcal{P}$ and on the abelian logarithm ℓ_σ, but this is irrelevant for our applications. The main feature of the Betti map is that $\beta_\sigma(s) \in \mathbb{Q}^{2g}$ if and only if s is a torsion value of σ, so it allows us to treat the study of the torsion values of an abelian scheme as a transcendental Diophantine problem. Note that we need a non-torsion section

σ otherwise β_σ would be obviously constant and equal to a rational point. Viceversa, we recall that as a consequence of Manin's "theorem of the kernel" (see [18] or [4]) if β_σ is locally constant then σ is torsion. Moreover, the fibers of β_σ are complex submanifolds of $S(\mathbb{C})$ (see [8, Proposition 2.1] or [2, Section 4.2]).

Remark 1.1. There exists a compact subset $D \subseteq U$ such that the Betti map β_σ restricted to D is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ (using the real charts). This follows for instance by using [23, Fact 4.3] and the fact that for $i = 1, \dots, 2g$ we have $\beta_i = \pi_i \circ \ell_\sigma$, where π_i is the projection on the i-th coordinate with respect to the period map.

The rank, in the sense of real differential geometry, of the Betti map at a point s is denoted by $\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma(s)$. It can be shown that it depends only on the point s (see for instance [2, Section 4.2.1] or [14, Section 4]). Moreover we define

$$\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma = \max_{s \in S(\mathbb{C})} \mathrm{rk}_{\mathbb{R}}\, \beta_\sigma(s) \tag{5}$$

and note that it obviously holds that $\mathrm{rk}_{\mathbb{R}}\, \beta_\sigma \leq 2\min(g, \dim S)$. We call a section $\sigma\colon S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ *non-degenerate* if $\mathrm{rk}\, \beta_\sigma = 2\dim S$. The following crucial proposition allows us to have a uniform control on the fibers of the Betti map, under certain conditions.

Proposition 1.2. *Let $2\dim S = 2g = \mathrm{rk}_{\mathbb{R}}\, \beta_\sigma$. There exist a non-empty Zariski open set U of $S(\mathbb{C})$ such that: for any $x \in U$ there is a compact subanalytic set $D \subseteq S(\mathbb{C})$ containing x and a constant $c = c(D)$ such that the Betti map $\beta_\sigma\colon D \to \mathbb{R}^{2g}$ has finite fibers of cardinality at most c.*

Proof. From the condition on the rank of the Betti map it follows immediately that there exists a non-empty Zariski open set $U \subseteq S(\mathbb{C})$ on which β_σ is a submersion. Pick any compact subanalytic D inside U and contained in a chart. Restrict the Betti map on D and identify the latter with an euclidean compact in $\mathbb{R}^{2g}$. Since β_σ is now a submersion, the fibers must have real codimension equal to $2g$ (see for instance [17, Corollary 5.13]), which means that the fibers are discrete, and hence finite (D is compact). It remains to prove the uniform bound on the cardinality. So consider the subanalytic set

$$Z := \{(z, \beta_\sigma(z))\colon z \in D\} \subset \mathbb{R}^{2g} \times \mathbb{R}^{2g}\,,$$

Let $\pi_2 : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}^{2g}$ the projection on the second factor, then for any $p \in \mathbb{R}^{2g}$ we obviously have

$$Z \cap \pi_2^{-1}(p) = \beta_\sigma^{-1}(p)\,.$$

By Gabrielov theorem (see [30, Theorem A.4] or [5, Theorem 3.14]) $Z \cap \pi_2^{-1}(p)$ has at most c connected components, hence $\beta_\sigma^{-1}(p)$ has cardinality at most c. $\square$

1.2 Height bounds

In this short subsection we use the same notation of Section 1.1. Let $\mathcal{L}$ be a relative f-ample and symmetric line bundle on $\mathcal{A}$, then we define $\hat{h} : \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ to be the fiberwise Néron-Tate height i.e. $\hat{h}(p) := \hat{h}_{\mathcal{L}_s}(p)$ with $s = f(p)$. Moreover we consider a height function $h : S(\overline{\mathbb{Q}}) \to \mathbb{R}$ on the base. The following height inequality proved in [12, Theorem B.1] (see also [29, Theorem 5.3.5] for a more general approach) is a crucial result that relates the values of $\hat{h}$ and h:

Theorem 1.3. *Let X be an irreducible and non-degenerate[2] subvariety of $\mathcal{A}$ that dominates S. Then there exist two constants $c_1 > 0$ and $c_2 \geq 0$ and a Zariski non-empty open subset $V \subseteq X$ with*

$$\hat{h}(p) \geq c_1 h(f(p)) - c_2 \quad \textit{for all } p \in V\left(\overline{\mathbb{Q}}\right).$$

Proof. See [12, Theorem B.1]. $\square$

Corollary 1.4. *Assume that $f\colon \mathcal{A} \to S$ is endowed with a non-degenerate section $\sigma : S(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$. Then there exists a constant $C \geq 0$ and a non-empty Zariski open subset $V \subseteq S$ such that*

$$h(s) \leq C \quad \textit{for all } s \in V(\overline{\mathbb{Q}}) \cap \sigma^{-1}(\mathcal{A}_{\mathrm{tor}}). \tag{6}$$

[2]The references [12] and [16] use a slightly different (but equivalent) definition of Betti map and they have a notion of non-degenerate subvariety. A section σ is non-degenerate in our sense if and only if the subvariety $\sigma(S(\mathbb{C}))$ of $\mathcal{A}$ is non-degenerate in the sense of Dimitrov, Gao, Habbegger.

Remark 1.5. Note that Corollary 1.4 doesn't claim the maximality of the open set V. For instance assume that Equation (6) is satisfied, and fix a number field K whose set of $\mathbb{C}$-embeddings is denoted by Σ. Consider the open set:

$$W = \bigcup_{\tau \in \Sigma} V^{\tau}$$

then the inequality $h(s) \leq C$ (the same C as above) holds for any $s \in W(\overline{\mathbb{Q}}) \cap \sigma^{-1}(\mathcal{A}_{\mathrm{tor}})$ because of the invariance of the height with the respect to conjugation.

1.3 Torsion bounds

Let's quickly recall the definition of the stable Faltings height. Let A be a g-dimensional abelian variety over a number field K. Consider a finite extension $L \supseteq K$ such that $A \otimes L$ is semistable; moreover let $\mathcal{A} \to S := \operatorname{Spec} O_L$ be the connected component of the Neron model of $A \otimes L$ and denote with $\epsilon \colon S \to \mathcal{A}$ be the zero section. The sheaf of relative differentials $\Omega^g_{\mathcal{A}/S}$ pulls back on the base S through ϵ and we put $\omega_{\mathcal{A}/S} := \epsilon^* \Omega^g_{\mathcal{A}/S}$. The *stable Faltings height of* A is defined as:

$$h_F(A) := \frac{1}{[L : \mathbb{Q}]} \widehat{\deg}\left(\omega_{\mathcal{A}/S}\right)$$

where $\widehat{\deg}$ is the Arakelov degree calculated on $\omega_{\mathcal{A}/S}$ seen as hermitian line bundle on the base. It can be shown that h_F doesn't depend on the field extension (for details check [13]).

Let's recall an important property of the stable Faltings height. If $\phi \colon A \to A'$ is a K-isogeny between abelian varieties over K, then [26, Corollary 2.1.4] says that the stable Faltings heights of A and A' are related in the following way:

$$|h_F(A) - h_F(A')| \leq \frac{1}{2} \log \deg(\phi) \tag{7}$$

Moreover the stable Faltings height can be used to bound the exponent and the cardinality of the group of rational torsion points. The result is due to Rémond:

Proposition 1.6. *Let A be a principally polarized abelian variety of dimension g defined over a number field K. The finite group $A(K)_{\mathrm{tor}}$ has exponent at most $\kappa(A)^{\frac{35}{16}}$ and cardinality at most $\kappa(A)^{4g+1}$, where $d = [K \colon \mathbb{Q}]$ and $\kappa(A) = \left((14g)^{64g^2} d \max(1, h_F(A), \log d)^2\right)^{1024g^3}$.*

Proof. See [27, Proposition 2.9]. ☐

For a slightly weaker result involving the semistable Faltings height see [19, Proposition 7.1]. Let $\mathfrak{A}_g$ be the coarse moduli space over $\mathbb{C}$ of g-dimensional principally polarized abelian schemes. It is known that $\mathfrak{A}_g$ is a quasi-projective variety defined over $\mathbb{Q}$ and moreover there is a canonical projective embedding which induces a height function[3] $h_{\mathrm{mod}} \colon \mathfrak{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$ (see for instance [13, §3]). There is a close relationship between h_{mod} and the stable Faltings height h_F, in fact if $x \in \mathfrak{A}_g(K)$ is the point corresponding to a semistable abelian variety A over a number field K, then there exists a constant C independent from A and K such that:

$$|h_{\mathrm{mod}}(x) - r h_F(A)| \leq C \tag{8}$$

where r is a certain positive integer. For the proof of this deep result see [13, Theorem 3.1].

Proposition 1.7. *Let $f : \mathcal{A} \to S$ be a g-dimensional abelian scheme (induced by a morphism of varieties) admitting a non-torsion section $\sigma : S \to \mathcal{A}$. Let K be the field of definition of S, let s be a torsion value for σ and put $d(s) := [K(s) : \mathbb{Q}]$. Then there exists real constant $C = C(g)$ (so independent from the point s) such that*

$$\operatorname{ord}(\sigma(s)) \leq \left((14g)^{64g^2} d(s) \max\left(1,\, h(s) + C,\, \log d(s)\right)^2\right)^{\frac{35840g^3}{16}}.$$

[3]There is no general agreement on the notation of this height function on $\mathfrak{A}_g$. Some authors for instance denote it as h_{geo} and use h_{mod} for the Faltings height instead.

Proof. Recall that $\mathcal{A}_s$ is an abelian variety over the number field $K(s) \supseteq K$. The first step consists in reducing to the principally polarized case. The explicit construction is explained in [12, Poof of Theorem B.1 (Fourth devissage)], here we just recall the result: there is a quasi-finite dominant étale morphism $\rho : S' \to S$ with S' irreducible and a principally polarized abelian scheme $g\colon \mathcal{A}' \to S'$ such that there exists a S'-isogeny

$$\phi : \mathcal{A}' \to \mathcal{A}'' := \mathcal{A} \times_S S' .$$

Note that if $s' \in S'$ is a point lying above $s \in S$, then $\mathcal{A}''_{s'} = \mathcal{A}_s \otimes K(s')$, thus $h_F(\mathcal{A}_s) = h_F(\mathcal{A}''_{s'})$. By Equation (7) we have that $h_F(\mathcal{A}''_{s'}) \leq h_F(\mathcal{A}'_{s'}) + \deg(\phi_{s'})$, but notice that $\deg(\phi_{s'})$ doesn't depend on s', therefore we can just write:

$$h_F(\mathcal{A}_s) \leq h_F(\mathcal{A}'_{s'}) + C_1 . \tag{9}$$

Consider the induced morphism

$$\begin{aligned} m_g : S' &\to \mathfrak{A}_g \\ s' &\mapsto [\mathcal{A}'_{s'}] =: x_{s'} . \end{aligned}$$

The stable Faltings height of $\mathcal{A}'_{s'}$ is calculated over a finite extension $L \supseteq K(s')$ such that $\mathcal{A}'_{s'} \otimes L$ is semistable, in other words $h_F(\mathcal{A}'_{s'}) = h_F(\mathcal{A}'_{s'} \otimes L)$. From this fact and Equation (8) we obtain

$$h_F(\mathcal{A}'_{s'}) < C_2 + h_{\mathrm{mod}}(x_{s'}) . \tag{10}$$

On the other hand, by the usual functorial properties of the Weil height we have

$$|h(s') - h_{\mathrm{mod}}(x_{s'})| < C_3 \tag{11}$$

for a constant C_3 and for any height function h on S'. Finally the claim follows after putting together Equations (9) to (11) and Proposition 1.6 applied to the fiber $\mathcal{A}_s$. $\square$

1.4 Control on conjugate points

Let's fix an affine variety $Y(\mathbb{C}) \subseteq \mathbb{A}^N(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C})$ defined over a number field K. For any point $p \in Y(\mathbb{C})$ we denote by $K(p)$ the field generated by the coordinates of p; this is the same as the residue field of p when the latter is seen as an abstract point of Y. With the letter h we denote both the absolute height on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and $\mathbb{A}^1(\overline{\mathbb{Q}})$, since the formal meaning is clear from the argument of h. Further, we denote by $\|\cdot\|$ the euclidean norm in $\mathbb{A}^N(\mathbb{C})$. We fix a closed subvariety B' of Y and we define

$$W'_\delta := \{x \in Y(\mathbb{C})\colon d(x, B'(\mathbb{C})) < \delta\}, \quad \text{for } \delta \in \mathbb{R}_{>0}$$

where

$$d(x, B'(\mathbb{C})) := \inf_{b \in B(\mathbb{C})} \|x - b\| .$$

Moreover let's consider the set $C'_\delta := Y(\mathbb{C}) \setminus W'_\delta$.

Lemma 1.8. *Let H be a subset of $Y(\mathbb{C})$ and let C be a compact subset of H. Fixed $p \in Y(\mathbb{C}) \setminus H$, there exists a constant c (uniform with respect to $b \in C$) such that*

$$d(p, H) \geq c \cdot \|p - b\| \quad \textit{for each } b \in C.$$

Proof. For each $b \in C$, let us consider a constant a_b which satisfies $0 < a_b < \frac{d(p,H)}{\|p-b\|}$ (note that it exists since $p \notin H$). Observe that a_b is a constant which depends on b and such that

$$d(p, H) - a_b \cdot \|p - b\| > 0.$$

Then there exists an open (analytic) neighbourhood N_b of b such that

$$d(p, H) - a_b \cdot \|p - b'\| > 0 \qquad \text{for each } b' \in N_b.$$

The family $\{N_b : b \in H\}$ is an open covering of the compact set C. Thus there exists a finite subcovering $\{N_{b_i} : i = 1, \ldots, n\}$. The constant $c := \min_{1 \leq i \leq n}(a_{b_i})$ works uniformly on C. In fact for each $b \in C$ we have

$$c \cdot \|p - b\| \leq a_b \cdot \|p - b\| < d(p, H).$$

$\square$

Proposition 1.9. *Let K be a number field which contains the field of definition of the subvariety B'. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in Y(\overline{\mathbb{Q}}) \setminus B'(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{1}{2}[K(\alpha) : K]$ different K-embeddings $\tau \colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C'_δ.*

Proof. Fix $\beta = (\beta_1, \ldots, \beta_N) \in B'(\overline{\mathbb{Q}})$ such that there exists an index i with $\beta_i \in K(\alpha)$ (observe that such a β always exists); and write $\alpha := (\alpha_1, \ldots, \alpha_N)$. Clearly $h(\alpha) \geq h(\alpha_i)$ and $h(\beta) \geq h(\beta_i)$. This implies

$$h(\alpha_i - \beta_i) \leq h(\alpha_i) + h(\beta_i) + \log(2) \leq h(\alpha) + h(\beta) + \log(2) \tag{12}$$

Now, define

$$\Sigma := \{\tau : K(\alpha) \hookrightarrow \mathbb{C} \colon \mathrm{id} = \tau_{|K} \text{ and } \alpha^\tau \notin C'_\delta\}$$

and denote by k the cardinality of Σ. Since τ is a K-embedding we have $\beta^\tau \in B'(\overline{\mathbb{Q}})$. Moreover observe that, given $\tau \in \Sigma$, we have $\alpha^\tau \notin B'(\mathbb{C})$. Thus, by Lemma 1.8 for $p = \alpha^\tau, H = B'(\mathbb{C})$ and $C = \{\beta^\tau \colon \tau \in \Sigma\}$, and since $\alpha^\tau \notin C'_\delta$ (by definition of Σ) there exists a constant c_τ such that

$$\frac{1}{|\alpha_i^\tau - \beta_i^\tau|} \geq \frac{1}{\|\alpha^\tau - \beta^\tau\|} \geq \frac{c_\tau}{d(\alpha^\tau, B(\mathbb{C}))} > \frac{c_\tau}{\delta}.$$

Considering $c := \min_{\tau\in\Sigma}(c_\tau)$ we obtain a constant c such that:

$$\frac{1}{|\alpha_i^\tau - \beta_i^\tau|} \geq \frac{c}{\delta} \quad \text{for fixed } i \text{ and for all } \tau \in \Sigma.$$

Then for δ small enough we obtain

$$\begin{aligned} h(\alpha_i - \beta_i) &\geq \frac{1}{[K(\alpha) : \mathbb{Q}]} \sum_\nu \log\max\left(1, \left|\frac{1}{\alpha_i - \beta_i}\right|_\nu\right) \geq \\ &\geq \frac{1}{[K(\alpha) : \mathbb{Q}]} \sum_{\tau\in\Sigma} \log\max\left(1, \left|\frac{1}{\alpha_i^\tau - \beta_i^\tau}\right|\right) \geq \frac{k}{[K(\alpha) : \mathbb{Q}]} \log\left(\frac{c}{\delta}\right). \end{aligned} \tag{13}$$

By (12), (13) and the fact that α has bounded height we obtain

$$k \leq \frac{(a + h(\beta) + \log(2)) \cdot [K(\alpha) : \mathbb{Q}]}{\log(c/\delta)}.$$

For δ small enough we have

$$\frac{a + h(\beta) + \log(2)}{\log(c/\delta)} \leq \frac{1}{2[K : \mathbb{Q}]}.$$

Therefore

$$k \leq \frac{1}{2}[K(\alpha) : K].$$

□

Now let's fix a projectve variety X defined over K and a closed subvariety B of X. For any point $p = (x_0 : \ldots : x_N) \in X(\mathbb{C})$ pick any $x_i \neq 0$ and then put $K(p) := K\left(\frac{x_j}{x_i} \colon j = 0, \ldots, N\right)$. Note that $K(p)$ doesn't depend on the choice of x_i (i.e. the standard affine chart) and moreover $K(p)$ is the residue field of p when the latter is seen as an abstract point of X. We prove a higher dimensional generalization of a quite useful result already appeared for the projective line in [19, 20, 21, Lemma 8.2]. Roughly speaking the result claims the following: K is the field of definition of B, $a \in \mathbb{R}$ and $\alpha \in X(\overline{\mathbb{Q}})$ is any point not contained in $B(\mathbb{C})$ with height at most a; then we can give an explicit lower bound, depending only on $[K(\alpha) : K]$, on the number of $K(\alpha)$ conjugates of α that lie in a "big enough" compact not intersecting $B(\mathbb{C})$.

We first construct the compact subset. Denote by $U_0, \ldots, U_N$ the standard affine charts of the projective space. Let's define

$$W_{i,\delta} := \{x \in X(\mathbb{C}) \cap U_i : d(x, B(\mathbb{C}) \cap U_i) < \delta\} \qquad \text{for fixed } \delta \in \mathbb{R}_{>0} \text{ and } i = 1, \ldots, N. \tag{14}$$

Then we put $W_\delta := \bigcup_{i=0}^N W_{i,\delta}$ and note that it is an open subset of $X(\mathbb{C})$ containing $B(\mathbb{C})$. Therefore $C_\delta := X(\mathbb{C}) \setminus W_\delta$ is a compact set not intersecting $B(\mathbb{C})$.

Proposition 1.10. *Let K be a number field which contains the field of definition of the subvariety B. Given a real constant $a > 0$, there exists a real constant $\delta = \delta(K, a) > 0$ with the following property: for any $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$, there are at least $\frac{1}{2}[K(\alpha) : K]$ different K-embeddings $\tau \colon K(\alpha) \hookrightarrow \mathbb{C}$ such that α^τ lies in C_δ.*

Proof. Fix $\alpha \in X(\overline{\mathbb{Q}}) \setminus B(\mathbb{C})$ with $h(\alpha) \leq a$ and fix a chart U_i such that $\alpha \in U_i$. Since the chart is invariant under the action of each τ, we can apply Proposition 1.9 for $Y(\mathbb{C}) = X(\mathbb{C}) \cap U_i$, $B'(\mathbb{C}) = Y(\mathbb{C}) \cap B(\mathbb{C})$ and $C'_\delta = C_\delta \cap U_i$. Therefore, we obtain a real number δ_i which only depends on K, a and U_i and which satisfies the statement for $\alpha \in U_i$. We can repeat the argument for any standard chart and after defining $\delta := \min_{0 \leq i \leq N}(\delta_i)$, we can conclude. □

Remark 1.11. Observe that the the intersection of C_δ with each standard chart U_i is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$. In fact, first of all let's identify $U_i \cap X(\mathbb{C})$ with $\mathbb{R}^{2N}$, then the map $\mathbb{R}^{2N} \ni p \mapsto d(p, B(\mathbb{C}) \cap U_i)$ is a globally subanalytic function (see for instance [3, Example 2.10]). At this point we apply [28, §1 Lemma 2.3] to conclude that the set $W_{i,\delta} = U_i \cap W_\delta$ is globally subanalytic for any $\delta > 0$. Finally, note that the intersection $C_\delta \cap U_i$ is the complement set $(U_i \cap X(\mathbb{C})) \setminus (U_i \cap W_{i,\delta})$, so it is also globally subanalytic.

2 The main theorem

2.1 Proof

In this section we prove Theorem 0.2. The proof is rather long and technical; it will be eventually split in two cases after a common setup. We use the same notations fixed in the introduction.

2.1.1 Setup of the proof

We recall that it is enough to work with the conditions given by Equation (2). Let's keep in mind the assumptions $1) - 4)$, in particular recall we are assuming two fibers $\mathcal{A}_{1,s_1}$ and $\mathcal{A}_{2,s_2}$ to be equal at most over a proper Zariski-closed subset $E \subset \overline{S}_i$ for fixed i, say $i = 1$. Notice that f_1 defines a rational map when restricted to $\mathcal{A}_2$, analogously for f_2 when restricted to $\mathcal{A}_1$; we denote by $\text{Ind}(f_1, f_2) := \mathcal{A}_1 \cup \mathcal{A}_2 \setminus (\mathcal{A}_1 \cap \mathcal{A}_2)$ the union of indeterminacy loci of the previous maps. Since $\mathcal{A}_1 \cap \mathcal{A}_2$ has dimension $2g$, the closed set $\text{Ind}(f_1, f_2)$ is finite. We denote with $\mathcal{C}(\beta_1)$ the locus of critical points of the Betti map β_1; it is the proper subset of S_1 on which the Betti map β_1 is not a submersion. Moreover, by Corollary 1.4 there exist open subsets $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$ where the σ_1-torsion values and the σ_2-torsion values have bounded height by the same constant, respectively. We fix a number field K containing all the fields of definitions of $\overline{\mathcal{A}}, \overline{S}_1, \overline{S}_1, f_1, f_2, \sigma_1, \sigma_2, \Delta_1, \Delta_2, \mathcal{C}(\beta_1), \text{Ind}(f_1, f_2)$; let's define

$$\mathcal{A}' := \mathcal{A}_2 \cap f_1^{-1}(S'_1) \cap f_2^{-1}(S'_2) \cap f_1^{-1}(\overline{S}_1 \setminus E) \cap f_1^{-1}(S_1 \setminus \mathcal{C}(\beta_1)) \,. \tag{15}$$

By Remark 0.5 it's enough to prove Theorem 0.2 for $\mathfrak{F} \cap \mathcal{A}'$. We denote with Σ_K the set of complex embeddings of K. By Remark 1.5 we can also ensure that S'_2 is invariant under the action of Σ_K; S'_1 is also invariant but this is not relevant for us. Consider $p \in \mathfrak{F} \cap \mathcal{A}'$ and let's put $b := f_2(p)$. Let m be the order of $\sigma_2(b)$ in $\mathcal{A}_{2,b}$, in symbols we set $m = m(b) := \text{ord}(\sigma_2(b))$. If the set

$$\mathfrak{O} := \{\text{ord}(\sigma_2(b)) \colon b \in f_2(\mathfrak{F} \cap \mathcal{A}')\} \subseteq \mathbb{N} \tag{16}$$

is bounded by a constant C, then

$$\{f_2(p) : p \in \mathfrak{F} \cap \mathcal{A}'\} \subseteq \{b \in S'_2 \colon \text{ord}(\sigma_2(b)) \leq C\} \subseteq \sigma_2^{-1}\left(\bigcup_{N \leq C} \mathcal{A}_2[N]\right),$$

so Theorem 0.2 follows. Thus the strategy of the proof is the following:

$$\boxed{\text{We prove that } \mathfrak{O} \text{ is uniformly bounded i.e. } m \text{ is uniformly bounded.}}$$

In particular, we will partition $\mathfrak{O}$ in two subsets and show that each of them contains a finite number of elements.

For any $b \in S'_2$, let $\tau \colon K(b) \hookrightarrow \mathbb{C}$ be any K-embedding. Recalling that S'_2 is invariant under the action of Σ_K, in our proof we can replace b by b^τ (for any n we have $n\sigma_2(b^\tau) = n\sigma_2(b)$). So, since

b has bounded height, we can apply Proposition 1.10 on $\overline{S}_2$ and Δ_2 (that play the role of X and B respectively) and conclude that there exists an analytic compact set $\Delta \subseteq S_2(\mathbb{C})$ such that $b \in \Delta$. With the notation adopted in Proposition 1.10 we have $\Delta = C_\delta$ for $\delta > 0$ small enough. Roughly speaking we have just explained that we can assume that b lies in a "big enough" compact set of $\overline{S}_2(\mathbb{C})$ that avoids the discriminant locus of f_2. Moreover, by Remark 1.11 the compact set Δ has the property that the intersection $\Delta \cap U_i$ with each standard chart is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$.

Before starting with the actual proof we need introduce some other objects. Let $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $f_2(p) = b$ as above, and let's consider the points $p_r := t_2^r(p) = p + r\sigma_2(b)$ for $r = 0, 1, \ldots, m-1$. Then define:

$$n_r := \operatorname{ord} \sigma_1(f_1(p_r)) .$$

Note that n_r is finite since $O(p)$ is finite. Moreover, n_r is only defined when p_r does not lie in the indeterminacy locus of f_1.

We also need to construct an auxiliary abelian scheme that will play a crucial role in the whole proof. Consider the variety $F_b := \mathcal{A}_{2,b} \cap \mathcal{A}_1 \setminus \left(f_1^{-1}(\Delta_1 \cup \mathcal{C}(\beta_1))\right)$; it can be seen as the base of an abelian scheme $\mathcal{X} \to F_b$ by defining its fibers as $\mathcal{X}_z := \mathcal{A}_{1,f_1(z)}$. Clearly such fibers are all smooth since we have removed the discriminant locus of f_1. In addition, this abelian scheme is endowed with a non-torsion section $s_{\mathcal{X}} := \sigma_1 \circ f_1$. Finally, with Proposition 1.7 in mind, we also fix the constant

$$c' = c'(g) := 3 \cdot \frac{35840g^3}{16}. \tag{17}$$

After the following section in which we will deal with 'translates of points', we will distinguish two cases in the proof, each of them dealing with a subset of $\mathfrak{O}$.

2.1.2 Control on translates and their heights

Given a point $p \in \mathfrak{F} \cap \mathcal{A}'$ we are interested in points of the type $p + r\sigma_2(b)$, which we call "translates" of p, conjugates of them and their images by f_1. Notice that some translates could lie into the indeterminacy locus of f_1, but this is not a problem since we can assume m so big that many translates avoid this indeterminacy locus. However, in order to avoid long and redundant comments we write the f_1-images of all translates tacitly ignoring points where f_1 is not defined. At a first glance, these points seems to be "wild" with respect to the property of lying in $\mathcal{A}_1$ or having bounded height (i.e. lying in $\mathcal{A}'$). We now show that under our hypotheses it is actually possible to have a certain degree of control on such properties:

Proposition 2.1. *Let $b \in \Delta$ and $p \in \mathfrak{F} \cap \mathcal{A}'$. Let $h : \mathcal{A}_{2,b}(\overline{\mathbb{Q}}) \to \mathbb{R}$ and $h' : \overline{S}_1(\overline{\mathbb{Q}}) \to \mathbb{R}$ be any two height functions. Then there exists a constant $C \geq 0$ such that*

$$h(p + r\sigma_2(b)) \leq C, \; h'(f_1(p + r\sigma_2(b))) \leq C \qquad \text{for each } r = 0, \ldots, m-1.$$

Proof. Denote by $\hat{h} : \mathcal{A}_{2,b}(\overline{\mathbb{Q}}) \to \mathbb{R}$ the Néron-Tate height on the fiber and recall that the following relation holds:

$$|h(x) - \hat{h}(x)| < c_1 \qquad \text{for any } x \in \mathcal{A}_{2,b}(\overline{\mathbb{Q}}).$$

Moreover, the Néron-Tate height satisfies the parallelogram law

$$\hat{h}(x+y) + \hat{h}(x-y) = 2\hat{h}(x) + 2\hat{h}(y) \qquad \text{for } x, y \in \mathcal{A}_{2,b}(\overline{\mathbb{Q}}).$$

Now: we can assume $p \in \mathcal{A}' \cap \mathcal{A}_1$ so that it maps through f_1 on a torsion value, and moreover $r\sigma_2(b)$ is a torsion point of $\mathcal{A}_{2,b}$. So we get $\hat{h}(p) \leq C'$ (where the constant is uniform with respect to b, p) and $\hat{h}(r\sigma_2(b)) = 0$. Thus, by choosing $x = p$ and $y = r\sigma_2(b)$ for any $r = 0, \ldots, m-1$ in the parallelogram law, we obtain

$$\hat{h}(p + r\sigma_2(b)) \leq \hat{h}(p + r\sigma_2(b)) + \hat{h}(p - r\sigma_2(b)) = 2\hat{h}(p) \leq C_2.$$

In other words, we get $h(p + r\sigma_2(b)) \leq C''$, i.e. each point of the type $p + r\sigma_2(b)$ has uniformly bounded height. The full claim then follows from the functorial properties of the Weil height with respect to the morphism $f_1 : \mathcal{A}_{2,b} \to S_1$. ☐

Let's embed the fiber $\mathcal{A}_{2,b}(\mathbb{C})$ inside some $\mathbb{P}^N(\mathbb{C})$ and let $U_0, \ldots, U_N \subseteq \mathbb{P}^N(\mathbb{C})$ be as usual the standard charts. The f_2-fibers could contain part of some 'problematic' Zariski-closed set X_b which comes from a Zariski-closed subset $Y \subseteq S_1$ via f_1, in the sense that

$$X_b = \mathcal{A}_{2,b}(\mathbb{C}) \cap f_1^{-1}(Y(\mathbb{C})).$$

In general, in order to allow our tools to work well we need to remove X_b from the fiber, but preserving some properties we are interested in. To be more precise, we want to show that X_b is contained in a small enough open subset $V_b \subseteq \mathcal{A}_{2,b}(\mathbb{C})$ whose intersection $V_b \cap U_i$ with each standard chart of $\mathbb{P}^N(\mathbb{C})$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$: first of all let's identify $\mathcal{A}_{2,b}(\mathbb{C}) \cap U_i$ with $\mathbb{R}^{2N}$; then by repeating the construction in Equation (14), consider the globally subanalytic sets $V_{i,\delta} := \{z \in \mathcal{A}_{2,b}(\mathbb{C}) \cap U_i \colon d(z, X_b \cap U_i) < \delta\}$ for any $\delta > 0$ small enough and define

$$V_b = V_b^\delta := \bigcup_{i=0}^{N} V_{i,\delta}. \tag{18}$$

Now, denote by $U_0, \dots, U_M$ the standard affine charts on $\overline{S}_1(\mathbb{C})$. Analogously, we can encircle Y with a small enough open set of which we can control the size (chart-by-chart), so let us consider the sets

$$W_{i,\delta} := \{z \in S_1(\mathbb{C}) \cap U_i \colon d(z, Y \cap U_i) < \delta\}$$

for any $\delta > 0$ small enough, and define $W := \bigcup_{i=0}^{M} W_{i,\delta}$. We can carry out the construction of V_b and W such that $f_1(V_b) \subseteq W$, so that their size is controlled via the same δ.

Further, we can decompose the compact set $\Delta \subseteq S_2(\mathbb{C})$ as a finite union of small definable compact sets A_i. We work in one of those compact sets that contains b and we shall call it A. We are interested in controlling the translates of a point $p \in \mathfrak{F} \cap \mathcal{A}'$ or one of its conjugates with respect to the just described open sets V_b and W; moreover, we want A to be preserved by the said conjugation.

To this end suppose that the Zariski-closed set Y, and consequently also X_b, is defined over K (or alternatively enlarge the field to make it so) and denote by Σ_p the set of complex K-embeddings of the field $K(p)$; given $\tau \in \Sigma_p$, we get $f_2(p^\tau) = b^\tau$ but observe that two conjugates of b might coincide since $K(p)$ properly contains $K(b)$ in general. Each element of Σ_p induces by restriction a complex K-embedding of $K(b)$ in a surjective way. We can apply Proposition 1.10 to b and conclude that the number of $K(p)$-conjugates of b contained in A is $\gg d_1$ where $d_1 := [K(p) \colon K]$ and the implicit constant is independent of p and b (we are using the fact that the number of A_i's is fixed). Denote by $\Sigma_{p,A}$ the subset of Σ_p given by the K-embeddings τ which satisfy the further condition $b^\tau \in A$; by the previous discussion we get $\#\Sigma_{p,A} \gg d_1$ (say $\geq c_1 d_1$).

Proposition 2.2. *Let X_b, Y, V_b, W be as above and let $p \in \mathfrak{F} \cap \mathcal{A}'$. For $m = \operatorname{ord}(\sigma_2(b))$ large enough there exists $\tau \in \Sigma_{p,A}$ with the following property: there are at least $m/2$ elements of the type $f_1(p^\tau + r\sigma_2(b^\tau))$ which don't lie in $f_1(V_{b^\tau})$.*

Proof. Denote again by $U_0, \dots, U_M$ the standard affine charts on $\overline{S}_1(\mathbb{C})$. First of all, consider the compact set Y: cover the intersection $U_i \cap Y$ with euclidean disks centered at each point and with fixed radius R. The union of these disks gives an open covering of Y, so we can fix a finite subcovering $\mathcal{O}$ with the property that each element of $\mathcal{O}$ is an open disk contained in some chart U_i. Define $N_{\mathcal{O}} := \#\mathcal{O}$.

Observe that the action of $\tau \in \Sigma_{p,A}$ fixes the charts. Thus, for b, p given, the points $f_1(p^\tau + r\sigma_2(b^\tau))$ varying τ are contained into the same chart; but the same points varying r are not necessarily contained into the same chart. Fixed $b \in \Delta, p \in \mathfrak{F} \cap \mathcal{A}'$, we introduce the set

$$C = C^{(b,p,\tau)} := \{f_1(p^\tau + r\sigma_2(b^\tau)) \colon r = 0, \dots, m-1\}$$

and prove the claim by contradiction: so let's assume that for all $\tau \in \Sigma_{p,A}$ there are at least $m/2$ elements of C that lie in $f_1(V_{b^\tau})$. Denote by N' the (finite) number of connected components of W. Then for each τ there exist an index $i(\tau)$ and a connected component W_τ of W such that there are at least $m' := \lceil m/(2N'(M+1)) \rceil$ elements of the type $f_1(p^\tau + r\sigma_2(b^\tau))$ that lie in $W_\tau \cap U_{i(\tau)}$ (recall that m can be thought big enough).

Step 1. First of all, prove that for m large enough and for each τ we can assume to have at least $m'/2$ values of r such that

$$f_1(p^\tau + r\sigma_2(b^\tau)) \in (W_\tau \cap U_{i(\tau)}) \setminus Y.$$

In order to ease the notation, we will write $a_r^{(b,p,\tau)} := f_1(p^\tau + r\sigma_2(b^\tau))$ for $k = 0, \dots, m-1$ and $\tau \in \Sigma_p$; moreover, fix j and denote by $\{\}_j$ the j-component of a point in each chart U_i. To prove the fact just claimed, fix m and suppose that there exist $\tau \in \Sigma_{p,A}$ and $\geq m'/2$ values of r such that

$$a_r^{(b,p,\tau)} \in \left(W_\tau \cap U_{i(\tau)}\right) \cap Y;$$

in what follows, we remove the explicit dependence on b, p, τ since they all are fixed. Note that, for the said values of r we have

$$a_r^\gamma \in (W_\tau \cap U_i) \cap Y \qquad \text{for each } \gamma \in \Sigma_p\,.$$

Observe that the connected component W_τ is not preserved by the action of γ a priori, but in this case this is true since it encircles a (finite) union of irriducible components of Y (which are preserved by γ). For each $\gamma \in \Sigma_p$ there exists $O_\gamma \in \mathcal{O}$ which contains at least $m'' := \lceil m'/2N_{\mathcal{O}} \rceil$ elements of the type a_r^γ with varying r; observe that O_τ is contained in some chart which can be $\neq U_i$. Fix a small $\varepsilon > 0$ and observe that for fixed m large enough there exist two elements $a_{r_{1,\gamma}}^\gamma, a_{r_{2,\gamma}}^\gamma \in O_\gamma$ such that

$$|\{a_{r_{1,\gamma}}^\gamma - a_{r_{2,\gamma}}^\gamma\}_j| < \varepsilon.$$

To prove the previous assertion, fix $\varepsilon > 0$. Since O_γ has compact closure in the corresponding chart, we can cover it with a finite number of small disks with fixed radius ε. If we call N_ε the cardinality of the covering, for $m'' > N_\varepsilon$ we obtain the thesis. Now, for each γ define

$$S_\gamma := \{r \in \{0, \dots, m-1\} \colon a_r^\gamma \in O_\gamma\}.$$

Denote by $D_\gamma \subseteq S_\gamma^2$ the diagonal of S_γ and define $\mathcal{S}_\gamma := S_\gamma^2 \setminus D_\gamma$. Observe that for fixed γ and $(r, r') \in \mathcal{S}_\gamma$ we get

$$|\{a_r^\gamma - a_{r'}^\gamma\}_j| \le |\{a_r^\gamma - a_{r_{1,\gamma}}^\gamma\}_j| + |\{a_{r_{1,\gamma}}^\gamma - a_{r_{2,\gamma}}^\gamma\}_j| + |\{a_{r_{2,\gamma}}^\gamma - a_{r'}^\gamma\}_j| < 2R + \varepsilon.$$

We have

$$\begin{aligned} &\frac{1}{[K(p):\mathbb{Q}]} \sum_{\gamma \in \Sigma_{p,A}} \sum_{(r,r') \in \mathcal{S}_\gamma} \max\log\left(1, \frac{1}{|\{a_r^\gamma - a_{r'}^\gamma\}_j|}\right) \ge \\ &\ge \frac{1}{[K(p):\mathbb{Q}]} c_1 d_1 m''(m''-1) \log\frac{1}{2R+\varepsilon}. \end{aligned}$$

By Proposition 2.1 we have $h(a_r^\gamma - a_{r'}^\gamma) \le c_2$ for each r. On the other hand we have

$$\begin{aligned} &\frac{1}{[K(p):\mathbb{Q}]} \sum_{\gamma \in \Sigma_{p,A}} \sum_{(r,r') \in \mathcal{S}_\gamma} \max\log\left(1, \frac{1}{|\{a_r^\gamma - a_{r'}^\gamma\}_j|}\right) \le \\ &\sum_{r \neq r'} \frac{1}{[K(p):\mathbb{Q}]} \sum_{\gamma \in \Sigma_p} \max\log\left(1, \frac{1}{|\{a_r^\gamma - a_{r'}^\gamma\}_j|}\right) \le \\ &\sum_{r \neq r'} h(a_r - a_{r'}) \le c_2 \cdot (m-1)^2. \end{aligned}$$

In the previous lines we have used the following facts:

(i) the property $f_1(p^\tau + r\sigma_2(b^\tau)) = f_1(p + r\sigma_2(b))^\tau$ which is valid since the fiberwise group law is given by a globally regular map (defined over K);

(ii) some elementary height properties exactly like in Equation (13);

(iii) the bounded height of $f_1(p + r\sigma_2(b))$. This is the crucial point which allows to obtain the contradiction.

Therefore, we finally obtain

$$\log\frac{1}{2R+\varepsilon} \le c_3.$$

For m big enough, we obtain a sufficiently small ε as to give a contradiction.

Step 2. Now fix m large enough for the claim of Step 1 to be satisfied. Similarly to the strategy we have adopted for Y, we can cover the compact closure of $W_\tau \setminus Y$ with finitely many open disks, each contained in some U_i. Call Q_i the compact closure of such disks and let $N_{\mathcal{Q}}$ be the cardinality of such a finite covering; note that the intersection $Q_i \cap Y$ is not necessarily empty. Put

$$R_\tau := \left\{r \in \{0, \dots, m-1\} \colon f_1(p^\tau + r\sigma_2(b^\tau)) \in Q_{j(\tau)} \cap (W_\tau \setminus Y)\right\}.$$

where $j(\tau)$ is chosen so that $\#R_\tau \ge m'/(2N_{\mathcal{Q}})$: it exists by Step 1. Denote by $E_\tau \subseteq R_\tau^2$ the diagonal of R_τ and define $\mathcal{R}_\tau := R_\tau^2 \setminus E_\tau$. Note that

$$\#\mathcal{R}_\tau \ge \left\lceil \frac{m'}{2N_{\mathcal{Q}}} \right\rceil \cdot \left(\left\lceil \frac{m'}{2N_{\mathcal{Q}}} \right\rceil - 1 \right).$$

Fix $\beta \in \left(\bigcap_{i=0}^{M} U_i\right) \setminus \left(\bigcup_{i=1}^{N_{\mathbb{Q}}} Q_i\right)$ and keep the same notation as above. By Lemma 1.8 for $p = \beta$ and $C = H = Q_{j(\tau)}$, there exists a constant $C_{j(\tau)}$ such that

$$\|\beta - \alpha\| \leq C_{j(\tau)} d(\beta, Q_{j(\tau)}) \qquad \text{for each } \alpha \in Q_{j(\tau)}.$$

With a similar argument as in the proof of Lemma 1.8, since the set $\{a_r^{(b,p,\tau)} \mid r \in R_\tau\}$ is compact and its elements lie outside Y we obtain a constant $C'_{j(\tau)}$ such that

$$d(\beta, Q_{j(\tau)}) \leq C'_{j(\tau)} d(a_r^{(b,p,\tau)}, Y \cap U_{j(\tau)}) \qquad \text{for each } a_r^{(b,p,\tau)} \text{ with } r \in R_\tau.$$

Taking the maximum of all constants involved and passing to the j-component, we finally obtain a uniform constant C such that for each $(r, r') \in \mathcal{R}_\tau$ we have

$$\begin{aligned} &|\{a_r^{(b,p,\tau)} - a_{r'}^{(b,p,\tau)}\}_j| \leq |\{a_r^{(b,p,\tau)} - \beta\}_j| + |\{a_{r'}^{(b,p,\tau)} - \beta\}_j| \leq \\ &\leq C \cdot \left(d(a_r^{(b,p,\tau)}, Y \cap U_{j(\tau)}) + d(a_{r'}^{(b,p,\tau)}, Y \cap U_{j(\tau)})\right) \leq 2C\delta. \end{aligned}$$

Now, with a similar argument as above, we obtain

$$\begin{aligned} &\frac{1}{[K(p):\mathbb{Q}]} \sum_{\tau \in \Sigma_{p,A}} \sum_{(r,r') \in \mathcal{R}_\tau} \max \log \left(1, \frac{1}{|\{a_r^{(b,p,\tau)} - a_{r'}^{(b,p,\tau)}\}_j|}\right) \geq \\ &\geq \frac{1}{[K(p):\mathbb{Q}]} c_1 d_1 \left\lceil \frac{m}{4N_{\mathbb{Q}} N'(M+1)} \right\rceil \left(\left\lceil \frac{m}{4N_{\mathbb{Q}} N'(M+1)} \right\rceil - 1\right) \log \frac{1}{2C\delta}. \end{aligned} \tag{19}$$

On the other hand, we get

$$\begin{aligned} &\frac{1}{[K(p):\mathbb{Q}]} \sum_{\tau \in \Sigma_{p,A}} \sum_{(r,r') \in \mathcal{R}_\tau} \max \log \left(1, \frac{1}{|\{a_r^{(b,p,\tau)} - a_{r'}^{(b,p,\tau)}\}_j|}\right) \leq \\ &\sum_{r \neq r'} \frac{1}{[K(p):\mathbb{Q}]} \sum_{\tau \in \Sigma_p} \max \log \left(1, \frac{1}{|\{f_1(p^\tau + r_1 \sigma_2(b^\tau)) - f_1(p^\tau + r_2 \sigma_2(b^\tau))\}_j|}\right) \leq \\ &\sum_{r \neq r'} \frac{1}{[K(p):\mathbb{Q}]} \sum_{\tau \in \Sigma_p} \max \log \left(1, \frac{1}{|\{f_1(p + r_1 \sigma_2(b))\}_j^\tau - \{f_1(p + r_2 \sigma_2(b))\}_j^\tau|}\right) \leq \\ &\leq (m-1)^2 \cdot (h(f_1(p + r_1 \sigma_2(b))) + h(f_1(p + r_2 \sigma_2(b))) + \log 2) \leq \\ &\leq (m-1)^2 \cdot C. \end{aligned} \tag{20}$$

Hence, comparing Equations (19) and (20) we finally have

$$\log \frac{1}{2C\delta} \leq \frac{C \cdot (m-1)^2 [K:\mathbb{Q}]}{c_1 \left\lceil \frac{m}{4N_{\mathbb{Q}} N'(M+1)} \right\rceil \cdot \left(\left\lceil \frac{m}{4N_{\mathbb{Q}} N'(M+1)} \right\rceil - 1\right)}.$$

When $m \to +\infty$ the latter equation is

$$\log \frac{1}{2C\delta} \leq \frac{O(m^2)}{O(m^2)}.$$

This is a contradiction, since the implicit constant is uniform but we are allowed to take $\delta > 0$ arbitrarily small. □

2.1.3 First case

Let's define

$$\mathfrak{O}' := \left\{m \in \mathfrak{O} \colon \exists p_r \in \mathcal{A}_{2,b} \setminus f_1^{-1}(\mathcal{C}(\beta_1)) \text{ such that } n_r > m^{g(2c'+1)}\right\}.$$

We prove that the set $\mathfrak{O}'$ is finite giving an upper bound for $m \in \mathfrak{O}'$. We keep all the notations introduced above and in addition we put for simplicity $L := f_1^{-1}(\mathcal{C}(\beta_1))$.

Suppose $b \in \Delta$ and $m \in \mathfrak{O}'$. Let $p_r = p + r\sigma_2(b) \in \mathcal{A}_{2,b} \setminus L$ be any of the m points $p_0, \ldots, p_{m-1}$ such that $n_r > m^{2c'+1}$. Similarly to what has been done in Equation (18) for the Zariski-closed set X_b, we can

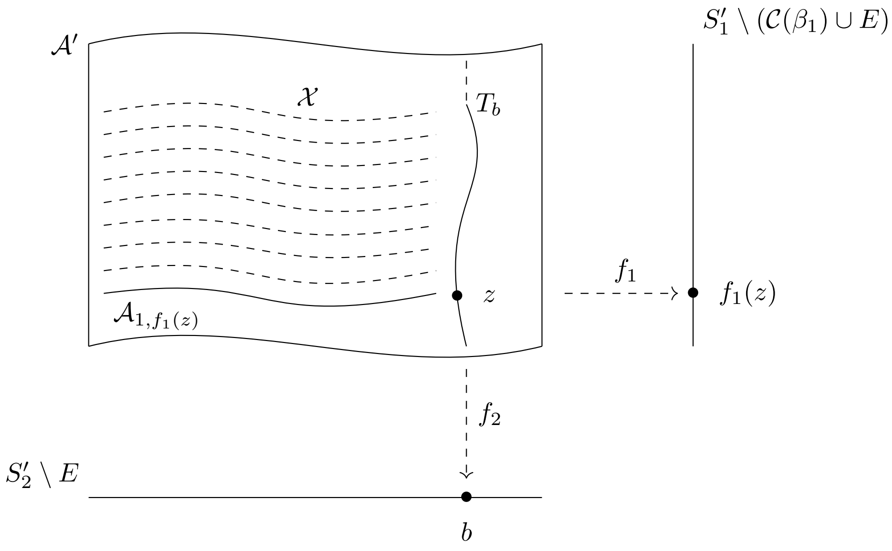

Figure 1: A schematization of the family $\mathcal{X} \to T_b$.

construct an open set $V_b \subset \mathcal{A}_{2,b}(\mathbb{C})$ which contains the locus $\mathcal{A}_{2,b} \cap (f_1^{-1}(\Delta_1) \cup L)$ and whose intersection $V_b \cap U_i$ with each standard chart of $\mathbb{P}^N(\mathbb{C})$ is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$. Moreover, we tacitly include in V_b small open sets which encircle points of the indeterminacy locus of f_1 which lie in $\mathcal{A}_{2,b}(\mathbb{C})$, leaving unchanged the properties of V_b. Define

$$T_b := \mathcal{A}_{2,b}(\mathbb{C}) \setminus V_b\,.$$

By choosing in a suitable way the size of V_b, we can assume that $p_r \in T_b$. Now, look at the abelian scheme $\mathcal{X} \to F_b$ and fix $z \in F_b(\mathbb{C})$. As explained in Equation (3) and Equation (4), there exists a simply connected open set $U_z' \subseteq F_b(\mathbb{C})$ in the complex topology containing z and a period map on U_z':

$$\mathcal{P}_{\mathcal{X}}^{(b)} = \left(\omega_{1,\mathcal{X}}^{(b)}, \ldots, \omega_{2g,\mathcal{X}}^{(b)}\right);$$

in other words we have holomorphic functions $\omega_{i,\mathcal{X}}^{(b)} : U_z' \to \mathbb{C}^g$ for $i = 1, \ldots, 2g$ which fix a basis of the corresponding lattice $\Lambda_{z'}$ for each $z' \in U_z'$. Thus, the family of open simply connected sets $\{U_z' : z \in T_b\}$ is a covering of T_b. Fixing a standard chart U_i which contains z, we can consider a simply connected open definable subset $U_z \subseteq U_z' \cap U_i$ which contains z and whose analytic closure D_z is contained in $U_z' \cap U_i$. In other words, we can consider an open covering $\{U_z \colon z \in T_b\}$, where each U_z is a simply connected open set with the following properties: its analytic closure $D_z \subseteq F_b \cap U_i$ is a definable compact set in the o-minimal structure $\mathbb{R}_{\text{an}}$ and all the period functions $\omega_{i,\mathcal{X}}^{(b)}$ with $i = 1, \ldots, 2g$ are well-defined as holomorphic functions on D_z. Since T_b is compact, it can be covered with finitely many small compact simply-connected sets of the type D_z.

Since $U_z' \subseteq F_b(\mathbb{C})$ is simply connected, we obtain notions of abelian logarithm $\ell_{\mathcal{X}}^{(b)}$ and Betti map $\beta_{\mathcal{X}}^{(b)} = \left(\beta_{1,\mathcal{X}}^{(b)}, \ldots, \beta_{2g,\mathcal{X}}^{(b)}\right)$ of the section $s_{\mathcal{X}}$ on each U_z' as explained in Equation (4); note that the abelian logarithm is a holomorphic function on each compact set D_z and the Betti map is described by the equation

$$\ell_{\mathcal{X}}^{(b)}(z) = \beta_{1,\mathcal{X}}^{(b)}(z)\omega_{1,\mathcal{X}}^{(b)}(z) + \cdots + \beta_{2g,\mathcal{X}}^{(b)}(z)\omega_{2g,\mathcal{X}}^{(b)}(z),$$

where the Betti coordinates $\beta_{i,\mathcal{X}}^{(b)}$ are real-analytic functions on each compact set D_z. In addition note that $\beta_{\mathcal{X}}^{(b)}$ doesn't have any critical points on T_b by construction (we have expressly removed them).

Summarizing: we have obtained the existence of finitely many simply connected compact sets D_i with $i = 1, \ldots, n_{\text{comp}}$ which are definable in the o-minimal structure $\mathbb{R}_{\text{an}}$ and where the Betti map $\beta_{\mathcal{X}}^{(b)}$ is definable in $\mathbb{R}_{\text{an}}$ and a submersion.

Remark 2.3. Fix $z \in T_b$. Observe that all the relevant functions (i.e. period functions, logarithms and Betti maps of $\mathcal{X} \to F_b$) are constant on each fiber $\mathcal{A}_{1,f_1(z)}$; in other words, they are independent of b. As a consequence, the number n_{comp} of compact sets D_i just constructed can be supposed to be constant with respect to b: in fact the D_i's are projected by f_1 onto a finite number of compact disks in S_1, where the Betti coordinates are real-analytic.

Let's proceed with some useful relabelling in order to simplify the notations, we put $\zeta := p_r$ and $n := n_r = \mathrm{ord}(s_{\mathcal{X}}(\zeta))$. Fix one of the previous compact sets, say D, which contains ζ. Note that T_b is not closed under the addition, but ζ is ensured to be contained in T_b by our previous discussions. Moreover, since $\mathcal{A}_{2,b} \cap \mathcal{A}'$ is not closed under the addition then ζ need not to have uniformly bounded height a priori. Anyway, bounded height was ensured by Proposition 2.1. By applying Proposition 1.7 to $\mathcal{X} \to F_b$ with $s = \zeta$, there exist two constants $c = c(g), c' = c'(g)$ which only depend on g such that

$$n \leq c[K(\zeta) : \mathbb{Q}]^{c'}.$$

Note that c' is exactly the constant defined in Equation (17). Up to multiplying c by $[K : \mathbb{Q}]^{c'}$ we obtain

$$n^{\frac{1}{c'}} \ll [K(\zeta) : K]. \tag{21}$$

On the other hand, recalling that the degree of the isogeny induced by the multiplication by m is m^{2g} and the condition $m^{g(2c'+1)} < n$, we deduce

$$[K(b) : K] = [K(\sigma_2(b)) : K] \leq m^{2g} \ll n^{\frac{2}{2c'+1}}. \tag{22}$$

By (21) and (22) we obtain

$$[K(\zeta) : K(b)] = \frac{[K(\zeta) : K]}{[K(b) : K]} \gg \frac{n^{\frac{1}{c'}}}{n^{\frac{2}{2c'+1}}} = n^{\frac{1}{c'(2c'+1)}}.$$

In this case of the proof we are now going to define a series of positive constants $c_0, c_1, \dots$ that we need keep until the end. Let's start with $c_0 := c'(2c' + 1)$. We can then take n_1 conjugates ζ_j with $j = 1, \dots, n_1 \gg n^{\frac{1}{c_0}}$ of ζ over $K(b)$. In this way, since $s_{\mathcal{X}}$ is defined over K we obtain torsion values $s_{\mathcal{X}}(\zeta_j)$ on $\mathcal{X}$ for all $j = 1, \dots, n_1$. Note that each ζ_j inherits the same properties as ζ: for example, it has uniformly bounded height since ζ does. Moreover, up to reduce the size of the open set V_b we can assume that all ζ_j's lie in T_b. Therefore, since the number n_{comp} of compact sets D_i is prescribed since the beginning (see Remark 2.3), by Proposition 1.10 we may assume that there exists a positive number $c_1 > 0$ depending only on the original data (it can be taken for instance equal to $1/(2n_{\mathrm{comp}})$) such that at least $c_1 n_1$ of these conjugates lie in a same compact set of the type D_i. From now on, we will denote by $\Omega = \Omega_b \subseteq A_{2,b}(\mathbb{C})$ the compact set (among the D_i's) just described. Hence, we may assume that Ω contains ζ_j for $j = 1, \dots, \lambda$ with

$$\lambda > c_2 n^{\frac{1}{c_0}}, \tag{23}$$

where the constant c_2 is uniform with respect to b. Recall that the Betti coordinates are well-defined and real-analytic in Ω.

With the same reasoning we carried out for the D_i's, we can decompose the compact set $\Delta \subseteq S_2(\mathbb{C})$ as a finite union of small definable compact sets A_j. Recall that the Betti coordinates $\beta^{(b)}_{i,\mathcal{X}}$ are real-analytic with respect to a variable z which varies in the corresponding compact set Ω_b. With the following construction we want to make the Betti coordinates real-analytic with respect to b too. To this regard, observe that for each point $\widetilde{b} \in S_2'$ and each point $\widetilde{p} \in \mathfrak{F} \cap \mathcal{A}'$ which satisfies $f_2(\widetilde{p}) = \widetilde{b}$ we can realize the same construction as above, thus we have corresponding objects for which we keep the analogous notations: for example we have corresponding integers numbers $\widetilde{n}, \widetilde{m}$, point $\widetilde{\zeta}$ and sets $\widetilde{\Omega}, T_{\widetilde{b}}$. Fix A_j and define the set

$$I_j := \left\{ \widetilde{b} \in A_j : \text{there exists } \widetilde{p} \in T_{\widetilde{b}} \text{ with } \widetilde{n} > \widetilde{m}^{g(2c'+1)} \right\}.$$

For each $\widetilde{b} \in I_j$ there exists a definable simply connected compact set $\widetilde{\Omega} \subseteq T_{\widetilde{b}}$ which contains $\gg n^{\frac{1}{c_0}}$ conjugates (with implicit constant independent of b). Fix b and $\widetilde{b}$ and take an analytic path $\alpha : [0,1] \to \mathcal{A}' \cap \mathcal{A}_1$ such that

$$\alpha(0) = \zeta, \qquad \alpha(1) = \widetilde{\zeta}.$$

For $t \in [0,1]$ denote by E_z a disk in $T_{f_2(\alpha(t))}$ centred in $\alpha(t)$ where $\beta^{(f_2(\alpha(t)))}_{i,\mathcal{X}}$ are real-analytic. Choose α such that $f_2(\alpha([0,1])) \subseteq A_j$. Since the Betti coordinates are uniform with respect to f_1-fibers (see Remark 2.3), the condition which ensures the Betti coordinates to be real-analytic with respect to both

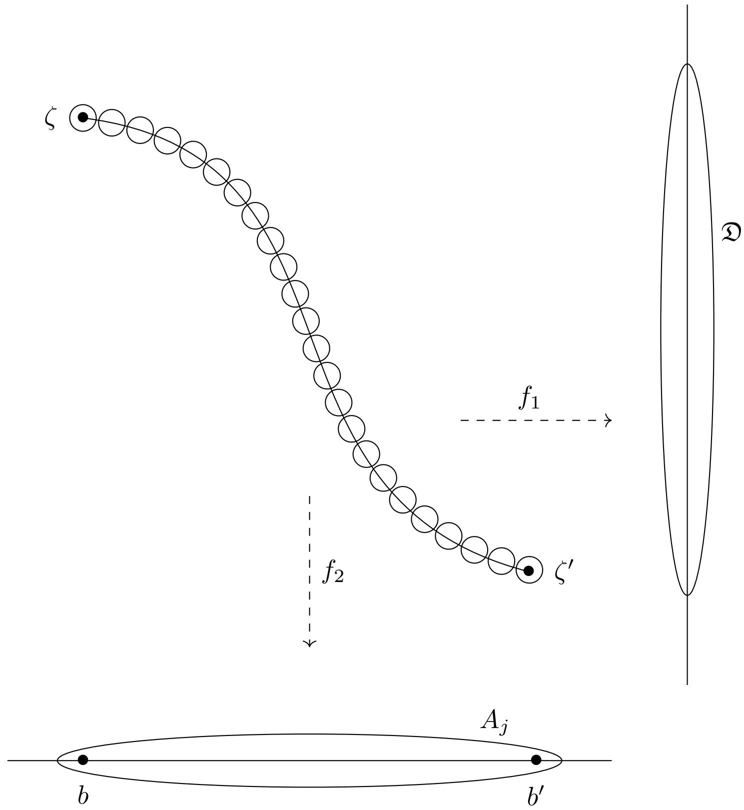

Figure 2: A visualization of the analytic path $\alpha\colon [0,1] \to \mathcal{A}' \cap \mathcal{A}_1$.

variables b and z is expressed by requiring that there exists a simply connected set $\mathfrak{D} \subseteq S_1(\mathbb{C})$ such that $f_1(\alpha([0,1])) \subseteq \mathfrak{D}$. By this motivation, fix $b \in A_j$ and consider the open set

$$\mathcal{U}_b := \left\{ b' \in A_j \colon \begin{array}{l} \exists \text{ analytic path } \alpha \text{ in } \mathcal{A}' \cap \mathcal{A}_1 \text{ such that} \\ \alpha(0) = \zeta,\ \alpha(1) = \zeta',\ f_1(\alpha([0,1])) \subseteq \mathfrak{D},\ f_2(\alpha([0,1])) \subseteq A_j \end{array} \right\},$$

where $\mathfrak{D} \subseteq S_1(\mathbb{C})$ is a fixed simply connected open set which makes $\mathcal{U}_b$ non-empty. We can replace the compact covering $\{A_j\}$ of Δ by a (finite) compact covering made with definable sets contained in $\mathcal{U}_b$ for any b.

Roughly speaking, we can assume that the compact sets are such that the Betti coordinates are real-analytic in the union

$$\bigcup_{b \in A_j} \{b\} \times \Omega_b \subseteq A_j \times \Omega_b \subseteq \mathbb{R}^{2g} \times \mathbb{R}^{2g}. \tag{24}$$

Note that if $b \notin f_2(\mathfrak{F} \cap \mathcal{A}')$, by Ω_b we mean one of the disks E_z defined above. Let us now consider the real-analytic variety defined in $\mathbb{R}^{2g}$ by

$$Z_b := \{\beta(z) \colon z \in \Omega_b\},$$

where $\beta(z) := \beta^{(b)}(z) := (\beta^{(b)}_{1,\mathcal{X}}(z), \ldots, \beta^{(b)}_{2g,\mathcal{X}}(z))$. Observe that thanks to the main property of the Betti map each ζ_j, for $j = 1, \ldots, \lambda$, gives a rational point $\beta(\zeta_j)$ of denominator $\geq n > m^{\frac{1}{c_0}}$ on Z_b. Some of these rational points might coincide, but since the ζ_j's lie in $\mathcal{A}_{2,b} \setminus L$ we can apply Proposition 1.2 and conclude that we have a number of distinct rational points which is $\gg \lambda$, say $\geq c_3\lambda$. Moreover they have height $\ll n$, say $\leq c_4 n$. The constants c_3, c_4 depends only on the involved compact sets, which were previously fixed.

Remark 2.4. Let's explain more in detail why c_4 is uniform: on each compact D_z the Betti map attains a maximum, but the denominators of $\beta(\zeta_j)$ are bounded, hence we get a uniform constant for each compact. Since the number of compact sets was previously fixed we get a uniform constant c_4.

Since the Betti coordinates are real-analytic in the union described in Equation (24) and by Remark 1.1, for each j we have a definable family

$$Z := \bigcup_{b\in A_j} \{b\} \times Z_b \subseteq \mathbb{R}^{2g} \times \mathbb{R}^{2g},$$

where Z_b are the fibers. Denote by H the usual absolute height on the projective space and define

$$Z_b(\mathbb{Q}, T) = \{p \in Z_b(\mathbb{Q}) \mid H(p) \leq T\}, \qquad N(Z_b, T) := \#Z_b(\mathbb{Q}, T).$$

Therefore by Equation (23) we have

$$N(Z_b, c_4 n) \geq c_2 c_3 n^{\frac{1}{c_0}}. \tag{25}$$

On the other hand by [25, Theorem 1.9], for any $\varepsilon > 0$ there exists a constant $c(Z, \varepsilon)$ such that

$$N(Z_b - Z_b^{\mathrm{alg}}, T) \leq c(Z, \varepsilon) T^{\varepsilon}, \tag{26}$$

where Z_b^{alg} and $Z_b - Z_b^{\mathrm{alg}}$ are the algebraic and the transcendental part of Z_b, respectively. Observe that the constant is independent of $b \in A_j$.

We now show that the algebraic part of Z_b is empty. This is a rather standard procedure that employs the algebraic independence of coordinates of the logarithm with respect to the periods (see for instance [21, Lemma 6.2]). Anyway, we recall the main steps for the sake of clarity, provided having the following important elucidation in mind.

Remark 2.5. We point out that the argument described below works only for $g \geq 2$ since we need at least two components of the abelian logarithm. Nevertheless, the case $g = 1$ can be treated with small modifications in the construction of the family Z: indeed it is enough to consider two auxiliary abelian schemes instead of $\mathcal{X}$ only. In this way we have two Betti maps and two logarithms (each of them with one component). Then we apply the same procedure described above on the new definable family Z that now lives in $\mathbb{R}^2 \times \mathbb{R}^4$. For the details of the case $g = 1$ the reader can check directly [9, Theorem 1.1] where, what we have just described in this remark, is exactly the technique carried out.

If the algebraic part is non-empty there is a real-algebraic arc γ contained in Z_b^{alg}. In what follows we omit the dependence on b and $\mathcal{X}$ to simplify the notation. Consider the real-analytic set $U := \beta^{-1}(\gamma) \subseteq \Omega$. Since γ is a real algebraic arc and the points $\beta(z)$ with $z \in U$ satisfy the defining real algebraic equations of γ, then the Betti coordinates β_i are algebraically dependent over $\mathbb{C}(S)$ when restricted to U. Moreover, this also implies that the field generated by the $2g$ Betti coordinates (when restricted to U) over $\mathbb{C}(S)$ has transcendence degree at most 1; in other words, any two of the Betti coordinates verify an algebraic equation over $\mathbb{C}(S)$. Thus, we have two cases: either the $2g$ Betti coordinates restricted to U all depend algebraically on any of them which is not constant, or otherwise they are all constant.

In the first case: let's denote with t the transcendence degree over $\mathbb{C}(S)$ of the coordinates of the period functions $\omega_i = (\omega_{i1}, \ldots, \omega_{ig})$, for $i = 1, \ldots, 2g$; clearly $t \leq 2g^2$. Here, all functions are intended to be restricted to U, unless otherwise specified. Therefore, the field generated by ω_i, β_i over $\mathbb{C}(S)$ has transcendence degree at most $t+1$ and contains coordinates of the abelian logarithm ℓ. This implies that coordinates of the abelian logarithm are algebraically dependent over $\mathbb{C}(S)(\{\omega_{ij}\})$. However all these functions are locally holomorphic, so the dependence would hold identically on their domain Ω, which violates the independence result [1, Theorem 3] of André (see also [21, Lemma 5.1]).

In the second case, i.e. when the Betti coordinates are all constant when restricted to U, they are constant on their domain Ω by the same principle as above. This implies that the corresponding sections are identically torsion, which is a contradiction.

Finally, consider the set

$$Z_b(\mathbb{Q}, c_4 n) = \{p \in Z_b(\mathbb{Q}) \colon H(p) \leq c_4 n\},$$

where c_4 is as above. Taking $\varepsilon = 1/(2c_0)$, by Equation (25) and Equation (26), we obtain

$$c_2 c_3 n^{\frac{1}{c_0}} \leq N(Z_b, c_4 n) \leq c(Z)(c_4 n)^{\frac{1}{2c_0}}$$

where all constants $c(Z), c_2, c_3, c_4$ are uniform with respect to $b \in A_j$. This implies $n^{\frac{1}{2c_0}} \leq c_5$, that is $n^{\frac{1}{2c'+1}} \leq c_5^{2c'}$. In particular, this implies

$$m < n^{\frac{1}{g(2c'+1)}} \leq c_5^{\frac{2c'}{g}}.$$

This estimate holds uniformly with respect to $b \in A_j$. Since we have a finite number of compact sets A_j which cover Δ, we obtain a global bound for $m \in \mathfrak{O}'$ on Δ. By Proposition 1.10, each torsion value of S_2' has at least a conjugate in Δ and this implies that the last estimate holds uniformly for $b \in f_2(\mathfrak{F} \cap \mathcal{A}')$.

2.1.4 Second case

Let's define

$$\mathfrak{O}'' := \{m \in \mathfrak{O} \colon n_r \leq m^{g(2c'+1)} \ \forall p_r \in \mathcal{A}_{2,b} \setminus f_1^{-1}(\mathcal{C}(\beta_1))\} .$$

We prove that the set $\mathfrak{O}''$ is finite. Suppose by contradiction they are infinitely many. Here, we complete our sequence of positive constants by c_6, c_7.

Let's continue with our point $p \in \mathfrak{F} \cap \mathcal{A}'$ such that $b = f_2(p) \in \Delta$, but now suppose $m \in \mathfrak{O}''$. Consider the above covering $\{A_j\}$ of Δ: we work again in one of those compact sets that contains b, which we now call A. Consider again the abelian scheme $\mathcal{X} \to F_b$ and decompose T_b as a finite union of compact subsets $\{D_i\}$ as above and consider the analogous definable family Z with fibers Z_b. Keeping the same notation as above, by Proposition 2.2 and since we are supposing $\mathfrak{O}''$ to be unbounded, there exists $m \in \mathfrak{O}''$ and $\tau \in \Sigma_{p,A}$ such that there are at least $m/2$ elements of the type $f_1(p^\tau + r\sigma_2(b^\tau))$ which don't lie in $f_1(V_{b^\tau})$; fix this K-embedding $\tau \in \Sigma_{p,A}$.

For any $r = 0, \ldots, m-1$ define $z_r := p^\tau + r\sigma_2(b^\tau)$. Then, for the previous $\geq m/2$ values of r the point z_r is a torsion value of the scheme $\mathcal{X} \to F_b$ whose order is n_r. Precisely, in the counting of the $\geq m/2$ values we have to exclude points which lie into the indeterminacy locus of f_1, but they are a finite number which only depends on the initial data. Equivalently, there is a subset $J \subset \{0, \ldots, m-1\}$ with $\#J \gg m$ such that for any $r \in J$ the coordinates $\beta^{(b)}_{1,\mathcal{X}}(z_r), \ldots, \beta^{(b)}_{2g,\mathcal{X}}(z_r)$ are rational numbers, but with denominator $\leq m^{g(2c'+1)}$ thanks to the hypotheses of the "second case".

As above, taking δ small enough we can assume $z_r \in T_b(\mathbb{C})$ for any $r \in J$. Hence, by applying Proposition 1.2, we obtain a number of distinct rational points $\beta_{\mathcal{X}}(z_r)$ which is $\gg m$, say $\geq c_6 m$. Again, they have height $\ll m^{g(2c'+1)}$, say $\leq c_7 m^{g(2c'+1)}$ for a uniform constant (see Remark 2.4). Therefore we get

$$N(Z_b, c_7 m^{g(2c'+1)}) \geq c_6 m. \tag{27}$$

On the other hand by [25, Theorem 1.9]:

$$N(Z_b - Z_b^{\mathrm{alg}}, c_7 m^{g(2c'+1)})) \leq c(Z,\varepsilon) c_7^{\varepsilon} m^{\varepsilon g(2c'+1)}, \quad \text{with } \varepsilon < \frac{1}{g(2c'+1)} . \tag{28}$$

At this point by reasoning exactly as in the previous case it is possible to show that Z_b^{alg} is empty. Also here we have to appeal to Remark 2.5: the case $g = 1$ needs a slightly different approach with a definable family in $\mathbb{R}^2 \times \mathbb{R}^4$; again, all the details are in [9]. Therefore from Equations (27) and (28) (and the choice of ε) we finally obtain:

$$m \leq \left(\frac{c(Z,\varepsilon) c_7^{\varepsilon}}{c_6}\right)^{\frac{1}{1-\varepsilon g(2c'+1)}} .$$

This bound holds uniformly on A and since $\{A_j\}$ is a fixed finite covering, then we get a uniform bound for $m \in \mathfrak{O}''$ on the whole Δ. By Proposition 1.10 applied as at the end of the first case, we get the contradiction.

2.2 Some comments on the shape of Z_1 and Z_2

We list some subsets that are contained in the sets Z_1 and Z_2 of Theorem 0.2. They are essentially the closed subsets that already show up in Equation (15). We removed those sets at the beginning of the proof (see Remark 0.5), so they consequently fall inside Z_1 and Z_2:

(i) The locus of $\overline{S}_1(\mathbb{C})$ on which the two families have coinciding fibers lies in Z_1 by assumptions (see beginning of Section 2.1.1).

(ii) The locus of $\overline{S}_1(\mathbb{C})$ on which the Betti map is not a submersion is in Z_1.

(iii) Let $C_i := S_i \setminus S_i'$ be the complementary sets of the open dense sets with uniform bounded height arising from the height inequality. Then $C_i \subseteq Z_i$.

(iv) The subset Δ_i is contained in Z_i.

Remark 2.6. Thanks to the previous considerations, we get explicit expressions of Z_1 and Z_2 as it follows:

$$Z_1 = \Delta_1 \cup C_1 \cup E \cup \mathcal{C}(\beta_1) \cup f_1(\mathcal{A}_1 \setminus \mathcal{A}_2), \qquad Z_2 = \Delta_2 \cup C_2 \cup \mathfrak{O}.$$

In turns, $\mathfrak{O}$ is contained in

$$\sigma_2^{-1}\left(\bigcup_{N\leq C}\mathcal{A}_2[N]\right)$$

but unfortunately the constant C is implicit.

When $1=\dim\overline{S}_1=\dim\overline{S}_2=g$ we get $\overline{S}_1=\overline{S}_2=\mathbb{P}^1$, then we denote both bases simply by S. In this case the subsets C_i are actually empty for obvious reasons and the locus $f_1^{-1}(E)$ can be equivalently taken as a finite set of f_2-fibers. Moreover also the closed set of the item (iii) doesn't need to be removed, in fact since the Betti map is not constant and the base S is an irreducible curve, even if there are critical points, the fibers of β_1 are all finite, hence Gabrielov theorem holds everywhere.

Finally the following proposition shows that, still in the case $1=\dim S=g$, all points of $\mathfrak{F}\cap f_1^{-1}(\Delta_1)$ are contained in some $f_2^{-1}(Z)$ for a proper Zariski closed subset Z. In other words we recover the stronger result $\mathfrak{F}\subseteq f_2^{-1}(Z)$ of [9].

Proposition 2.7. *Let $1=\dim S=g$, then there exists a proper closed Zariski subset $Z\subset S(\mathbb{C})$ such that:*

$$\mathfrak{F}\cap f_1^{-1}(\Delta_1)\subseteq f_2^{-1}(Z).$$

Proof. Assume that Δ_1 has cardinality n. Put $Y=\mathfrak{F}\cap f_2^{-1}(Z_2)$ where Z_2 is the proper Zariski closed subset arising from Theorem 0.2. So there exists a finite set $W\subset S$ such that $f_2(Y)=W$. By Bézout theorem we know that $\#(\mathcal{A}_{2,s}(\mathbb{C})\cap f_1^{-1}(\Delta_1))\leq 9n$. Let's put $H=\mathfrak{F}\cap f_1^{-1}(\Delta_1)$ and let's consider the following partition of H:

$$H_1:=\{p\in H:\#(O(p))\leq 9n\},\qquad H_2:=\{p\in H:\#(O(p))>9n\}.$$

Note that

$$f_2(H_1)\subseteq\sigma_2^{-1}\left(\bigcup_{N=1}^{9n}\mathcal{A}[N]\right)$$

which is a finite set W_1; so let us focus on H_2. Fix $p\in H_2$ and observe that there exists $r\in\mathbb{N}$ such that $t_2^r(p)\notin f_1^{-1}(\Delta_1)$. If not, we would have $O(p)=\{t_2^r(p):r\in\mathbb{N}\}\subseteq f_1^{-1}(\Delta_1)\cap\mathcal{A}_{2,s}(\mathbb{C})$ where $\#(O(p))>9n$ and this is a contradiction. So $f_2(t_2^r(p))=f_2(p)\in W$ since t_2 acts on the f_2-fibers and $t_2^r(p)\in Y$. The claim follows if we put $Z=W\cup W_1$. □

References

[1] Y. André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Mathematica*, 82(1):1–24, 1992.

[2] Y. André, P. Corvaja, and U. Zannier. The Betti map associated to a section of an abelian scheme. *Invent. Math.*, 222(1):161–202, 2020. With an appendix by Z. Gao.

[3] J. L. Barbosa, L. Birbrair, M. do Carmo, and A. Fernandes. Globally subanalytic CMC surfaces in $\mathbb{R}^3$. *Electron. Res. Announc. Math. Sci.*, 21:186–192, 2014.

[4] D. Bertrand. Revisiting Manin's theorem of the kernel. *Ann. Fac. Sci. Toulouse Math. (6)*, 29(5):1301–1318, 2020.

[5] E. Bierstone and P.D. Milman. Semianalytic and subanalytic sets. *Inst. Hautes Études Sci. Publ. Math.*, (67):5–42, 1988.

[6] S. Cantat and R. Dujardin. Finite orbits for large groups of automorphisms of projective surfaces. *Compos. Math.*, 160(1):120–175, 2024.

[7] S. Cantat, Z. Gao, P. Habegger, and J. Xie. The geometric Bogomolov conjecture. *Duke Math. J.*, 170(2):247–277, 2021.

[8] P. Corvaja, D. Masser, and U. Zannier. Torsion hypersurfaces on abelian schemes and Betti coordinates. *Math. Ann.*, 371(3-4):1013–1045, 2018.

[9] P. Corvaja, J. Tsimerman, and U. Zannier. Finite orbits in surfaces with a double elliptic fibration and torsion values of sections, 2023, 2302.00859.

[10] P. Corvaja and U. Zannier. Unramified sections of the Legendre scheme and modular forms. *J. Geom. Phys.*, 166:Paper No. 104266, 26, 2021.

[11] J. F. Davis and P. Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[12] V. Dimitrov, Z. Gao, and P. Habegger. Uniformity in Mordell-Lang for curves. *Ann. of Math. (2)*, 194(1):237–298, 2021.

[13] G. Faltings, G. Wüstholz, F. Grunewald, N. Schappacher, and U. Stuhler. *Rational points*. Aspects of Mathematics, E6. Friedr. Vieweg & Sohn, Braunschweig, third edition, 1992.

[14] Z. Gao. Generic rank of Betti map and unlikely intersections. *Compos. Math.*, 156(12):2469–2509, 2020.

[15] Z. Gao and P. Habegger. Heights in families of abelian varieties and the geometric Bogomolov conjecture. *Ann. of Math. (2)*, 189(2):527–604, 2019.

[16] Z. Gao and P. Habegger. The relative Manin-Mumford conjecture, 2023, 2303.05045.

[17] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[18] Yu. I. Manin. Rational points on algebraic curves over function fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(2):447–448, 1989.

[19] D. Masser and U. Zannier. Torsion points on families of squares of elliptic curves. *Math. Ann.*, 352(2):453–484, 2012.

[20] D. Masser and U. Zannier. Torsion points on families of products of elliptic curves. *Advances in Mathematics*, 259:116–133, 2014.

[21] D. Masser and U. Zannier. Torsion points, Pell's equation, and integration in elementary terms. *Acta Math.*, 225(2):227–313, 2020.

[22] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, third edition, 1994.

[23] Y. Peterzil and S. Starchenko. Uniform definability of the Weierstrass $\wp$ functions and generalized tori of dimension one. *Selecta Math. (N.S.)*, 10(4):525–550, 2004.

[24] J. Pila. Rational points of definable sets and results of André-Oort-Manin-Mumford type. *Int. Math. Res. Not. IMRN*, (13):2476–2507, 2009.

[25] J. Pila and A. Wilkie. The rational points of a definable set. *Duke Mathematical Journal*, 133, 06 2006.

[26] M. Raynaud. Hauteurs et isogénies. *Astérisque*, (127):199–234, 1985. Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell.

[27] G. Rémond. Conjectures uniformes sur les variétés abéliennes. *Q. J. Math.*, 69(2):459–486, 2018.

[28] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[29] X. Yuan and S.-W. Zhang. Adelic line bundles on quasi-projective varieties, 2023, 2105.13587.

[30] U. Zannier. *Some problems of unlikely intersections in arithmetic and geometry*, volume 181 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012. With appendixes by David Masser.

P. Dolce, Westlake University, China
E-mail address: `dolce@westlake.edu.cn`

F. Tropeano, Università della Calabria, Italy
E-mail address: `francesco.tropeano@unical.it`